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A New Bivariate Shock Model Covering All Degrees of Dependencies

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Abstract

This paper presents a bivariate distribution that improves the Marshall-Olkin exponential shock model. The new construction method enhances the model's capacity to include a common joint shock across components, making it especially suitable for reliability and credit risk assessments. The model features a single component and supports negative dependence structures. We investigate the key dependence properties and conduct a stress-strength analysis. After evaluating the performance of the parameter estimator, chemical engineering data is analyzed.

Keywords: Dependence; Marshall-Olkin model; Shock model.
Mathematics Subject Classification (2010): 60E05, 60E15, 62N05.

Introduction

The univariate exponential distribution is known for its applications in different fields such as reliability, telecommunication, hydrology, medical sciences and environmental science; see, e.g., Balakrishnan (1). Several bivariate and multivariate extensions have been proposed in the literature (Lai and Balakrishnan (2), Chapter 10), with a significant multivariate extension introduced Marshall and Olkin (3) through a shock model. For three independent exponential random variables $T_1 \sim E(\theta_1)$, $T_2 \sim E(\theta_2)$, and $T_{12} \sim E(\theta_{12})$, the Marshall-Olkin (MO) shock model is derived from the stochastic representation

$$(X, Y) = (\min\{T_1, T_{12}\}, \min\{T_2, T_{12}\}), \quad (1)$$

with the joint survival function given by

$$S(x, y) = \exp\{-\theta_1 x - \theta_2 y - \theta_{12}\max(x, y)\}. \quad (2)$$

Due to the common shock identified by T_{12} in (1), we have $P(X = Y) = \frac{\theta_{12}}{\theta_1+\theta_2+\theta_{12}}$, if $\theta_{12} > 0$, the distribution (1), has a singular component along the line $\{x = y\}$. Therefore, the MO exponential distribution has both singular and continuous parts in its density and covers a positive dependence structure.

Numerous studies have examined models and modifications based on the foundational work of Marshall and Olkin (3), particularly in reliability, finance, actuarial science, and credit risk (e.g., Cherubini et al. (4), Lindskog and McNeil (5)). Recently, Cherubini and Mulinacci (6) emphasized the MO model's importance and adaptability for analyzing systemic crises in credit risk and financial contexts. They noted that the MO model captures unobserved shocks affecting individuals either independently or collectively, using common shocks to explain simultaneous defaults within clusters influenced by the same factor. The model also maintains the marginal exponential distribution for observed default times.

Several bivariate and multivariate extensions of the exponential distribution have been developed for reliability applications. For instance, Esary and Marshall

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(7) characterized a multivariate exponential distribution and established conditions for positive dependence among distributions with exponential minima. Raftery (8) introduced a continuous multivariate exponential distribution that accommodates various correlation structures while achieving Fréchet bounds in the bivariate case. A multivariate exponential distribution based on the limiting behavior of normalized maxima or minima was introduced by Tawn (9). Lin et al. (10) examined a shared-load model of the multivariate exponential distribution for dependent redundancies. Constant failure rate multivariate exponential distributions were defined by Basu and Sun (11), while Gomez et al. (12) introduced the multivariate power exponential distribution. An analytical method for assessing the reliability of coherent systems with dependent components based on the MO model was proposed by Cui and Li (13). Additionally, Fan et al. (14) developed a multivariate exponential survival tree procedure utilizing a score test statistic from a parametric exponential frailty model, and Kundu and Gupta (15) presented parameter estimation for a new bivariate exponential distribution using an EM algorithm. Generalized bivariate MO distributions, with the common MO model as a special case, were investigated by Li and Pellerey (16). Bayesian estimation for the MO bivariate Weibull distribution was conducted by Kundu and Gupta (17), and Bayramoglu and Ozkut (18) applied the MO model considering system structure. A multivariate proportional reversed hazard model derived from the MO copula was discussed by Kundu et al. (19), while Cha and Badia (20) proposed a multivariate exponential distribution model based on dependent dynamic shock models. A multivariate weighted exponential distribution for failure time data analysis was developed by Al-Mutairi et al. (21). Recently, Mohtashami-Borzadaran et al. (22) enhanced the MO shock model by incorporating a distortion function, broadening its applicability. Additionally, El Ktaibi et al. (23) introduced a bivariate copula based on a bivariate exponential distribution with negative dependence, Bentoumi et al. (24) developed by using the counter-monotonic shock model. Lee and Cha (25) developed a new class of continuous bivariate distributions based on a shock model. Agrawal et al. (26) proposed a bivariate distribution for modelling competing risks data with singularity originating from a shock model. A variant of the bivariate Poisson common shock model was presented in Genest et al. (27).

From a shock model perspective, the MO model (1) has limitations due to the assumption of shock equality in the common shock T_{12}, suggesting that T_{12} is likely equal for components X_1 and X_2. Our new model addresses this limitation by allowing for a random percentage of common stock on each component. Most existing bivariate and multivariate exponential distribution extensions exhibit positive dependence structures, with negative dependence structures being rare.

In this paper, which was first posted at arXiv Mohtashami-Borzadaran et al. (28), we propose a new bivariate exponential shock model that accommodates negative dependence structures as well. Section 2 presents the new shock model and its flexibility compared to the MO model. Section 3 outlines the model's main properties, including dependence structure, association measures, tail dependence measures, and stress-strength index. Section 4 focuses on parameter estimation for the new model, which poses challenges due to its singular component, followed by a performance analysis of the estimators. Section 5 applies the model to real data, demonstrating its superior fit.

Materials and Methods

Consider three independent exponential random variables $T_1 \sim E(\theta_1)$, $T_2 \sim E(\theta_2)$, and $T_{12} \sim E(\theta_{12})$, along with an independent uniform random variable $U \sim U(0,1)$, which is independent of T_1, T_2, and T_{12}. Let α_{12} (taking values ± 1) be the dependence structure of the model where $\alpha_{12} = +1$ concludes positive and $\alpha_{12} = -1$ gives negative dependence structure. When $\alpha_{12} = +1$, set $T_{12}^*(\alpha_{12}) = T_{21}^*(\alpha_{12}) = F_{T_{12}}^{-1}(U)$ or $T_{12}^*(\alpha_{12}) = T_{21}^*(\alpha_{12}) = F_{T_{12}}^{-1}(1-U)$. If $\alpha_{12} = -1$, put $T_{12}^*(\alpha_{12}) = F_{T_{12}}^{-1}(U), T_{21}^*(\alpha_{12}) = F_{T_{12}}^{-1}(1-U)$ or $T_{12}^*(\alpha_{12}) = F_{T_{12}}^{-1}(1-U), T_{21}^*(\alpha_{12}) = F_{T_{12}}^{-1}(U)$ where $F_{T_{12}}$ is the corresponding distribution function of T_{12}. Then, the bivariate MO random vector (R,S) covering all degree of dependence is

$$(R,S) = (\min\{T_1, T_{12}^*(\alpha_{12})\}, \min\{T_2, T_{21}^*(\alpha_{12})\}). \quad (3)$$

Clearly, when $\alpha_{12} = +1$, the vector (R,S) reduces to

$$(R,S) = (\min\{T_1, T_{12}\}, \min\{T_2, T_{12}\}),$$

which is the well-known MO model given in Marshall and Olkin (3) that has positive dependence structure. When $\alpha_{12} = -1$, the random vector (R,S) gives a new MO model with negative dependence structure (see Proposition 2.2) which is called the bivariate negative MO model, denoted by $BNMO(\theta_1, \theta_2, \theta_{12})$. This model is obtained by

$$(R,S) = \left(\min\{T_1, F_{T_{12}}^{-1}(U)\}, \min\{T_2, F_{T_{12}}^{-1}(1-U)\}\right), \quad (4)$$

or

$$(R,S) = \left(\min\{T_1, F_{T_{12}}^{-1}(1-U)\}, \min\{T_2, F_{T_{12}}^{-1}(U)\}\right). \quad (5)$$

Throughout this paper, we focus on the (R,S) given in (4).

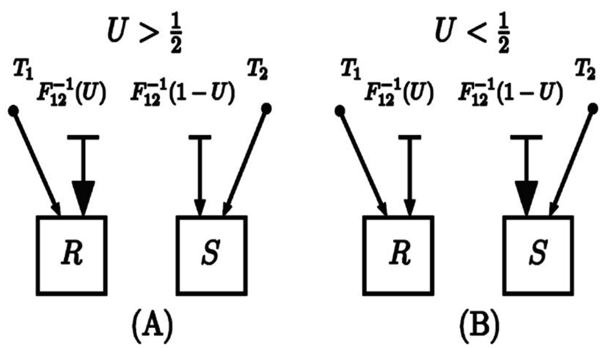


Figure 1. Shock models based on the construction in (4)

A similar construction for a bivariate Poisson model has been given by Genest et al. (27).

The interpretation for this construction is different from the well-known MO model. Consider Figure 1 given based on the relation (4). If $U > \frac{1}{2}$, then the dependent shock is more likely to be powerful on the first component R (Figure 1 (**A**)) and, if $U < \frac{1}{2}$, the dependent shock is more likely to be powerful on S (Figure 1 (**B**)). The survival function of both vectors (4) and (5) for $\theta_1, \theta_2, \theta_{12} > 0$ is

$$\begin{aligned} \bar{F}_{R,S}(r,s) &= P\left(T_1 > r, T_2 > s, F_{T_{12}}(r) < U < 1 - F_{T_{12}}(s)\right), \\ &= e^{-\theta_1 r} e^{-\theta_2 s} \max\{e^{-\theta_{12} r} + e^{-\theta_{12} s} - 1, 0\}. \end{aligned} \tag{6}$$

This model has a singular part at $e^{-\theta_{12} r} + e^{-\theta_{12} s} = 1$. The probability density function of (4) when $e^{-\theta_{12} r} + e^{-\theta_{12} s} > 1$ is

$$f_{R,S}(r,s) = e^{-\theta_1 r - \theta_2 s}\left(\theta_2(\theta_1 + \theta_{12}) e^{-\theta_{12} r} + \theta_1(\theta_2 + \theta_{12}) e^{-\theta_{12} s} - \theta_1 \theta_2\right). \tag{7}$$

The following statement gives the probability of the singular part.

Proposition 2.1. Suppose $(R,S) \sim BNMO(\theta_1, \theta_2, \theta_{12})$ and let $\alpha := \frac{\theta_{12}}{\theta_1 + \theta_{12}}$, $\beta := \frac{\theta_{12}}{\theta_2 + \theta_{12}}$. Then

$$P\left(e^{-\theta_{12} R} + e^{-\theta_{12} S} = 1\right) = Beta\left(\frac{1}{\beta}, \frac{1}{\alpha}\right),$$

where $Beta(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$.

Proof. Based on (Joe (29), Theorem 1.1, p. 15), for $e^{-\theta_{12} r} + e^{-\theta_{12} s} = 1$, we have

$$1 - \int_0^{g(r)} f_{S|R}(s|r) ds = \frac{\theta_{12}}{\theta_1 + \theta_{12}} (1 - \exp\{-\theta_{12} r\})^{\theta_2/\theta_{12}},$$

where $g(r) = \frac{-1}{\theta_{12}} \ln(1 - \exp\{-\theta_{12} r\})$. Let $h(r) = 1 - \int_0^{g(r)} f_{S|R}(s|r) ds$. By using the construction in (4) and Theorem 1.1 in Joe (27), we get

$$\begin{aligned} P\left(e^{-\theta_{12} R} + e^{-\theta_{12} S} = 1\right) &= \int_0^{+\infty} h(r) f_R(r) dr \\ &= \int_0^{+\infty} \theta_{12} \left[1 - exp(-\theta_{12} r)\right]^{\frac{\theta_2}{\theta_{12}}} \exp(-(\theta_1 + \theta_{12}) r) dr \\ &= Beta\left(\frac{1}{\beta}, \frac{1}{\alpha}\right). \end{aligned} \tag{8}$$

Remark 2.1. If $\theta_1 = \theta_2 = \theta_{12}$ or equivalently $\alpha = \beta = \frac{1}{2}$, then

$$P\left(e^{-\theta_{12} R} + e^{-\theta_{12} S} = 1\right) = \frac{1}{6}.$$

The survival copula associated with (6) is achieved by

$$\hat{C}(u,v) = u^{1-\alpha} v^{1-\beta} \max\{u^\alpha + v^\beta - 1, 0\}, \quad \alpha, \beta, u, v \in (0,1), \tag{9}$$

and the corresponding copula density is given as

$$\begin{aligned} c(u,v) &= (1-\alpha) u^{-\alpha} + (1-\beta) v^{-\beta} - (1-\alpha)(1-\beta) u^{-\alpha} v^{-\beta} I_{\{u^\alpha + v^\beta - 1 > 0\}}(u,v) \\ &\quad + Beta\left(\frac{1}{\alpha}, \frac{1}{\beta}\right) I_{\{u^\alpha + v^\beta - 1 = 0\}}(u,v), \quad \alpha, \beta, u, v \in (0,1), \end{aligned} \tag{10}$$

where $I_A(u,v)$ is an indicator function getting 1 if $(u,v) \in A$ and zero elsewhere.

Remark 2.2. El Ktaibi et al. (23) introduced only the survival copula (9) in parallel with this paper in an almost similar way to the present study by defining $\theta := \alpha = \beta$. They used four independent exponential random variables that have a common parameter, which is the factor that creates the dependency. One part of this parameter ultimately appears as the dependence parameter of the survival copula. Instead, this paper uses three independent exponential random variables with different parameters, and one of them is included as a factor for establishing dependence in the model (1). Therefore, unlike, El Ktaibi et al. (23) the present study uses a common random variable instead of a parameter to develop the dependency. So, the results of El Ktaibi et al. (23) are a special case of the present study with only one dependence parameter.

1. Some distributional properties

In this section, we present some properties of BNMO model such as dependence structure, association measures, tail dependence measures and stress-strength index.

1.1. Dependence structure

Let (X,Y) be a random vector with a survival function $\bar{F}$. The pair (X,Y) is said to be the right corner set decreasing, denoted by $RCSD(X,Y)$, whenever for any $x_1 < x_2$ and $y_1 < y_2$ we have

$$\bar{F}(x_1,y_1)\bar{F}(x_2,y_2) - \bar{F}(x_1,y_2)\bar{F}(x_2,y_1) \leq 0,$$

that is equivalent to

$$\frac{\partial^2}{\partial r\,\partial s}\ln\left(\bar{F}(r,s)\right) \leq 0.$$

$RCSD(X,Y)$ implies negative dependence structures like $RTD(X|Y)$, $RTD(Y|X)$ and $NQD(X,Y)$ (for more information see Nelsen (30)). The following statement specifies the dependence structure of the proposed model.

Proposition 2.2. If $(R,S) \sim BNMO(\theta_1,\theta_2,\theta_{12})$, then we have $RCSD(R,S)$.

Proof. For all $\theta_1,\theta_2,\theta_{12} \in R$, we obtain

$$\frac{\partial^2}{\partial r\,\partial s}\ln\left(\bar{F}_{R,S}(r,s)\right) = \frac{-\theta_{12}^2 e^{-\theta_{12}r-\theta_{12}s}}{(1-e^{-\theta_{12}r}-e^{-\theta_{12}s})^2} \leq 0,$$

which implies $RCSD(R,S)$ and the proof is complete.

The random vectors (X_1,Y_1) and (X_2,Y_2) can be compared in terms of their dependence structure via the upper orthant (UO) order. For any two vectors such as $(X_1,Y_1),(X_2,Y_2)$, we say (X_1,Y_1) is less than (X_2,Y_2) in UO order and write $(X_1,Y_1) \prec_{UO} (X_2,Y_2)$ whenever $\bar{F}_{X_1,Y_1}(x,y) \leq \bar{F}_{X_2,Y_2}(x,y)$ for all x,y.

Proposition 2.3. Let $(R,S) \sim BNMO(\theta_1,\theta_2,\theta_{12})$ and $(R',S') \sim BNMO(\theta_1,\theta_2,\theta_{12}')$. If $\theta_{12} \leq \theta_{12}'$ then $(R',S') \prec_{UO} (R,S)$.

Proof. For every $r,s,\theta_1,\theta_2 > 0$ and $\theta_{12} \leq \theta_{12}'$, we have

$$\begin{aligned}\bar{F}_{R,S}(r,s) &= e^{-\theta_1 r}e^{-\theta_2 s}\left(e^{-\theta_{12}r}+e^{-\theta_{12}s}-1\right)\\ &\geq e^{-\theta_1 r}e^{-\theta_2 s}\left(e^{-\theta_{12}'r}+e^{-\theta_{12}'s}-1\right)\\ &\geq \bar{F}_{R',S'}(r,s).\end{aligned}$$

Hence, $(R',S') \prec_{UO} (R,S)$ and this completes the proof.

1.2. Association measures and tail dependence

Two common measures of concordance between continuous random variables X and Y are Kendall's tau (τ) and Spearman's rho (ρ_s). In the following, after some elementary (but tedious) algebra, we provide explicit expressions for these measures based on the survival function (6).

Proposition 2.4. If $(R,S) \sim BNMO(\theta_1,\theta_2,\theta_{12})$ with the survival function $\bar{F}_{R,S}$ in (6), then we have

$$\begin{aligned}\tau &= 4E\left(\bar{F}_{R,S}(R,S)\right)-1\\ &= -2ab,\end{aligned}$$

and

$$\begin{aligned}\rho_s &= 12\int_{(0,\infty)^2}\left[\bar{F}_{R,S}(r,s)-\bar{F}_R(r)\bar{F}_S(s)\right]f_R(r)f_S(s)\,dr\,ds\\ &= -3ab,\end{aligned}$$

where $a = \frac{\theta_{12}}{2\theta_1+\theta_{12}}$ and $b = \frac{\theta_{12}}{2\theta_2+\theta_{12}}$.

The plots of τ and ρ_s with respect to a and b are

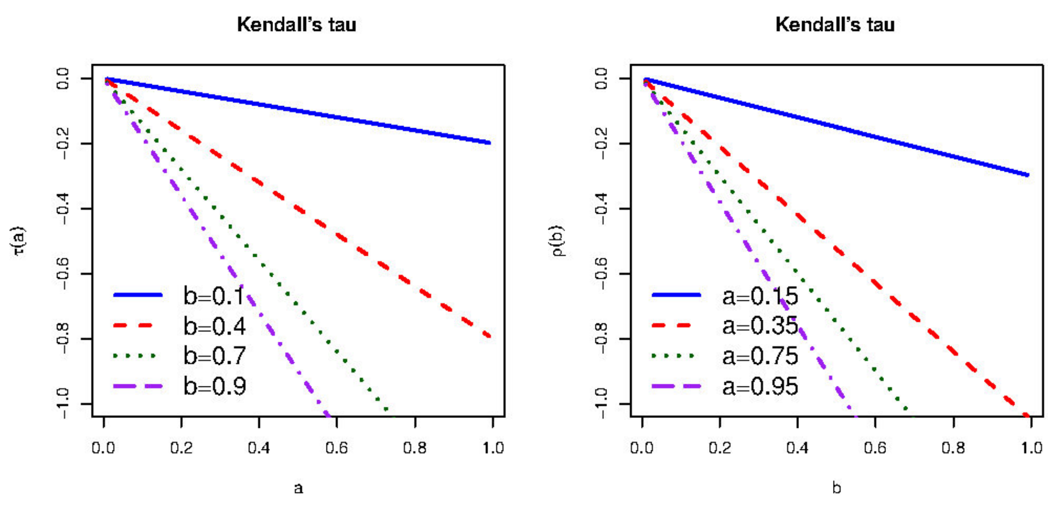


Figure 2. Kendall's τ plots against a and b

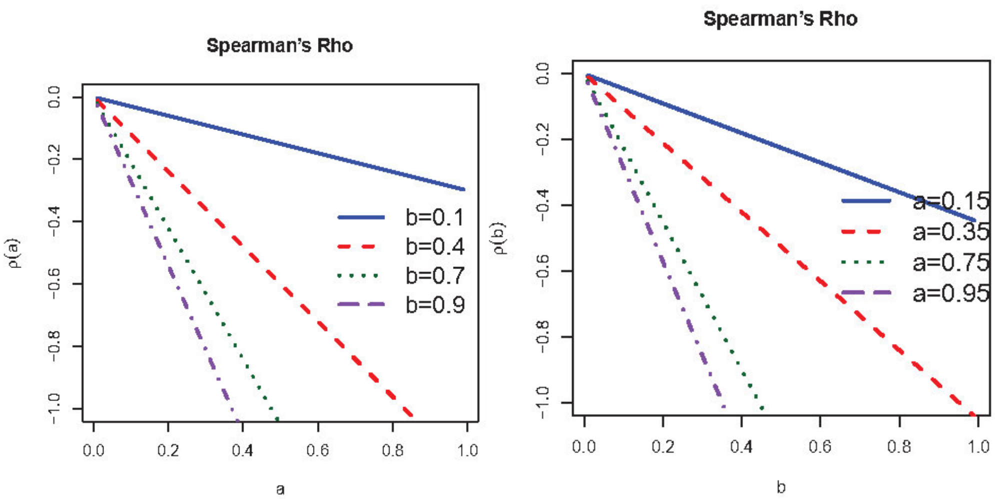


Figure 3. Spearman's ρ_s plots against a and b

shown in Figures 2 and 3, respectively. Based on Figure 2, as the value of a and b tend to 1, the value of dependence measure τ decreases to -1 and the dependency becomes stronger. Also, Figure 3 illustrates that the strength of dependence increases to $\rho_s = -1$ as a and b become large.

The lower and upper tail dependence coefficients λ_L and λ_U are another dependence measures that are defined by $\lambda_L = \lim_{t\to 0^+} P\left(R \le F_R^{-1}(t) | S \le F_S^{-1}(t)\right)$ and $\lambda_U = \lim_{t\to 1^-} P\left(R > F_R^{-1}(t) | S > F_S^{-1}(t)\right)$, respectively (see Nelsen (28)). The following proposition presents the tail dependence coefficients for $\breve{\bar{F}}_{R,S}$ in (6).

Proposition 2.5. If $(R,S) \sim BNMO(\theta_1, \theta_2, \theta_{12})$, then $\lambda_L = \lambda_U = 0$.

Proof. Let $\alpha = \frac{\theta_{12}}{\theta_1+\theta_{12}}$ and $\beta = \frac{\theta_{12}}{\theta_2+\theta_{12}}$ as defined in Proposition 2.1. For every $\alpha, \beta \in (0,1)$, we have

$$\breve{\bar{F}}_{R,S}\left(F_R^{-1}(t), F_S^{-1}(t)\right) = (1-t)^{2-\alpha-\beta}\left((1-t)^{\alpha} + (1-t)^{\beta} - 1\right).$$

So,

$$\begin{aligned}\lambda_L &= \lim_{t\to 0^+} P\left(R \le F_R^{-1}(t) | S \le F_S^{-1}(t)\right)\\ &= \lim_{t\to 0^+} \frac{1}{t}\left(2t - 1 + \breve{\bar{F}}_{R,S}\left(F_R^{-1}(t), F_S^{-1}(t)\right)\right)\\ &= \lim_{y\to 1^-} \frac{1}{1-y}\left(1 - 2y + y^{2-\alpha-\beta}\left(y^{\alpha} + y^{\beta} - 1\right)\right) = 0.\end{aligned}$$

Also,

$$\begin{aligned}\lambda_U &= \lim_{t\to 1^-} P\left(R > F_R^{-1}(t) | S > F_S^{-1}(t)\right)\\ &= \lim_{t\to 1^-} \frac{1}{1-t}\left(\breve{\bar{F}}_{R,S}(F_R^{-1}(t), F_S^{-1}(t)\right)\\ &= \lim_{y\to 0^+} \frac{1}{y}\left(y^{2-\alpha-\beta}\left(y^{\alpha} + y^{\beta} - 1\right)\right) = 0.\end{aligned}$$

So, the proof is complete.

1.3. Stress-strength index

In reliability analysis, the stress-strength model evaluates a system's reliability using random variables R for stress (supply) and S for strength (demand). The system fails when stress exceeds strength, leading to the reliability expression $P(R < S)$. The stress-strength index can be calculated using competing risk models, where failure times R and S are considered latent. We define the failure time as $T = \min(R,S)$ and the failure cause $C = 1$, if $R < S$ and $C = 2$, if $R > S$. The corresponding sub-distribution functions are given by:

$$F^*(1,t) = P(C = 1, T \le t) = \int_0^t f^*(1,z)dz,$$

and

$$F^*(2,t) = P(C = 2, T \le t) = \int_0^t f^*(2,z)dz,$$

where $f^*(1,t) = -\partial\breve{\bar{F}}(x,y)/\partial x|_{x=y=t}$ and $f^*(2,t) = -\partial\breve{\bar{F}}(x,y)/\partial y|_{x=y=t}$ are sub-density functions. Consequently, the stress-strength index is defined as $P(R < S) = F^*(1,\infty)$ and $P(R > S) = F^*(2,\infty)$. According to the competing risk model, the stress-strength index for the proposed model is expressed as follows.

Proposition 2.6. Let $(R,S) \sim BNMO(\theta_1, \theta_2, \theta_{12})$, then

$$P(R<S)=\frac{2\theta_1+\theta_{12}}{\theta_1+\theta_2+\theta_{12}}-\frac{\theta_1}{\theta_1+\theta_2},$$

or equivalently

$$P(R<S)=\frac{2\beta-\alpha\beta}{\beta+\alpha-\alpha\beta}-\frac{\beta-\alpha\beta}{\beta+\alpha-2\alpha\beta}.$$

Proof. Based on (6), we have

$$f^*(1,t)=-\frac{\partial\bar{F}_{R,S}(r,s)}{\partial r}|_{r=s=t}$$
$$=\theta_1 e^{-(\theta_1+\theta_2)t}\left(2e^{-\theta_{12}t}-1\right)+\theta_{12}e^{-(\theta_1+\theta_2+\theta_{12})t}.$$

Thus,

$$F^*(1,t)=\int_0^t f^*(1,u)du$$
$$=\frac{2\theta_1+\theta_{12}}{\theta_1+\theta_2+\theta_{12}}\left(1-e^{-(\theta_1+\theta_2+\theta_{12})t}\right)-\frac{\theta_1}{\theta_1+\theta_2}\left(1-e^{-(\theta_1+\theta_2)t}\right).$$

Therefore,

$$P(R<S)=\lim_{t\to+\infty}F^*(1,t)=\frac{2\theta_1+\theta_{12}}{\theta_1+\theta_2+\theta_{12}}-\frac{\theta_1}{\theta_1+\theta_2}.$$

Using $\alpha=\frac{\theta_{12}}{\theta_1+\theta_{12}}$ and $\beta=\frac{\theta_{12}}{\theta_2+\theta_{12}}$, we obtain the second statement.

As a result, the obtained stress-strength parameter can be estimated as follows:

$$\hat{P}(R<S)=\frac{2\hat{\theta}_1+\hat{\theta}_{12}}{\hat{\theta}_1+\hat{\theta}_2+\hat{\theta}_{12}}-\frac{\hat{\theta}_1}{\hat{\theta}_1+\hat{\theta}_2}.$$

Remark 2.3. If $\theta_1=\theta_2$ or equivalently $\alpha=\beta$, then $P(R<S)=\frac{1}{2}$.

Figure 4 illustrates the stress-strength index for different values of α and β. According to this figure, we conclude that as α increases, the estimated stress-strength index decreases approximately. Also, if β increases, then $\hat{P}(R<S)$ increases approximately.

Results and Discussion

In this section, we introduce the generation of random data from the proposed model $BNMO(\theta_1,\theta_2,\theta_{12})$ and estimate of its parameters.

1. Random number generation

Simulating random numbers is essential for understanding the behavior of a model. To generate the random numbers from $BNMO(\theta_1,\theta_2,\theta_{12})$, the following algorithm is introduced.

Algorithm 3.1. Random number generation from $BNMO(\theta_1,\theta_2,\theta_{12})$

The algorithm is carried out in the following three steps:

- Step 1: Generate three independent random variables $T_i\sim E(\theta_i)$ for $i=1,2$ and $U\sim U(0,1)$.
- Step 2: Set $R=\min\{T_1,F_{T_{12}}^{-1}(U)\}$ and $S=\min\{T_2,F_{T_{12}}^{-1}(1-U)\}$, where $F_{T_{12}}^{-1}(.)$ is the quantile function of $T_{12}\sim E(\theta_{12})$.
- Step 3: The desired pair is (R,S).

Figure 5 shows scatterplots of 750 generated data from Algorithm 3.1. As the dependence parameter θ_{12} increases, the dependence increases.

2. Estimation method

Here, we estimate the parameters using the maximum likelihood (ML) method. Consider the random sample of size m, namely $\{(r_1,s_1),\dots,(r_m,s_m)\}$ distributed from $BNMO(\theta_1,\theta_2,\theta_{12})$. Let m_1 and m_2 be the number of observations for which $e^{-\theta_{12}r}+e^{-\theta_{12}s}>1$ and $e^{-\theta_{12}r}+e^{-\theta_{12}s}=1$, respectively, such that $m_1+m_2=m$. Then, the log-likelihood function for a given sample of observations is obtained by

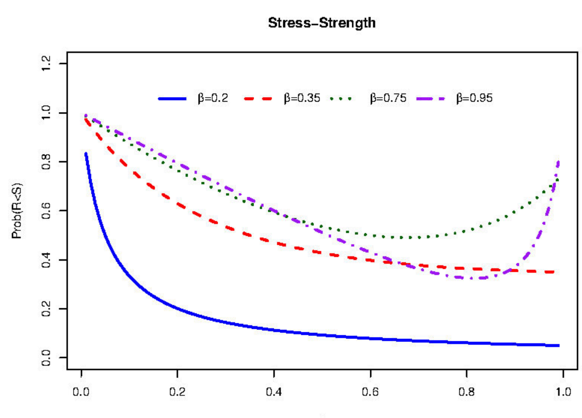


Figure 4. Stress-strength index for varying dependence parameters, α and β

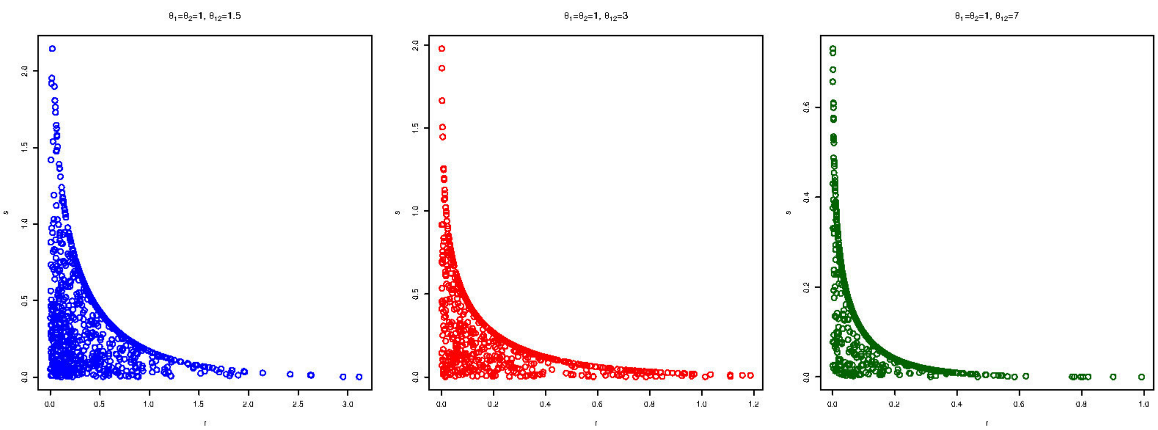


Figure 5. Scatterplot of 750 generated data using Algorithm 3.1 for different values of dependence parameter $\theta_{12} = 1.5,3,7$ (from left to right) and fixed marginal parameters $\theta_1 = \theta_2 = 1$

$$l(\theta_1,\theta_2,\theta_{12}) = -\theta_1 \sum_{j=1}^{m_1} r_j - \theta_2 \sum_{j=1}^{m_1} s_j + \sum_{j=1}^{m_1} \ln\left(\theta_2(\theta_1+\theta_{12})e^{-\theta_{12}r_j} + \theta_1(\theta_2+\theta_{12})e^{-\theta_{12}s_j} - \theta_1\theta_2\right) + m_2 \ln\left(\frac{\theta_{12}}{\theta_1+\theta_{12}}\right) + \frac{\theta_2}{\theta_{12}} \sum_{j=m_1+1}^{m} \ln\left(1-e^{-\theta_{12}r_j}\right), \quad (11)$$

where the observations are classified such that $\{(r_1,s_1),\dots,(r_{m_1},s_{m_1})\} \in A$ and $\{(r_{m_1+1},s_{m_1+1}),\dots,(r_m,s_m)\} \in A^c$ and $A = \{(r_i,s_i) | e^{-\theta_{12}r} + e^{-\theta_{12}s} > 1\}$.
Based on the normal equations (given in the Appendix), if either m_1 or m_2 are zero, then the ML estimator may lack uniqueness. However, this will not pose a problem since

$$P(m_1 = 0) = \left[P\left(e^{-\theta_{12}R} + e^{-\theta_{12}S} > 1\right)\right]^m \to 0 \;\; as \;\; m \to \infty,$$

and

$$P(m_2 = 0) = \left[P\left(e^{-\theta_{12}R} + e^{-\theta_{12}S} = 1\right)\right]^m \to 0 \;\; as \;\; m \to \infty.$$

For moderate sample size m, the events $[m_1 = 0]$ and $[m_2 = 0]$ are rare. When $m_1, m_2 > 0$, the normal equations (detailed in the Appendix) cannot be solved analytically, necessitating numerical methods. However, we found these methods to be less efficient than directly maximizing the log-likelihood function in (11). This maximization can be executed using the optim function in R. Initial values for optimization are obtained through the global non-linear optimization package "Rsolnp" in R version 3.6.1 (Ghalanos et al. (31)). We consider the constraints $\theta_1, \theta_2, \theta_{12} > 0$ and identified local maxima for various values of $\theta_1, \theta_2, \theta_{12}$. The global maximum is selected using the relation:

$$\left(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_{12}\right) = \text{argmax}_{\theta_1,\theta_2,\theta_{12} \in \Theta}\, l(\theta_1,\theta_2,\theta_{12}). \quad (12)$$

3. Performance analysis

Next, a finite sample performance of the estimators for marginal parameters (θ_1, θ_2) and dependence parameter θ_{12} is given. The performance is evaluated according to the bias and mean squared error (MSE) of the ML estimators introduced in the previous section. A specific sample size m has been taken from $BNMO(1,3,0.8)$ and MSEs have been calculated based on 10,000 iterations. The results are shown in Figure 6. The ML estimator performs very well for small sample sizes. Notably, after some fluctuations, the bias values stabilize around zero as the sample size increases. It is important to mention that the MLE consistently found a unique global maximum that did not lie on the boundary of the parameter space. The computational time to determine this global maximum, after testing all initial value combinations, was under 7 hours.

4. A real data analysis

This section illustrates the results of applying the BNGM distribution to a dataset on mercury (Hg) concentration in large-mouth bass, as explored by Mohsin et al. (32). Data were collected from 53 Florida lakes to study the factors affecting mercury levels in bass. Water samples were collected from the surface middle of each lake on specific dates, where measurements of alkalinity (mg/l), calcium (mg/l), and chlorophyll (mg/l) were taken, using averages from August and March. Fish samples, ranging from 4 to 44 individuals per lake, were then analyzed for minimum mercury concentration (μ g/g). Lange et al. (33) noted that mercury bioaccumulation in large-mouth bass is significantly influenced by the lakes' chemical characteristics, making calcium and minimum mercury concentration key variables of interest. We apply the proposed distribution

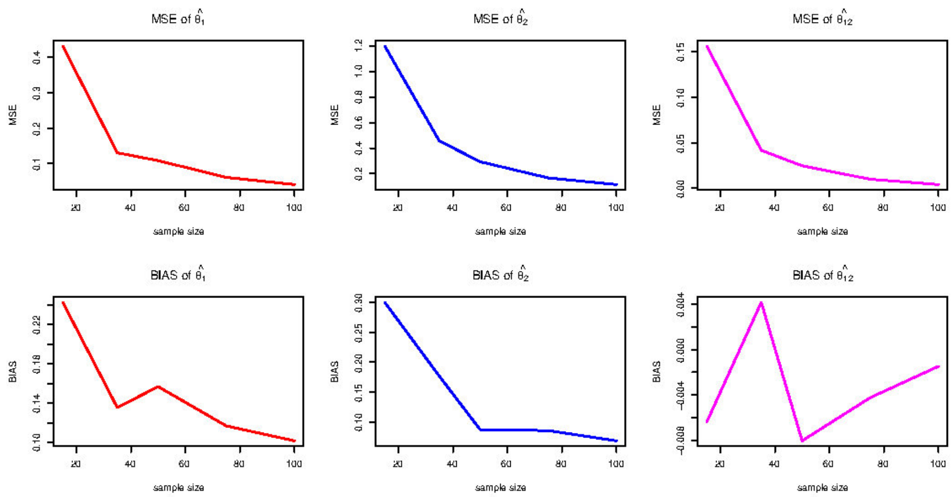


Figure 6. Performance analysis of ML estimators based on MSE and bias for $(\theta_1, \theta_2, \theta_{12})$=(1,3,0.8) using 10,000 independent replications

to model this data, noting that the 40^{th} row, considered an outlier, was omitted as stated by Mohsin et al. (32). A summary of the data is presented in Table 1. The dependency values of ρ_s and τ indicate a moderate relationship between the variables. We have fitted an exponential distribution to the marginal data, which are summarized in Table 2 and illustrated in Figure 7. Clearly the marginal distributions are well fitted to the data. With the confirmation that the marginal data follows an exponential distribution, we will fit the joint model to the Mercury and Calcium data and compare it to the results in Mohsin et al. (32). These results are shown in Table 3. The BNMO model outperforms the BALE model from Mohsin et al. (32). Both models fit the data well according to the Kolmogorov-Smirnov goodness-of-fit criteria. Figure 8 illustrates the scatter plot of actual versus simulated Mercury and Calcium data derived from the fitted BNMO model.

Conclusion

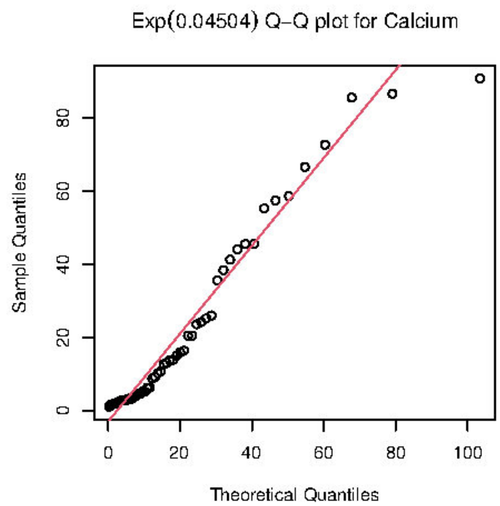
In practical applications, systems of components are frequently subjected to various shocks, which impact their reliability. According to the well-known MO bivariate shock model in (1), it is challenging to assign the probability of a common shock (T_{12}) to each component (X_1 and X_2). We address this limitation by proposing a new MO shock model, which offers beneficial properties such as dependency characteristics and a closed-form formula for the strength parameter, enhancing its applicability. There are few bivariate exponential distributions with a negative dependence structure, making our model particularly appealing. However, the presence of a singular component complicates parameter estimation. We have developed an estimation method and conducted a performance analysis to assess its effectiveness. Applying our model to real data demonstrates that it outperforms existing models.

Table 1. Descriptive statistics of data vectors Mercury and Calcium

Statistics	Mercury	Calcium
Minimum	0.04	1.1
1^{st}-Quantile	0.09	3.3
Median	0.25	12.6
Mean	0.27	22.2
3^{rd}-Quantile	0.33	35.6
Maximum	0.92	90.7
SD	0.22	24.93
Spearman's rho	-0.536	
Kendall's tau	-0.392	

Table 2. Marginal goodness-of-fit for Mercury and Calcium

Variables	Distribution	MLE	Log-likelihood	K-S p-value
Mercury	Exponential	3.573	14.502	0.195
Calcium	Exponential	0.045	-217.309	0.232



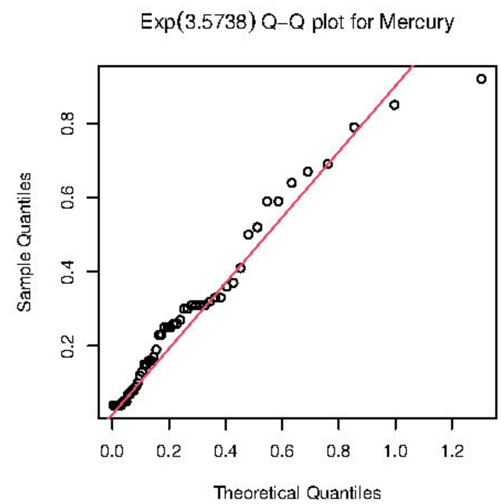


Figure 7. Q-Q plots of Calcium (left) and Mercury (right) for their fitted distributions

Table 3. Goodness-of-fit for the joint vector (Mercury, Calcium)

Model	MLE	Log-Likelihood	K-S p-value
BNMO	$\hat{\theta}_1 = 0.01, \hat{\theta}_2 = 3.67, \hat{\theta}_{12} = 0.038$	-194.0028	0.28
BALE (Mohsin et al. (32))	$\hat{\alpha} = 3.63, \hat{\beta} = 0.01, \hat{\gamma} = 0.25$	-3887.665	0.16

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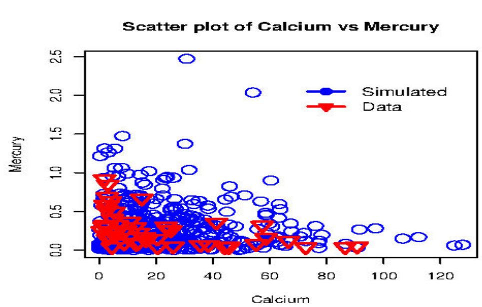

Figure 8. Scatterplot of Mercury versus Calcium for generated data, which are from the best BNMO model

References

1. Balakrishnan, K. Exponential Distribution: Theory, Methods and Applications. Taylor and Francis, Routledge. 2018.
2. Lai, CD, and Balakrishnan, N. Continuous Bivariate Distributions. Springer-Verlag, New York. 2009.
3. Marshall, AW, and Olkin, I. A multivariate exponential distribution. Journal of the American Statistical Association. 1967;62(317):30–44.
4. Cherubini, U, Durante, F, and Mulinacci, S. Marshall-Olkin distributions. in Advances in Theory and Applications. Springer Proceedings in Mathematics and Statistics, Springer. 2015.
5. Lindskog, F, and McNeil, A.J. Common Poisson shock models: Applications to insurance and credit risk modeling. ASTIN Bulletin: The Journal of the IAA. 2003;33(2):209–

238.
6. Cherubini, U, and Mulinacci, S. The Gumbel-Marshall-Olkin distribution. In Copulas and Dependence Models with Applications, Springer. 2017.
7. Esary, JD, and Marshall, AW. Multivariate distributions with exponential minimums. The Annals of Statistics. 1974;2(1):84–98.
8. Raftery, AE. A continuous multivariate exponential distribution. Communications in Statistics-Theory and Methods. 1984;13(8):947–965.
9. Tawn, JA. Modeling multivariate extreme value distributions. Biometrika. 1990;77(2):245–253.
10. Lin, H-H, Chen, K, and Wang, R-T. A multivariant exponential shared-load model. IEEE Transactions on Reliability. 1993;42(1):165–171.
11. Basu, AP, and Sun, K. Multivariate exponential distributions with constant failure rates. Journal of Multivariate Analysis. 1997;61(2):159–170.
12. Gómez, E, Gomez-Viilegas, M, and Marin, J. A multivariate generalization of the power exponential family of distributions. Communications in Statistics-Theory and Methods. 1998;27(3):589–600.
13. Cui, L, and Li, H. Analytical method for reliability and MTTF assessment of coherent systems with dependent components. Reliability Engineering and System Safety. 2007;92(3):300–307.
14. Fan, J, Nunn, ME, and Su, X. Multivariate exponential survival trees and their application to tooth prognosis. Computational Statistics and Data Analysis. 2009;53(4):1110–1121.
15. Kundu, D, and Gupta, RD. Bivariate generalized exponential distribution. Journal of Multivariate Analysis. 2009;100(4):581–593.
16. Li, X, and Pellerey, F. Generalized Marshall–Olkin distributions and related bivariate aging properties. Journal of Multivariate Analysis. 2011;102(10):1399–1409.
17. Kundu, D, and Gupta, AK. Bayes estimation for the Marshall–Olkin bivariate Weibull distribution. Computational Statistics and Data Analysis. 2013;57(1):271–281.
18. Bayramoglu, I, and Ozkut, M. The reliability of coherent systems subjected to Marshall–Olkin type shocks. IEEE Transactions on Reliability. 2014;64(1):435–443.
19. Kundu, D, Franco, M, and Vivo, J-M. Multivariate distributions with proportional reversed hazard marginals. Computational Statistics and Data Analysis. 2014;77:98–112.
20. Cha, JH, and Badía, F. Multivariate reliability modeling based on dependent dynamic shock models. Applied Mathematical Modelling. 2017;51:199–216.
21. Al-Mutairi, D, Ghitany, M, and Kundu, D. Weighted Weibull distribution: Bivariate and multivariate cases. Brazilian Journal of Probability and Statistics. 2018;32(1):20–43.
22. Mohtashami-Borzadaran, HA, Jabbari, H, and Amini, M. Bivariate Marshall–Olkin exponential shock model. Probability in the Engineering and Informational Sciences. 2021;35(3):745–765.
23. El Ktaibi, F, Bentoumi, R, Sottocornola, N, and Mesfioui, M. Bivariate copulas based on counter-monotonic shock method. Risks. 2022; 10:202–222.
24. Bentoumi, R, El Ktaibi, F, and Mesfioui, M. A new family of bivariate exponential distributions with negative dependence based on counter-monotonic shock method. Entropy. 2021;23:548–565.
25. Lee, H, and Cha, JH. New continuous bivariate distributions generated from shock models. Statistics. 2024;58(2):437–449.
26. Agrawal, A, Ganguly, A, and Mitra, D. Modelling competing risks data by a bivariate model with singularity originating from a shock model using the Lehmann family of distributions. Research in Statistics. 2025;3(1):1–13.
27. Genest, C, Mesfioui, M, and Schulz, J. A new bivariate Poisson common shock model covering all possible degrees of dependence. Statistics and Probability Letters. 2018;140:202–209.
28. Mohtashami-Borzadaran, HA, Amini, M, Jabbari, H, and Dolati, A. Marshall-Olkin exponential shock model covering all ranges of dependence. arXiv preprint arXiv: 2004.11241. 2020.
29. Joe, H. Multivariate Models and Multivariate Dependence Concepts. Chapman and Hall/CRC, New York. 1997.
30. Nelsen, RB. An Introduction to Copulas. Springer-Verlag, New York, 2007.
31. Ghalanos, A, Theussl, S, and Ghalanos, MA. Package "Rsolnp". 2012.
32. Mohsin, M, Kazianka, H, Pilz, J, and Gebhardt, A. A new bivariate exponential distribution for modeling moderately negative dependence. Statistical Methods and Applications. 2014;23(1):123–148.
33. Lange, TR, Royals, H, and Connor, LL. Influence of water chemistry on mercury concentration in largemouth bass from Florida lakes. Transactions of the American Fisheries Society. 1993;122(1):74–84.

Appendix

Let $\tilde{\theta} = (\theta_1, \theta_2, \theta_{12})$ and for all j:

$\Delta_j = \theta_2(\theta_1 + \theta_{12})e^{-\theta_{12} r_j} + \theta_1(\theta_2 + \theta_{12})e^{-\theta_{12} s_j} - \theta_1\theta_2$.

The normal equations for estimating parameters are as follows:

$$\frac{\partial l(\tilde{\theta})}{\partial \theta_1} = -\sum_{j=1}^{m_1} r_j + \sum_{j=1}^{m_1} \frac{1}{\Delta_j}\left(\theta_2 e^{-\theta_{12} r_j} + (\theta_2 + \theta_{12})e^{-\theta_{12} s_j} - \theta_2\right) - \frac{m_2}{\theta_1 + \theta_{12}},$$

$$\frac{\partial l(\tilde{\theta})}{\partial \theta_2} = -\sum_{j=1}^{m_1} s_j + \sum_{j=1}^{m_1} \frac{1}{\Delta_j}\left((\theta_1 + \theta_{12})e^{-\theta_{12} r_j} + \theta_1 e^{-\theta_{12} s_j} + \theta_1\right) + \frac{1}{\theta_{12}} \sum_{j=m_1+1}^{m} \log\left(1 - \exp\{-\theta_{12} r_j\}\right),$$

and

$$\frac{\partial l(\tilde{\theta})}{\partial \theta_{12}} = \sum_{j=1}^{m_1} \frac{-1}{\Delta_j}\left(\theta_2(\theta_1 + \theta_{12})e^{-\theta_{12} r_j} r_j + \theta_1(\theta_2 + \theta_{12})e^{-\theta_{12} s_j} s_j\right) + \frac{m_2}{\theta_{12}} - \frac{m_2}{\theta_1 + \theta_{12}} - \frac{\theta_2}{\theta_{12}^2} \sum_{j=m_1+1}^{m} \log\left(1 - \exp\{-\theta_{12} r_j\}\right) + \frac{\theta_2}{\theta_{12}} \sum_{j=m_1+1}^{m} \frac{r_j e^{-\theta_{12} r_j}}{1 - \exp\{-\theta_{12} r_j\}}.$$