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On Projection Invariant Rickart Modules

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Abstract

This study examined π-Rickart modules, a module-theoretic analog of π-Rickart rings, from the perspective of their endomorphism rings. It is shown that π-Rickart conditions are located between π-e. Baer and p.q.-Baer conditions, and it is established that the corresponding endomorphism ring possesses the appropriate π-Rickart property. Besides, the notion of π-e.AIP modules is presented. Furthermore, connections to the aforementioned concepts of π-Rickart, endo-AIP, and π-e.AIP modules are examined.

Keywords: Baer modules; Rickart ring; Endomorphism rings; Annihilators; Rickart modules.

Introduction

This study operated within the ring and module theory framework, where R represents a ring with a non-zero identity, and M is a unitary right R-module. The notation S signifies the ring of R-endomorphisms of M. We further define, $l_S(X)$ and $r_M(X)$ as the left and right annihilators of a X within S and M, respectively, and $I(R)$ as the subring of R generated by its idempotent elements.

Based on (1) and (2), a ring R is referred to as *(quasi-)Baer* if, for any nonempty subset (or ideal) Y of R, it holds that $r_R(Y) \leq_{\oplus} R_R$. Furthermore, R is designated right *Rickart* (3) if, given each $x \in R$, $r_R(x) \leq_{\oplus} R_R$. These classes of modules have applications in functional analysis. The concept of Rickart rings was initially introduced in (3) and has since been extensively studied by various researchers, including (4-9).

The aforementioned ring-theoretic concepts are naturally generalized to the module setting. Specifically, as delineated in (10), M_R is defined as (quasi-)Baer if, for every (fully invariant) submodule K of M_R, $l_S(K) \leq_{\oplus} {}_SS$. The notion of *p.q.-Baer* modules, as introduced in (11), pertains to M_R where $r_M(\psi S) \leq_{\oplus} M_R$ for every $\psi \in S$. Moreover, based on (12), a module M_R is classified as Rickart if, for each $\psi \in S$, $\mathrm{K}er\psi = \mathrm{r}_M(\psi) \leq_{\oplus} M_R$. The absence of symmetry in the Rickart ring property, unlike in the Baer and quasi-Baer conditions, motivates the introduction of $\mathfrak{L}$-Rickart modules. A module M_R is referred to as $\mathfrak{L}$-Rickart (13) if, for every $y \in M$, $l_S(y) \leq_{\oplus} {}_SS$.

A right (or left) ideal A in a ring R is called *projection invariant* if, for every element $e\ in\ R$ such that $e^2 = e$, ideal A remains unchanged when multiplied by e, i.e., $eA \subseteq A$. The concept of *π-Baer* rings is introduced in (14), is based on these kinds of ideals. A ring R is termed *π-Baer* if, for any projection-invariant left ideal X of R, $\mathrm{r}_R(X) \leq_{\oplus} R_R$. Moreover, this idea extends to modules, where a submodule P of M_R is projection invariant if, for all idempotent elements $g \in S$, submodule satisfies $g(P) \subseteq P$,meaning it is preserved under multiplication by g. A module M_R is defined as π-e.Baer (15) if every projection-invariant submodule P of M_R satisfies $l_S(P) \leq_{\oplus} {}_SS$.. In recent studies, a more generalized form of π-Baer rings, called *π-Rickart* rings, was introduced.

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As described in (16), a ring R is defined as left π-Rickart if, for any element $x \in R$, $l_R(I(R)x) \leq_{\oplus} {}_RR$. As with the Rickart ring condition, π-Rickart ring condition does not generally possess symmetry between the left and right aspects. For more results related to this concept, see (17-20).

Motivated by these studies, we aimed to explore the concepts of π-Rickart from rings to modules. We define M_R as *π-Rickart* if $r_M(\psi I(S)) \leq_{\oplus} M_R$ for all $\psi \in S$. It is apparent that R being a right π-Rickart ring is equivalent to R_R being a π-Rickart module. This new module classification is situated between p.q.-Baer modules and π-e.Baer modules. In our most recent study (21), we investigated π-endo.Rickart modules, which are an extension of the concept of left π-Rickart rings. The introduction of the concept of π-Rickart modules was motivated by the fact that the left and right π-Rickart properties are not necessarily symmetric, as previously mentioned. Thus, we found an interesting result stating that the endomorphism ring of a π-Rickart module is a right π-Rickart ring. However, this property does not generally hold for π-endo. Rickart modules.

In Section 1, we introduced π-Rickart modules and explored their fundamental properties. We established that a module M is π-Rickart if and only if, for every finitely generated left ideal Y of S, $r_M(YI(S)) \leq_{\oplus} M_R$ holds (Proposition 2.8). We showed that π-e.Baer modules are equivalent to π-Rickart modules satisfying FI-SSIP condition (Proposition 2.21). We investigate when the direct summand of a π-Rickart module retains this property (Theorem 2.10). Moreover, we established that the ring of endomorphisms of a π-Rickart module forms a right π-Rickart ring (Theorem 2.11). An analogous version of Chatters and Khuri's Theorem is derived for π-Rickart modules (Corollary 2.17). Therefore, for a right hereditary, right noetherian ring R, every injective right R-module M is π-e.Baer if and only if M is π-Rickart (Corollary 2.22). For an indecomposable artinian π-Rickart module M, the ring of endomorphisms of M is a division ring (Corollary 2.7). Furthermore, M is π-e.Baer if and only if M is π-Rickart and the set $\{Se | e \in S_r(S)\}$ is a complete lattice (Theorem 2.25).

In Section 2, we explore the concept of π-e.AIP modules, which encapsulated the definitions of π-Rickart and π-e.Baer modules, extending their applicability to a broader class of modules. The interconnections between π-Rickart, endo-AIP, and π-e.AIP modules are explored (Theorem 3.2). We investigated the conditions in which the characteristics of π-e.AIP, Rickart, and π-Rickart modules coincide (Proposition 3.4). Furthermore, we examine the theoretical characteristics of π-e.AIP modules. The characteristic of π-e.AIP is not preserved by direct summands or direct sums, as seen in Example 3.8. We therefore investigated the circumstances under which the aforementioned property was inherited by direct summands and direct sums (Theorems 3.6 and 3.9). We also show that the ring of endomorphisms of a π-e.AIP module is left π-AIP (Theorem 3.11).

The notations $N \subseteq M$, $N \leq M$, $N \trianglelefteq M$, $N \trianglelefteq_p M$, $N \leq_{\oplus} M$, and $N \leq^{ess} M$ signify that N is a subset, a right R-submodule, a fully invariant R-submodule, a projection invariant right R-submodule, a direct summand of M, and an essential submodule of M, respectively. Recall that an idempotent element $g \in R$ is termed *left (right) semicentral* if $tg = gtg$ ($gt = gtg$) for all $t \in R$. The sets of left and right semicentral idempotents are denoted as $S_l(R)$ and $S_r(R)$, respectively. A ring R is *abelian* if its idempotents commute with all elements of R, and a module is *abelian* if the ring of its endomorphisms is abelian.

Results and Discussion

1 π- Rickart modules

The discussion on π-Rickart modules was initiated in this section, with an emphasis on their key attributes. Given the connections between Baer and Rickart modules, our objective was to explore the connections between π-e.Baer and π-Rickart modules. Additionally, we investigate the connections between extending modules and nonsingular modules by analyzing the properties of projection-invariant extending modules and projection-invariant nonsingular modules. Furthermore, we investigate the endomorphism ring of π-Rickart modules. The results that were employed throughout the investigation are summarized below for the sake of comprehensiveness.

Lemma 2.1 [(22), Lemma 1.1] *The followings are equivalent for an idempotent element $f \in R$:*

(1) $f \in S_l(R)$.

(2) $1 - f \in S_r(R)$.

(3) $(1 - f)Rf = 0$.

(4) $fR \trianglelefteq R$.

(5) $R(1 - f) \trianglelefteq R$.

(ii) $S_l(R) \cap S_r(R) = B(R)$, where $B(R)$ is the set of central idempotents.

Lemma 2.2 [(15), Lemma 3.1](i) *Let $M = \bigoplus_{i \in I} M_i$ and $N \trianglelefteq_p M$. Then, $N = \bigoplus_{i \in I} N \cap M_i$ and $N \cap M_i \trianglelefteq_p M_i$ for all $i \in I$.*

(ii) Let M be a module. Then, $e \in S_l(S)$ if and only if $eM \trianglelefteq_p M$.

Definition 2.3 *We call M_R is π-Rickart, if for any $\eta \in S$, there exists an idempotent element $f \in S$ such that $r_M(\eta I(S)) = fM$.*

Note that the idempotent f in Definition 2.3 belongs to $S_l(S)$ by Lemma 2.2.

Example 2.4(i) Every abelian von Neumann regular (strongly regular) ring is also π-Rickart.

(ii) The module R_R is π-Rickart if and only if the ring R is a right π-Rickart ring.

(iii) The classes of semisimple modules, Baer modules, and π-e.Baer modules are each examples of π-Rickart modules.

(iv) Consider the ring R given by $R = \begin{pmatrix} A & H \\ 0 & \mathbb{C} \end{pmatrix}$, where A denotes a Banach subalgebra of the ring of bounded linear operators acting on a Hilbert space H, with the additional condition that A contains all rank 1 idempotents. As indicated in [(16), Example 3.12], R is a π-Baer ring and thus it is π-Rickart. Furthermore, by Theorem 2.10, it follows that eR also is a π-Rickart module, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(v) $\mathbb{Z}_{p^\infty}$ is an injective $\mathbb{Z}$-module, but it does not qualify as π-Rickart.

In the forthcoming theorem, we showed that π-Rickart modules constitute a discrete category situated between π-e.Baer and p.q.-Baer modules.

Proposition 2.5 *M_R is π-e.Baer $\Rightarrow$ M_R is π-Rickart $\Rightarrow$ M_R is p.q.-Baer.*

Proof. Assume M_R is a π-e.Baer module. For any $\psi \in S$, $S\psi I(S)$ constitutes a left ideal of S that is projection invariant. Consequently, we have $g^2 = g \in S$ for which $r_M(\psi I(S)) = r_M(S\psi I(S)) = gM$. Thus, M_R is π-Rickart. Now, suppose M_R is π-Rickart. Then, for any $\psi \in S$, we have $g^2 = g \in S$ such that $r_M(\psi I(S)) = gM$. As $\psi I(S) \subseteq \psi S$, it follows that $r_M(\psi S) \subseteq r_M(\psi I(S)) = gM$. Additionally, since $g \in S_l(S)$, we have $(\psi S)gM = (\psi g)(gSgM) \subseteq (\psi I(S))gM = 0$. Hence, $r_M(\varphi S) = gM$, and thus M is p.q.-Baer.

The subsequent example serves to show that the implications stated in Proposition 2.5 are not generally reversible.

Example 2.6 (i) Consider the subring T of $\prod_{n=1}^{\infty} A_n$, where $A_n = \mathbb{Z}$ for $n = 1,2,\cdots$, defined as $T = \{(a_n) \in \prod_{n=1}^{\infty} A_n | a_n is\ eventually\ constant\}$. Then T_T is a π-Rickart module, which is not π-e.Baer [(16), Example 1.6].

(ii) The ring of endomorphisms of a π-e.Baer module is a π-e.Baer ring by [(15), Theorem 2.5]. Let B be a π-Rickart ring that is not a π-Baer ring (see, [(16), Example Example 1.6]). Consider the ring $R = \begin{bmatrix} B & B \\ 0 & B \end{bmatrix}$ and idempotent $g = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$. Then, based on Theorem 2.10, $M_R = gR$ is a π-Rickart module. However, since $End_R(M) \cong B$ is not a π-Baer ring, M_R is not a π-e.Baer module.

(iii) Suppose R is a simple ring that only has trivial idempotents $\{0,1\}$, and is not a domain (see, (23)). Then, R is a quasi-Baer ring and therefore p.q-Baer. It can be easily verified that R does not satisfy the right π-Rickart property. Consequently, $M = R_R$ is not π-Rickart.

Proposition 2.7 (i) *For an indecomposable module M_R, being π-Rickart, Baer, and π-e.Baer are equivalent properties.*

(ii) If M_R is an indecomposable artinian π-Rickart module, then $End_R(M)$ is a division ring.

Proof. (i) It is straightforward, as M_R is indecomposable.

(ii) It follows from the part (i) and [(12), Corollary 4.11].

Proposition 2.8 *The following conditions are equivalent for M_R.*

1. M_R is π-Rickart.
2. For every finite subset $X = \{\varphi_1, \dots, \varphi_n\}$ of S, $r_M(XI(S)) \leq_{\oplus} M$.
3. For each finitely generated left ideal Y of S, $r_M(YI(S)) \leq_{\oplus} M$.

Proof. $(i) \Rightarrow (ii)$ It can be verified that $r_M(XI(S)) = r_M(\varphi_1 I(S)) \cap \ldots \cap r_M(\varphi_n I(S))$. Since M is π-Rickart, we can find elements $g_j \in S_l(S)$ such that $r_M(\varphi j I(S)) = g_j M$ for each $j \in \{1,2, \ldots\}$. Therefore, $r_M(XI(S)) = \bigcap_{j=1}^{n} g_j M = gM$, where $g = g_1 g_2 \cdots g_n \in S_l(S)$.

$(ii) \Rightarrow (iii)$ It is straightforward.

$(iii) \Rightarrow (i)$ It is evident because every principal left ideal is finitely generated.

The following example shows that a direct summand of a π-Rickart module may not necessarily be π-Rickart, in general.

Example 2.9 Consider a prime ring R where R_R is uniform and $Z(R_R) \neq 0$. Now, let's consider the free module A_R=$\bigoplus_{i=1}^{n} R_i$ where $R_i \cong R$ for each $1 \leq i \leq n$. Based on [(15), Example 4.1], we can deduce that A_R is π-e.Baer. Using Proposition 2.5, we can further deduce that A_R is π-Rickart. However, since each one sided ideal of R is projection invariant and R is not Rickart, we can conclude that R_R is not π-Rickart.

The forthcoming theorem establishes the conditions under which a direct summand of a π-Rickart module is π-Rickart.

Theorem 2.10 *Direct summands that are projection invariant in π-Rickart modules remain π-Rickart.*

Proof. Let M be π-Rickart and N be a projection invariant direct summand of M_R. Then, there exist $e^2 = e \in S$ such that $N = eM$ and $E \cong eSe$, where $E = End_R(N)$. Note that $e \in S_l(S)$, as $N \trianglelefteq_p M$. Observe that $\varphi = e\varphi e$, so $I(E) = eI(S)e$. For every $n \in r_N(\varphi I(E))$, $n = en$. So $\varphi I(S)n = e\varphi eI(S)en = \varphi(eI(S)e)n = \varphi I(E)n = 0$, and hence $r_N(\varphi I(E)) \subseteq r_M(\varphi I(S)) \cap N$. Now let $x \in r_M(\varphi I(S)) \cap N$ we have $x = ex \in N$ and

$\varphi I(E)x = \varphi eI(S)ex = \varphi I(S)x = 0$. This implies that $r_N(\varphi I(E)) = r_M(\varphi I(S)) \cap N$. Since M is π-Rickart, $r_M(\varphi I(S)) = fM$ for some $f^2 = f \in S$. Hence $r_M(\varphi I(S)) \cap N = fM \cap eM = (efe)eM$, and efe is an idempotent of eSe. Therefore $r_N(\varphi I(E)) = (efe)eM \leq_{\oplus} eM = N$.

Theorem 2.11 *The endomoprhism ring of a π-Rickart module is a right π-Rickart ring.*

Proof. Consider M as a π-Rickart module. For every $\psi \in S$, there is $g^2 = g \in S$ such that $r_M(\psi I(S)) = gM$. Consequently, $\psi I(S)g = 0$, implying $gS \subseteq r_S(\psi I(S))$. Now, let $\alpha \in r_S(\psi I(S))$. Hence, $\psi I(S)\alpha = 0$, which leads to $\alpha(M) \subseteq r_M(\psi I(S))$. This implies $\alpha = g\alpha$, so $r_S(\psi I(S)) = gS$. Therefore, S is right π-Rickart.

Corollary 2.12 *Let R be a right π-Rickart ring and $e \in S_l(R)$. Then eRe is also a right π-Rickart ring.*

Proof. Since $e \in S_l(R)$,it follows from Lemma 2.2 that $eR \trianglelefteq_p R$. Note that $End_R(eR) \cong eRe$. Thus, the conclusion is derived from Theorem 2.10 and 2.11.

The following example shows that the converse of Theorem 2.11 does not hold, in general.

Example 2.13 Let $\mathcal{C} = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $g = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{C}$. Consider $M_{\mathcal{C}} = g\mathcal{C}$. Then $End_{\mathcal{C}}(M)$ is a π-Rickart ring. However $M_{\mathcal{C}}$ is not a π-Rickart module.

As defined in [20], M_R is called *local-retractable* if, for each $\varnothing \neq A \subseteq S$ and for any $0 \neq m \in r_M(A)$, there exists a homomorphism $\psi_m: M \to r_M(A)$ with $m \in \psi_m(M) \subseteq r_M(A)$. Local-retractability paves the way for the establishment of the converse of Theorem 2.11.

Theorem 2.14 *Let M_R be local-retractable module. Then, M_R is a π-Rickart module if and only if S is a right π-Rickart ring.*

Proof. Let S be π-Rickart and $\varphi \in S$. Then, there exists $g^2 = g \in S$ such that $r_S(\varphi I(S)) = gS$. By [(13), Proposition 2.20], $r_M(\varphi I(S)) = r_S(\varphi I(S))(M)$. Thus, $r_M(\varphi I(S)) = gS(M) = gM$. Therefore, M is a π-Rickart module. The converse follows from Theorem 2.11.

Corollary 2.15 *Consider $\mathfrak{F}$ as a free R-module. Then, $\mathfrak{F}$ is π-Rickart if and only if $End_R(\mathfrak{F})$ is a right π-Rickart ring.*

Proof. It is clear from [(13), Lemma 2.9] and Theorem 2.14.

Based on (13), M_R is identified as *π-e.nonsingular* if $r_M(A) \leq^{ess} gM$ where $g^2 = g \in S$ and $A \trianglelefteq_p S$, leading to $r_M(A) = gM$. Thus, M_R is termed $\mathfrak{K}$-nonsingular in (13), if, for any $\varphi \in S$, $Ker\varphi \leq^{ess} M$ implies $\varphi = 0$.

Proposition 2.16 *Given a π-Rickart module M_R, we have the following properties:*

1. M_R is π-e.nonsingular.
2. S is a semiprime ring if and only if every left semicentral idempotents of S is central.
3. M_R is $\mathfrak{K}$-nonsingular, if every essential submodule of M is an essential extension of a projection invariant submodule.

Proof. (i) Let M_R be π-Rickart, and let P be a projection invariant left ideal in S. Assume $r_M(P) \leq^{ess} gM$, with $g^2 = g \in S$. Since $r_M(P) = \bigcap_{\varphi \in I} r_M(\varphi I(S))$, for any $\varphi \in P$, it follows that $r_M(P) \leq r_M(\varphi I(S)) \cap gM \leq^{ess} gM$. As M is π-Rickart, there exists $h \in S_l(S)$ such that $r_M(\varphi I(S)) = hM$. Hence $hM \cap gM = ghM$, $ghM \leq^{ess} gM$. Since gh is an idempotent in S, $ghM = gM$. Thus, $gM \leq r_M(\varphi I(S))$ leading to $gM \leq \bigcap_{\varphi \in I} r_M(\varphi I(S)) = r_M(P)$. Therefore, $gM = r_M(P)$, so M is π-e.nonsingular.

(ii) Clearly, each left semicentral idempotents of a semiprime ring is central. Let M be a π-Rickart module and all left semicentral idempotents in S be central. Suppose $\varphi \in S$ and $\varphi S \varphi = 0$. Then, $\varphi I(S)\varphi = 0$. Consequently, for every $m \in M$, $\varphi(m) \in r_M(\varphi I(S)) = gM$ for some $g \in S_l(S)$. This implies $\varphi g = 0$ and $(1 - g)\varphi(m) = 0$ for each $m \in M$. As g is a central, we have $\varphi = g\varphi + (1 - g)\varphi = 0$. Hence, S is a semiprime ring.

(iii) suppose $\psi \in S$ and $Ker\psi \leq^{ess} M$. By assumption, there exists $N \trianglelefteq_p M$ such that $N \leq^{ess} Ker\psi \leq^{ess} M$. Consequently, $\psi I(S)N = \psi N = 0$. Thus, $N \subseteq r_M(\psi I(S))$. Since M is π-Rickart, $r_M(\psi I(S)) = gM$ for some $g^2 = g \in S$. As $N \leq^{ess} M$, $g = 1$. Consequently, $\psi = 0$, implying that M is a $\mathfrak{K}$-nonsingular module.

In their important 1980 publication, Chatters and Khuri demonstrated that a right nonsingular, right extending ring is accurately defined as a right cononsingular Baer ring. The objective of the forthcoming discussion was to further explore analogues of Chatters–Khuri Theorem, with insights to be drawn from the results presented in this paper. Based on (24), M_R is termed *π-extending* if for any $N \trianglelefteq_p M$, we have $N \leq_{\oplus} gM$ where $g^2 = g$. As per (15), a module M_R satisfies the π-e.cononsingular property if, for all $P \trianglelefteq_p M$, the condition $r_M(l_S(P)) \leq_{\oplus} M$ leads to $P \leq^{ess} r_M(l_S(P))$.

Corollary 2.17

(i) Any abelian right π-Rickart ring is a semiprime ring.

(ii) Any π-extending π-Rickart module is π-e.cononsingular and π-e.Baer.

Proof. Proposition 2.16 and [6, Theorem 4.16] yield the result.

A module M_R has *IFP* (Insertion of Factors Property), if for any element $\varphi \in S$, we have $r_M(\varphi) \trianglelefteq M$ (25). Following this idea, we define *π-IFP* module as a generalization of *IFP* modules. This new class of modules strengthen the condition for modules with *IFP*.

Definition 2.18 *We call a module M_R is π-IFP, if for every $\phi \in S$, the submodule $r_M(\phi)$ is a projection invariant submodule of M. (or equivalently, for each $m \in M$, $l_S(m)$ is a projection invariant left ideal of S).*

Note that every ring with *IFP* has π*-IFP*, but the converse is not true. For example, every simple abelian ring which is not a domain (see, (23)) has π*-IFP* but does not have *IFP*. Therefore, every abelian ring has π*-IFP*, but it does not have *IFP* in general [(26), Example 14].

Example 2.19 *Let A be an abelian ring which does not have IFP. Consider $R = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ and $M_R = eR$ where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Since $S = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and A is abelian, M has π-IFP. On the other hand, since A does not satisfy IFP, there exist $x, y \in A$ such that $xy = 0$ and $xAy \neq 0$. Consequently, there exists $a \in A$ such that $xay \neq 0$. Set $m = \begin{pmatrix} y & y \\ 0 & 0 \end{pmatrix}$ and $\phi = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in S$. It follows that $\phi(m) = 0$, thus $\phi \in l_S(m)$. However, $\phi am \neq 0$ for $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in S$. Thus $l_S(m)$ is not a fully invariant left ideal of S. Hence, M fails to satisfy the IFP.*

An additional example of a π-IFP module that is not IFP can be constructed by taking A as a simple abelian ring that is not a domain, as presented in Example 2.19.

Proposition 2.20 *The following statements are equivalent.*

1. M_R is both Rickart and abelian.
2. M_R is π-Rickart and fulfills the π*-IFP*.

Proof. $(i) \Rightarrow (ii)$ It is evident that every abelian module satisfies π-IFP property. Let $\varphi \in S$. Since M_R is Rickart, there exists $f^2 = f \in S$ such that $r_M(\varphi) = fM$. For any $x \in r_M(\varphi I(S))$, we have $\varphi I(S)x = 0$. Thus, $x \in \ker(\varphi) = fM$. Since M is abelian, $\varphi I(S)fM = \varphi f I(S)M = 0$. Therefore, $r_M(\varphi I(S)) = fM$, indicating that M_R is π-Rickart.

$(ii) \Rightarrow (i)$ Let $\varphi \in S$ and $x \in r_M(\varphi)$. Since M satisfies the π*-IFP*, we have $I(S)x \subseteq r_M(\varphi)$. This implied that $x \in r_M(\varphi I(S))$. Therefore, we conclude that $r_M(\varphi) = r_M(\varphi I(S))$. Since M_R is π-Rickart, one can find an element $f \in S$ satisfying $f^2 = f$ and $r_M(\varphi) = fM$. Now, we proceed to prove that M_R is abelian. Since M has π*-IFP*, $r_M(g) = (1-g)M \trianglelefteq_p M$ for any idempotent $g \in S$. Consequently, $g \in S_l(S)$ and similarly $g \in S_r(S)$. Therefore S is an abelian ring.

M_R is said to possess the FI–strong summand intersection property (FI–SSIP) if every family of completely invariant direct summands of M_R has an intersection that is a direct summand of M_R. The following conclusion clarifies the conditions under which the π-e.Baer and π-Rickart modules are interchangeable.

Proposition 2.21 *M_R is π-e.Baer if and only if M_R is π-Rickart and has the FI-SSIP.*

Proof. Assume M is π-e.Baer and $\{N_\gamma\}_{\gamma\in\Gamma}$ is a family of fully invariant direct summands of M. For every $\gamma \in \Gamma$, there exist an element $e_\gamma \in S_l(S)$ such that $N_\gamma = e_\gamma M$. Let $J = \sum_{\gamma\in\Gamma} S(1-e_\gamma)$. Then J is a projection invariant left ideal of S, as $1 - e_\gamma \in S_r(S)$ for each $\gamma \in \Gamma$. Hence $r_M(J) \leq_\oplus M$. It follows that $\bigcap_{\gamma\in\Gamma} N_\gamma = \bigcap_{\gamma\in\Gamma} r_M(S(1-e_\gamma)) = r_M(J)$. Therefore, M has FI-SSIP. By Proposition 2.5, every π-e.Baer module is π-Rickart. Conversely, let A be a projection invariant left ideal of S. Then, we have $A = \sum_{\varphi_i\in A} \varphi_i I(S)$. So $r_M(A) = \bigcap_{\varphi_i\in A} r_M(\varphi_i I(S))$. Since M_R is π-Rickart, for each $\varphi_i \in A$, there is $e_{\varphi_i} \in S_l(S)$ such that $r_M(\varphi_i I(S)) = e_{\varphi_i} M$. As M has FI-SSIP, $r_M(A) = \bigcap_{\varphi_i\in A} e_{\varphi_i} M \leq_\oplus M$. Therefore M is π-e.Baer.

Corollary 2.22 *Let R be a right hereditary, right noetherian ring. Then every injective module M_R is π-e.Baer if and only if M_R is π-Rickart.*

Proof. [(27), Corollary 2.30] and Proposition 2.21 complete the result.

Lemma 2.23 *Suppose M_R is a π-Rickart module and J is a nonzero projection invariant left annihilator in S. Then J contains a nonzero idempotent.*

Proof. Let $0 \neq J = l_S(X)$ for some nonempty subset X of M and $\varphi \in J$. Due to π-Rickart property of M, $r_M(\varphi I(S)) = fM$, where $f^2 = f \in S$. Since J is projection invariant left ideal of S, it follows from (15) that $r_M(J) \trianglelefteq_p M$. Consequently, $r_M(J) \subseteq r_M(\varphi I(S)) = fM$. Define $e = 1 - f$. Hence, $er_M(J) \subseteq efM = 0$, so $e \in l_S(r_M(J)) = J$.

Theorem 2.24 *If S does not contain any infinite set of nonzero orthogonal idempotents, then M_R is the π-Rickart if and only if M_R is π-e.Baer if and only if M_R is π-e.Rickart.*

Proof. Suppose $N \trianglelefteq_p M$. By Lemma 2.23, $l_S(N)$ contains a nonzero idempotent. By assumption and [(12), Lemma 4.3], we can select an idempotent $f \in l_S(N)$ such that $S(1-f) = l_S(fM)$ is minimal. We aimed to prove that $l_S(N) \cap l_S(fM) = 0$. On the contrary, suppose $l_S(N) \cap l_S(fM) \neq 0$. Then $J = l_S(N \cup fM) \neq 0$. Now, by Lemma 2.23, J must contain a nonzero idempotent g. Since $gf = 0$, $h = f + (1-f)g$ is also an idempotent in $l_S(N)$. As $hg = g$, $h \neq 0$. Additionally, $l_S(hM) \subseteq l_S(fM)$. However, $gh = gf + g(1-f)g = g \neq 0$. Consequently, $l_S(hM) \subset l_S(fM)$, contradicting the choice of f. Therefore, $l_S(N) \cap l_S(fM) = 0$. Now, for any $\varphi \in l_S(N)$, we have $\varphi(1-f) = \varphi - \varphi f$. Since $f \in l_S(N)$, $\varphi(1-f) \in l_S(N) \cap l_S(fM) = 0$. Thus, $\varphi = \varphi f \in Sf$. This implies that $l_S(N) = Sf$, and hence M_R is π-e.Baer. The converse is deduced from Proposition 2.5. The other equivalent comes from [(21) Theorem 3.7].

Theorem 2.25 A *module M_R possesses the π-e.Baer property if and only if it is π-Rickart and the set $L =$*

$\{Se|e \in S_r(S)\}$ *forms a complete lattice in terms of inclusion.*

Proof. Let M_R be π-Rickart, $L = \{Se|e \in S_r(S)\}$ a complete lattice under inclusion, and X a projection invariant left ideal of S. Since X is projection invariant, $X = \sum_{\gamma\in\Gamma} x_\gamma I(S)$ for each $x_\gamma \in X$. Given M_R is π-Rickart, for each $\gamma \in \Gamma$, there exists an idempotent $e_\gamma \in S_l(S)$ such that $r_M(x_\gamma I(S)) = e_\gamma M$. Thus, we have $r_M(X) = \cap_{\gamma\in\Gamma} r_M(x_\gamma I(S))$, implying $r_M(X) = \cap_{\gamma\in\Gamma} e_\gamma M$. Hence, $l_S(r_M(x_\gamma I(S))) = S(1-e_\gamma)$ for each $\gamma \in \Gamma$. Since L is a complete lattice under inclusion, and $1 - e_\gamma \in S_r(S)$, there exists an element $e \in S_r(S)$ such that $Se \subseteq \cap_{\gamma\in\Gamma} S(1-e_\gamma)$. Consequently, $Se \subseteq l_S(r_M(X))$, yielding $r_M(X) = r_M(l_S(r_M(X))) \subseteq r_M(Se) = (1-e)M$. As $e \in l_S(r_M(x_\gamma I(S)))$ for each $\gamma \in \Gamma$, $em = 0$ for each $m \in r_M(X)$. Therefore, for each $m \in r_M(X)$, $m = (1-e)m$, implying $r_M(X) \subseteq (1-e)M$. Thus, M_R is π-e.Baer. Conversely, by Proposition 2.5, M_R is π-Rickart. Additionally, as M_R is π-e.Baer, by [(15), Theorem 5.1], S is a π-Baer ring and hence by [(16), Theorem 2.7], S is a complete lattice under inclusion.

2 *π-endo.AIP Modules*

A new class of modules, referred to as π-e.AIP, is introduced in this section. This class of modules extends its applicability to a considerably wider class by including the classes of π-Rickart and π-e.Baer modules. The interconnections between π–Rickart, endo-AIP, and π-e.AIP modules are explored. Moreover, the present work aimed to investigate potential connections between π-e.AIP module and the ring of its endomorphisms.

As defined in (28), a left ideal A of R is *right s-unital* if for every $x \in A$, there is some $y \in A$ for which $xy = x$.

Definition 3.1 *We say M_R is a π-endo.AIP module, denoted by π-e.AIP, if $l_S(L)$ forms a right s-unital ideal of S for all $L \trianglelefteq_p M$. A ring R is considered left π-AIP, if R_R is a π-e.AIP module.*

If $K \leq M$ and for all right R-module L, the map $L \otimes_R K \to L \otimes_R M$ is a monomorphism, then K is called pure. Whenever $l_S(K)$ stands as a pure left ideal for any fully invariant submodule K of M, it is recalled from (29) that M_R is termed *endo-AIP*.

Theorem 3.2 *The following implications holds true.*

(i) M_R is π-e.AIP $\Rightarrow$ M_R is endo-AIP.

(ii) M_R is Rickart $\Rightarrow$ M_R is π-e.AIP.

(iii) M_R is π-e.Baer $\Rightarrow$ M_R is π-Rickart $\Rightarrow$ M_R is π-e.AIP.

Proof. (i) Assume M is π-e.AIP and $K \trianglelefteq M$. For $x \in l_S(K)$, since $K \trianglelefteq_p M$, there exist $c \in l_S(K)$ such that $x = xc$. Consequently, $l_S(K)$ is right s-unital. According to [(30), Proposition 11.3.13], $l_S(K)$ is pure as a left ideal. Therefore, M_R is endo-AIP.

(ii) Suppose M is Rickart, $P \trianglelefteq_p M$ and $\varphi \in l_S(P)$. Then, $P \subseteq r_M(\varphi)$. As M_R is Rickart, there exists $g^2 = g \in S$ such that $r_M(\varphi) = gM$. Consequently, $1 - g \in l_S(P)$ and $\varphi(1-g) = \varphi$. Therefore, M is π-e.AIP.

(iii) By Proposition 2.5, π-e.Baer implies π-Rickart. Let M_R be π-Rickart, $P \trianglelefteq_p M$ and $\psi \in l_S(P)$. Then $\psi(P) = 0$, so $\psi I(S)P = 0$. Consequently, $P \subseteq r_M(\psi I(S))$. Since M is a π-Rickart module, there exists $c^2 = c \in S$ such that $P \subseteq r_M(\psi I(S)) = cM$. This implies that $1 - c \in l_S(P)$ and $\psi(1-c) = \psi$. Hence, M is π-e.AIP.

The following example illustrates that the converse of Theorem 3.2 does not necessarily hold in general.

Example 3.3 (i) endo-AIP $\nRightarrow$ π-e.AIP: Based on [(16), Example 1.6], there exists a right p.q.-Baer ring R with trivial idempotents that is not right π-Rickart. By [(29), Theorem 2.5], every right p.q.-Baer ring is endo-AIP. Thus, R_R is endo-AIP. Since R_R has π-IFP, R_R is π-e.AIP if and only if R_R is π-Rickart by Proposition 3.4. Therefore, R_R is not π-e.AIP.

(ii) π-e.AIP $\nRightarrow$ π-Rickart: Let R be the ring in [(22), Example 1.6]. Then, R is a right Rickart ring that is not right π-Rickart by[(16), p.5]. Consider $M = R_R$. Then, M is a π-e.AIP module, but not π-Rickart By Theorem 3.2(ii).

(iii) π-e.AIP $\nRightarrow$ Rickart: Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and $M = R_R$. By [(31), Proposition 3.12], M_R is endo-AIP. Note that $I(S) = S$, so M is π-e.AIP. However, M_R is not a Rickart module [(12), Example 2.9].

Proposition 3.4 *The following conditions are equivalent for module M with π-IFP.*

1. M_R is π-e.AIP.
2. M_R is Rickart.
3. M_R is π-Rickart.

Proof. (i) $\Rightarrow$ (ii) Consider $\eta \in S$ and $X = r_M(\eta)$. Given that M has π-*IFP*, $r_M(\eta) \trianglelefteq_p M_R$. Since M is π-e.AIP and $\eta \in l_S(X)$, there exists $\psi \in l_S(X)$ such that $\eta\psi = \varphi$. Consequently, $\eta(1-\psi) = 0$ and $r_M(\eta) \subseteq r_M(\psi)$. Hence, $\psi(1-\psi) = 0$, implying $\psi^2 = \psi \in S$. Let $e = 1 - \psi$, then $X = eM$. Thus, M_R is Rickart.

$(ii) \Rightarrow (iii)$ Let M_R be Rickart. Then, for any $\psi \in S$, there exists $g^2 = g \in S$ such that $r_M(\psi) = gM$. It is evident that $r_M(\psi I(S)) \subseteq r_M(\psi)$. As M has π-*IFP*, for every $m \in r_M(\psi)$, it follows that $m \in r_M(\psi I(S))$. Thus, we can deduce that $r_M(\psi I(S)) = gM$. Consequently, M_R is π-Rickart.

$(iii) \Rightarrow (i)$ follows directly from Theorem 3.2.

The necessity of π-*IFP* condition in Proposition 3.4 is illustrated by the following example. Notably, there is a module M within π-e.AIP class that lacks both the Rickart property and the π-*IFP* property.

Example 3.5 (i) Consider R and M_R as described in

Example 3.3 (ii). M_R is a π-e.AIP module but not Rickart. Now, let e_{ij} denote a 2×2 matrix with the element 1 in (i,j)-position and 0 elsewhere. Define φ as $\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and f as $e_{11} + e_{12}$. Note that $f^2 = f$. Since $\varphi e_{22} = 0$ while $\varphi f e_{22} \neq 0$, M fails to satisfy π-IFP.

(ii) Let $R = M_2(\mathbb{Z}[x])$ and $M = R_R$. since $\mathbb{Z}[x]$ is a domain, it is π-Rickart. Utilizing [(16), Proposition 3.10], we conclude that M is π-Rickart. Nonetheless, M is not Rickart as seen in [(32), Example 3.1.28]. Similar to the part (i), M_R fails to satisfy *π-IFP*.

A straight summand of a π-e.AIP module doesn't always inherit the π-e.AIP property. Example 2.9 shows that the module M_R meets the π-e.AIP requirement, which is supported by Theorem 3.2. However, the direct summand R_R does not fit this criterion, as shown in Example 3.3. The following discussion investigated the specific conditions under which a direct summand of a π-e.AIP module also retains the π-e.AIP property.

Theorem 3.*6* *In a π-e.AIP module, any direct summand that remains invariant under projection also fulfills the π-e.AIP condition.*

Proof. Consider M to be a π-e.AIP module, and let $P = gM \trianglelefteq_p M$ for some $g^2 = g \in S$, with $A \trianglelefteq_p P$. Then, $g \in S_l(S)$ and $E = \mathrm{End}_R(P) = gSg$. For any $\eta \in l_E(A)$, it follows that $\eta(A) = 0$, and one can find $\psi \in S$ with the property $\eta = g\psi g$. Observe from Lemma 2.2 that $A \trianglelefteq_p M$. Since $\eta \in l_S(A)$ and M is π-e.AIP, we have $a \in l_S(A)$ with $\eta = \eta a$. Note that $gag \in l_E(A)$ and $\eta(gag) = (g\psi g)(gag) = g\psi g = \eta$. It follows that P is π-e.AIP module.

Corollary 3.7 (i) *Given that M is a π-e.AIP module with an abelian endomorphism ring, it follows that all direct summands of M are also π-e.AIP.*

(ii) Let R_R be π-e.AIP and $e \in S_l(R)$. Then eR is a π-e.AIP module.

The example below shows that direct sums of π-e.AIP modules may not inherit the π-e.AIP property.

Example 3.8 Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_2$. It is evident that both $\mathbb{Z}$ and $\mathbb{Z}_2$ are π-e.Baer, implying they are π-e.AIP. Nevertheless, M itself is not π-e.AIP, as shown in [(29), Example 2.13].

Theorem 3.9 *Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ such that each M_γ meets π-e.AIP criterion while also being subisomorphic to M_η for all $\gamma \neq \eta \in \Gamma$. Then, M_R is a π-e.AIP module.*

Proof. For every $\gamma \in \Gamma$, let S_γ denote the ring of endomorphisms of M_γ. The ring of endomorphisms of M, denoted by S, is structured as a matrix ring. In this ring, the entry in the (γ,γ)-position comes from S_γ, and the entry in (γ,η)-position (for $\gamma,\eta \in \Gamma$ with $\gamma \neq \eta$) corresponds to a map from M_η to M_γ. Let $P \trianglelefteq_p M_R$. Then $P = \bigoplus_{\gamma \in \Gamma} P \cap M_\gamma = \bigoplus_{\gamma \in \Gamma} P_\gamma$ and $P_\gamma = P \cap M_\gamma \trianglelefteq_p M_\gamma$ for all $\gamma \in \Gamma$ by Lemma 2.2. Consider $\varphi \in l_S(P)$. since $\varphi(P) = 0$, it follows that $\varphi_{\gamma\gamma} \in l_{S_\gamma}(P_\gamma)$ for $\gamma \in \Gamma$. As M_η and M_γ are subisomorphic, there exists a monomorphism $\psi_{\gamma\eta}: M_\eta \to M_\gamma$. Clearly, for $\gamma \neq \eta \in \Gamma$, $\psi_{\gamma\eta}\varphi_{\eta\gamma}(P_\gamma) = 0$. Since M_γ satisfies the π-e.AIP property, $l_{S_\gamma}(P_\gamma)$ is right s-unital, and there exists a finite subset $\{\eta_1, \ldots \eta_n\}$ of Γ such that $\varphi_{\eta\gamma} \neq 0$. Hence there is $u_\gamma \in l_{S_\gamma}(P_\gamma)$ such that $\varphi_{\gamma\gamma}u_\gamma = \varphi_{\gamma\gamma}$ and $\psi_{\gamma\eta}\varphi_{\eta\gamma}u_\gamma = \psi_{\gamma\eta}\varphi_{\eta\gamma}$ for $\gamma \neq \eta \in \Gamma$. Since $\psi_{\gamma\eta}$ is a monomorphism for $\gamma \neq \eta \in \Gamma$, $\varphi_{\eta\gamma}u_\gamma = \varphi_{\eta\gamma}$. We construct an element $x = (u_{\gamma\eta})_{\gamma,\eta \in \Gamma}$, where $u_{\gamma\gamma} = u_\gamma$ and $u_{\gamma\eta} = 0$, if $\gamma \neq \eta$. Then $\varphi x = \varphi$. Thus, M exhibits the π-e.AIP property.

Corollary 3.10 *If a module is π-e.AIP, then its direct sum with any number of copies also preserves the π-e.AIP condition.*

Theorem 3.11 *For a π-e.AIP module M_R, then the ring of its endomorphisms, denoted $End_R(M)$, is a left π-AIP ring.*

Proof. Assume A is a projection invariat right ideal in S. For every $\varphi \in l_S(A)$, $\varphi A(M) = 0$. Then, $\varphi \in l_S(AM)$. Since $AM \trianglelefteq_p M$ and M is π-e.AIP, there is $\psi \in l_S(AM)$ such that $\varphi\psi = \varphi$. It follows that $\psi A = 0$, so $\psi \in l_S(A)$. Therefore, we can deduce that S is left π-AIP ring.

The subsequent example indicates that the converse of Theorem 3.11 is not necessarily valid. It shows that a module having a left π-AIP endomorphism ring does not ensure that the module itself is π-e.AIP.

Example 3.12 Let's consider $\mathbb{Z}$-module $M = \mathbb{Z}(p^\infty)$, where p is a prime number. The endomorphism ring $End(M_{\mathbb{Z}})$ is a commutative domain, specifically the ring of p-adic integers. Therefore, $End(M_{\mathbb{Z}})$ is a π-Baer ring, and by Theorem 3.2, it is π-AIP. From [(33), Theorem 1.2], it follows that $M_{\mathbb{Z}}$ being a duo module implies that it satisfies the π-IFP condition. However, as illustrated in [(12), Example 2.17], $M_{\mathbb{Z}}$ does not possess the Rickart property. Thus, based on Proposition 3.4, $M_{\mathbb{Z}}$ does not fulfill the condition of being π-e.AIP.

For the converse of Theorem 3.11 to be valid, the concept of being locally π-quasi-retractable is exactly the required condition.

Definition 3.13 *We say that M_R is locally π-quasi-retractable if for each $\gamma \in S$ where $r_M(S\gamma I(S)) \neq 0$, there is a nonzero element $\beta \in S$ satisfying $\beta(M) = r_M(S\gamma I(S))$.*

Proposition 3.14 *Given that M_R is locally π-quasi-retractable and S is a left π-AIP ring, it follows that M_R is π-e.AIP.*

Proof. Suppose $0 \neq X \trianglelefteq_p M$ and $\gamma \in l_S(X)$. Because $X \trianglelefteq_p M$, it follows that $S\gamma I(S) \subseteq l_S(X)$. Therefore, $0 \neq X \subseteq r_M(S\gamma I(S))$. Based on the concept of π- quasiretractability, there exists a nonzero $\beta \in S$ such that $\beta(M) = r_M(S\gamma I(S))$. Consequently, $X \subseteq r_M(S\gamma I(S)) = \beta(M)$ and $\gamma \in l_S(I(S)\beta S)$, where $I(S)\beta S$ is a projection invariant right ideal of S. Given that S is a left π-AIP ring, $l_S((I(S)\beta S)$ is right s-unital. Therefore, there exists $\gamma' \in l_S(I(S)\beta S)$ such that $\gamma = \gamma\gamma'$. Since $X \subseteq \beta(M)$, it follows that $\gamma'(I(S)X) \subseteq \gamma'(I(S)\beta(M)) = 0$. Hence, $\gamma' \in l_S(X)$, considering $X \trianglelefteq_p M_R$. Consequently, M is a π-e.AIP module.

According to [(16), Definition 2.1], a ring R is defined as *right ρ-regular* if, for each $\gamma \in R$ there exists $\eta = \eta^2 \in R$ such that $R\gamma I(R) = R\eta$.

Proposition 3.*15* *A right ρ-regular ring S ensures that M_R possesses the π-e.AIP condition.*

Proof. Based on [(16), Lemma 2.2], S is right π-Rickart. Consequently, S is is a left π-AIP ring by Theorem 3.2. For any $\gamma \in S$, since S is right ρ-regular, there exists a central idempotent $g \in S$ such that $S\gamma I(S) = gS$ as per [(16), Proposition 2.4]. Note that $g \neq 1$ and $0 \neq \beta = 1 - g \in S$. Given that g is central, we have $\beta(M) = r_M(S\gamma I(S))$. As a result, M is locally π -quasi-retractable. Consequently, Proposition 3.14 concludes the proof.

Corollary 3.16 *The ring of endomorphisms of a free module over a ρ-regular ring is a left π-AIP ring.*

Proof. Tis conclusion is derived from Corollary 3.10 and Theorem 3.11.

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References

1. Kaplansky I. Rings of Operators New York: Benjamin; 1965.
2. Clark WE. Twisted matrix units semigroup algebras. Duke Math J. 1967; 34(3): 417–23.
3. Maeda S. On a ring whose principal right ideals generated by idempotents form a lattice. J Sci Hiroshima Univ Ser A. 1960; 24: 509-25.
4. Armendariz EP. A note on extensions of Baer and P.P.-rings. J Austra Math Soc. 1947; 18: 470-73.
5. Berberian K. Baer *-Rings New York-Berlin: Springer Berlin, Heidelberg.
6. Bergman GM. Hereditary commutative rings and centres of hereditary rings. Proc London Math Soc. 1971; 23(4): 214-36.
7. Evans M. On commutative P.P. rings. Pacific J Math. 1972; 41(3): 687–97.
8. Hattori A. A foundation of torsion theory for modules over general rings. Nagoya Math J. 1960; 17: 147-58.
9. Jondrup S. p.p. rings and finitely generated flat ideals. Proc Ame. Math Soc. 1971; 28(2): 431-35.
10. Rizvi ST, Roman CS. Baer and quasi-Baer modules. Comm Algebra. 2004; 32(1): 103-23.
11. Lee G. Principally quasi-Baer modules and their generalizations. Comm Algebra. 2019; 47(10): 4077-94.
12. Lee G, Rizvi ST, Roman CS. Rickart Modules. Comm Algebra. 2010; 38(11): 4005-27.
13. Lee G, Rizvi ST, Roman CS. 𝔏-Rickart Modules. Comm Algebra. 2015; 43(5): 2124-51.
14. Birkenmeier GF, Kara Y, Tercan A. π-Baer rings. J Algebra Appl. 2018; 17(2).
15. Birkenmeier GF, Kara Y, Tercan A. π-endo Baer modules. Comm Algebra. 2020; 48(3): 1132-49.
16. Birkenmeier GF, Kara Y, Tercan A. π-Rickart rings. J Algebra Appl. 2021; 20(8).
17. Birkenmeier GF, Kara Y, Tercan A. Primitive and prime rings with s.Baer or related modules. J Algebra Appl. 2024; 23(03).
18. Birkenmeier GF, Kilic N, Mutlu , Tastan , Tercan , Yasar. Connections between Baer annihilator conditions and extending conditions for nearrings and rings. J Algebra Appl. 2025.
19. Kara Y. Weak π-Rickart rings. Comm Algebra. 2025;: 1-12.
20. KESKIN TÜTÜNCÜ D. A Study on the π-Dual Rickart Modules. J. Iran. Math. Soc. 2023 july; 4(2): 235-45.
21. Maleki Y, Moussavi A, Kara Y. On π-Endo.Rickart Modules. Iran J Sci. 2024.
22. Birkenmeier GF, Kim JY, Park JK. Principally quasi-Baer rings. Comm Algebra. 2001; 29(2): 638-60.
23. Lorenz M. K0 of Skew Group Rings and Simple Noetherian Rings without idempotents. J Lond Math Soc. 1985; 32(5): 41-50.
24. Birkenmeier GF, Tercan A, Yucel CC. The extending condition relative to sets of submodules. Commun Algebra. 2014; 42(2): 764-78.
25. Liu Q, Ouyang B, Wu T. Principally quasi-Baer modules. J Math Res Appl. 2009; 29(5): 823-30.
26. Huh C, Lee Y, Smoktunowicz A. Armendariz rings and semicommutative rings. Commun Algebra. 2002; 30(2): 751-61.
27. Lee G, Rizvi ST, Roman CS. Dual Rickart Modules. Comm Algebra. 2011; 39(11): 4036-58.
28. Tominaga H. On s-unital rings. Mathematical Journal of Okayama University. 1976; 18(2): 117-34.
29. Amirzadeh Dana P, Moussavi A. Modules in which the annihilator of a fully invariant submodule is pure. Comm Algebra. 2020; 48(11): 4875-88.
30. Wolfson KG. Baer rings of endomorphisms. Math Ann. 1961; 143(1): 19-28.
31. Majidinya A, Moussavi A, Paykan K. Rings in which the annihilator of an ideal is pure. Algebra Colloq. 2015; 22(01): 947-68.
32. Birkenmeier GF, Park JK, Rizvi ST. Extensions of Rings and Modules New York: Birkhauser; 2013.
33. Ozcan AC, Harmanci A, Smith PF. Duo modules. Glasg Math J. 2006; 48(03): 533-45.
34. Chatters AW, Khuri SM. Endomorphism rings of modules over nonsingular CS rings. J London Math Soc. 1980; 21(2): 434-44.
35. Birkenmeier GF, Kara Y, Tercan A. Primitive and prime rings with s.Baer or related modules. J Algebra Appl. 2024; 23(03).