Extension of Hardy Inequality on Weighted Sequence Spaces

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Abstract

Let \( p \geq 1 \) and \((w_n)\) be a sequence with non-negative entries. If \( T = (t_{n,k}) \geq 0 \), denote by \( \| T \|_{p,w} \) the infimum of those \( U \) satisfying the following inequality:

\[
\left( \sum_{n=1}^{\infty} W_n \left( \sum_{k=1}^{\infty} t_{n,k} a_k \right)^p \right)^{\frac{1}{p}} \leq U \left( \sum_{k=1}^{\infty} W_k a_k^p \right)^{\frac{1}{p}},
\]

whenever \((a_n) \in l_p(w)\). The purpose of this paper is to give an upper bound for the norm of operator \( T \) on weighted sequence spaces \( d(w,p) \) and \( l_p(w) \) and also \( e(w,\infty) \). We considered this problem for certain matrix operators such as Norlund, Weighted mean, Cesaro and Copson matrices. This problem is considered by some authors like Bennett, Jamson and the first author on sequence spaces \( l_p \) and weighted sequence spaces for some kind of matrix operators. Also, this study is an extension of paper by Chang-Pao Chen, Dah-Chin Luor and Zong-Yin Ou.

**Keywords:** Hardy inequality; Norlund matrix; Weighted mean matrix

Introduction

In this study we consider the norm of certain matrix operators on weighted sequence spaces \( l_p(w) \), \( e(w,\infty) \) and Lorentz sequence spaces \( d(w,p) \), \( p \geq 1 \), which is considered in [1] and [2] on \( l_p \) spaces and in [5-8] and [10] on \( l_p(w) \) and \( d(w,p) \) for some matrix operators such as Cesaro, Copson, Hausdorff and Hilbert operators.

Assume that \( l_p \) is the normed linear space of all sequences \( a = (a_k) \) with finite norm \( \| a \|_p \), where

\[
\| a \|_p = \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}}.
\]

Suppose that \( w = (w_n) \) is a sequence with non-negative entries. For \( p \geq 1 \), we define the weighted sequence space \( l_p(w) \) as follows:

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Let \( l_p(w) = \{ (a_n) : \sum_{n=1}^{\infty} w_n |a_n|^p < \infty \} \),
with norm \( \|a\|_{l_p(w)} \), where
\[
\|a\|_{l_p(w)} = \left( \sum_{n=1}^{\infty} w_n |a_n|^p \right)^{1/p}.
\]

Also, if \( (w_n) \) is a decreasing non-negative sequence such that \( \lim_{n \to \infty} w_n = 0 \) and
\[
\sum_{n=1}^{\infty} w_n = \infty,
\]
then the Lorentz sequence space \( d(w,p) \) is defined as follows:
\[
d(w,p) = \{ (a_n) : \sum_{n=1}^{\infty} w_n a_n^p < \infty \},
\]
where \( (a_n^*) \) is the decreasing rearrangement of \( (a_n) \).

In fact, \( d(w,p) \) is the space of null sequences \( a \) for which \( (a_n^*) \) is in \( l_p(w) \), with norm \( \|a\|_{l_p(w)} = \|a^*\|_{l_p(w)} \).

Let \( A_i^* = a_1^* + \cdots + a_i^* \) and \( W_k = w_1 + \cdots + w_k \), we define the weighted sequence space \( e(w,\infty) \) as follows:
\[
e(w,\infty) = \{ (a_n) : \sup_{i} A_i^*/W_i < \infty \},
\]
with norm \( \|a\|_{e(w,\infty)} \), which is defined as follows:
\[
\|a\|_{e(w,\infty)} = \sup_i A_i^*/W_i.
\]

Our objective in section 1 is to give a generalization of some results obtained by [1] and [2]. In section 2, we try to solve the problem of finding the norm of certain matrix operators on \( d(w,1) \) and \( e(w,\infty) \) and we deduce the existence of an upper bound for certain matrix operators such as Cesaro and Copson operators.

The problem of finding the lower bound of matrix operators on weighted sequence spaces is considered in [9].

**Results**

1. **Matrix Operators on \( d(w, p) \) and \( l_p(w) \)**

Now consider the operator \( T = (t_{i,j}) \) defined by \( Ta = b \), where \( b_i = \sum_{j} t_{i,j} a_j \). We write \( \|a\|_{l_p(w)} \) for the norm of \( T \) as an operator from \( l_p(w) \) into itself, and \( \|a\|_{d(w,p)} \) for the norm of \( T \) as an operator on \( d(w,p) \).

The following conditions is what we need to convert statements for \( l_p(w) \) to ones for \( d(w,p) \). We assume throughout that:

1. For all \( i,j \), \( t_{i,j} \geq 0 \).
2. For all \( i \), \( \lim_{j \to \infty} t_{i,j} = 0 \).
3. Either \( t_{i,j} \) decreases with \( j \) for each \( i \), or \( t_{i,j} \) decreases with \( i \) for each \( j \), and \( c_{a,j} = \sum_{i} t_{i,j} \) decreases with \( j \) for each \( m \).

Condition (1) implies that \( \|a\| \leq T \|a\| \) and hence the non-negative sequences are sufficient to determine norm of \( T \).

**Proposition 1.1.** ([5], Lemma 2.1). Let \( p \geq 1 \) and \( T = (t_{i,j}) \) be an operator with conditions (1), (2) and (3). Then
\[
\|T a\|_{d(w,p)} \leq \|a^*\|_{l_p(w)},
\]
for all non-negative elements \( a \) in \( d(w,p) \). Hence decreasing, non-negative elements are sufficient to determine norm of matrix operator \( T \).

In the following, we state some lemmas which are needed for main result. We set \( \xi^+ = \max(\xi,0) \) and \( \xi^- = \min(\xi,0) \) and also \( p^* = 1/p_{-1} \).

**Lemma 1.1.** ([2], Lemma 2.1). Assume that \( a,t \) are non-negative sequences. Then for all \( n \)
\[
\sum_{k=0}^{n} t_k a_k \leq \left( \max_{1 \leq i \leq n-k} \frac{1}{n-k+1} \sum_{j=i}^{n} t_j \right)^{1/p_{-1}} \sum_{k=0}^{n} (n-k+1)(t_k - t_{k-1})^*.
\]

**Lemma 1.2.** ([2], Lemma 2.2). Let \( N \geq 1 \) and \( a,t \) be non-negative sequences with \( a_n \geq a_{n+1} \geq \cdots \geq 0 \) and \( a_n = 0 \) for \( n < N \). Then for all \( n \)
\[
\sum_{k=0}^{n} t_k a_k \geq \left( \frac{1}{n} \sum_{j=1}^{n} d_j \right) \left( n \xi^+ \left( \frac{1}{n-N+1} \sum_{j=1}^{N} (n-k+1)(t_k - t_{k-1})^* \right) \right).
\]
Lemma 1.3. Suppose that \( u_n, v_n \) are non-negative numbers such that \( \sum_{n=1}^{\infty} u_n \) is divergent and \( \lim_{n \to \infty} v_n = 0 \). Then
\[
\sum_{n=1}^{m} \frac{u_n v_n}{u_n} \to 0 \text{ as } m \to \infty.
\]

Proof. If we take \( \varepsilon > 0 \), since \( \lim_{n \to \infty} v_n = 0 \), then there exists an integer \( N > 0 \) such that for all \( m > N \)
\[
\sum_{n=1}^{m} u_n v_n < \varepsilon \sum_{n=1}^{m} u_n + \frac{\varepsilon}{N} \sum_{n=1}^{N} u_n.
\]
Since \( \sum_{n=1}^{\infty} u_n \) is divergent, there exists an integer \( N > 0 \) such that for all \( m > N \) we have
\[
\sum_{n=1}^{m} u_n v_n < \frac{\varepsilon}{m} \sum_{n=1}^{m} u_n.
\]
Therefore
\[
\sum_{n=1}^{m} u_n v_n < 2 \varepsilon \sum_{n=1}^{m} u_n.
\]
If \( \varepsilon \to 0 \), we have the statement.

Proposition 1.2. ([5], Proposition 5.1). Let \( p > 1 \) and \((w_n)\) be a decreasing sequence with non-negative entries and let the matrix \( T = (t_{n,k}) \) be with the following entries:
\[
t_{n,k} = \begin{cases} 
\frac{1}{n} & \text{for } n \geq k \\
0 & \text{for } n < k.
\end{cases}
\]
Then \( \|T\|_{p,\infty} \leq p^\ast \).

Lemma 1.4. Let \( p > 1 \) and \((w_n)\) be a decreasing sequence with non-negative entries and also \( \sum_{n=1}^{\infty} w_n \) be divergent. Let \( N \geq 1 \) and the matrix \( C_N = (c_{n,k}^N) \) have the following entries:
\[
c_{n,k}^N = \begin{cases} 
1 & \text{for } n \geq k \\
0 & \text{for } n < k
\end{cases}
\]
Then \( \|C_N\|_{p,\infty} = p^\ast \).

Proof. \( C_1 \) is the Cesaro matrix and \( 0 \leq c_{n,k}^N \leq c_{n,k}^1 \) for all \( n,k \geq 1 \).
Since \((w_n)\) is a decreasing sequence, applying Proposition 1.2, we deduce that
\[
\|C_N\|_{p,\infty} \leq \|C_1\|_{p,\infty} \leq p^\ast.
\]
Fix \( m \) such that \( m \geq N \), and let
\[
a_n = \begin{cases} 
(n+m-1)^{\frac{1}{p^\ast}} & \text{for } 1 \leq n \leq m \\
0 & \text{for } n > m.
\end{cases}
\]
then \( \sum_{n=1}^{m} w_n a_n^p = \sum_{n=1}^{m} \frac{w_n}{n^{1/p^\ast}} \).

Also, for \( n \leq m \)
\[
A_n \geq \int_s^{(n+m-1)^{1/p^\ast}} ds = p^\ast \left( (n+m-1)^{1/p^\ast} - m^{1/p^\ast} \right),
\]
where \( A_n = a_1 + \cdots + a_n \).

So that
\[
b_n = \frac{A_n}{n+m-1} \geq \frac{p^\ast}{(n+m-1)^{1/p^\ast}} \left\{ 1 - \left( \frac{m}{n+m-1} \right)^{1/p^\ast} \right\}
\]
Since \((1-s)^p \geq 1 - ps \) for \( 0 < s < 1 \), we have
\[
b_n^p \geq \frac{(p^\ast)^p}{n+m-1} \left( 1 - p \left( \frac{m}{n+m-1} \right)^{1/p^\ast} \right),
\]
and hence
\[
\sum_{n=1}^{m} w_n b_n^p \geq (p^\ast)^p \sum_{n=1}^{m} \frac{w_n}{n+m-1}
\]
\[
- p \left( \frac{m}{n+m-1} \right)^{1/p^\ast} \sum_{n=1}^{m} \frac{w_n}{n+m-1} + \sum_{n=1}^{m} \frac{w_n}{n} \to \infty.
\]

Since \((w_n)\) is a decreasing sequence, \( w_n \geq w_{n+m-1} \) and so
\[
\sum_{n=1}^{m} w_n \geq \sum_{n=1}^{m} \frac{w_n}{n+m-1} \geq \sum_{n=1}^{m} \frac{w_n}{n} \to \infty.
\]

Therefore \( \sum_{n=1}^{m} \frac{w_n}{n+m-1} \) is divergent, setting \( x_n = \frac{w_n}{n+m-1}, \ y_n = \frac{1}{(n+m-1)^{1/p^\ast}} \) and apply Lemma 1.3, we have the statement.
In the following, we recall Theorem 8 of [3] which is needed for main result.

Theorem 1.1. ([3], Theorem 8). If \( p > 1 \) and \( x \) is a non-negative sequence, then
\[
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\[
\sum_{j=1}^{\infty} \left( \frac{1}{j-1+1} \sum_{k=1}^{\infty} x_k \right)^p \leq (p^*)^p \sum_{k=1}^{\infty} x_k^p.
\]

**Lemma 1.5.** If \( p > 1 \) and \( x, w \) are non-negative sequences and also \( w \) is decreasing, then
\[
\sum_{j=1}^{\infty} w_j \left( \frac{1}{j-1+1} \sum_{k=1}^{\infty} x_k \right)^p \leq (p^*)^p \sum_{k=1}^{\infty} w_k x_k^p.
\]

**Proof.** Applying Theorem 1.1, we have
\[
\sum_{j=1}^{\infty} w_j \left( \frac{1}{j-1+1} \sum_{k=1}^{\infty} x_k \right)^p \leq \sum_{j=1}^{\infty} w_j \left( \frac{1}{j-1+1} \sum_{k=j}^{\infty} x_k \right)^p
\]
\[
\leq (p^*)^p \sum_{k=1}^{\infty} w_k x_k^p.
\]

We set \( t_{n,0} = 0 \) for \( n \geq 1 \) and
\[
M_T = \sup_{n \geq 1} \left\{ \sum_{j=1}^{n} (n-k+1)(t_{n,k} - t_{n,k-1})^+ \right\},
\]
\[
m_T = \sup \left\{ \inf_{n \geq 1} \left\{ n \cdot \sum_{j=1}^{n} (n-k+1)(t_{n,k} - t_{n,k-1})^+ \right\} \right\}.
\]

We say that \( T = (t_{n,k}) \) is a lower triangular, if \( t_{n,k} = 0 \) for \( n < k \). We now introduce the first main result.

**Theorem 1.2.** Suppose \( p > 1 \) and \((w_k)\) is a decreasing sequence with non-negative entries. Let \( T = (t_{n,k}) \) be a lower triangular matrix with non-negative entries.

(i) \( \|f\|_{p,w} \leq p^* M_T \). Moreover, if \( M_T < \infty \), then \( T \) is bounded on \( l_p(w) \).

(ii) If \( \sum_{n=1}^{\infty} w_n = \infty \) and \((\frac{w_n}{n})\) is decreasing, then \( \|f\|_{p,w} \geq p^* m_T \).

Therefore if \((w_k)\) is a decreasing sequence with non-negative entries and \((\frac{w_n}{n})\) is decreasing and also \( \sum_{n=1}^{\infty} w_n = \infty \), then
\[
p^* m_T \leq \|f\|_{p,w} \leq p^* M_T.
\]

**Proof.** (i) Let \((a_n)\) be any sequence. By Lemma 1.1, we deduce that
\[
\sum_{k=1}^{\infty} t_{n,k} a_k \leq \max_{1 \leq s \leq n} \left\{ \sum_{j=1}^{s} \frac{1}{(n-k+1)j} \right\} \sum_{k=1}^{s} (n-k+1)(t_{n,k} - t_{n,k-1})^+
\]
\[
\leq M_T \max_{1 \leq s \leq n} \left\{ \sum_{j=1}^{s} \frac{1}{n-k+1} \right\}.
\]

Applying Lemma 1.5 and the maximal theorem of Hardy and Littlewood, we have
\[
\sum_{a=1}^{\infty} w_a \left( \sum_{k=1}^{\infty} t_{n,k} a_k \right)^p \leq M_T^p \sum_{a=1}^{\infty} w_a \max_{1 \leq k \leq a} \left( \sum_{j=1}^{k} \frac{1}{n-k+1} \right)^p
\]
\[
\leq (p^* M_T)^p \sum_{k=1}^{\infty} w_k a_k^p.
\]

This implies that
\[
\|f\|_{p,w} \leq p^* M_T.
\]

(ii) We have \( m_T = \sup_{N \geq 1} \beta_N \), where
\[
\beta_N = \inf_{n \geq N} \left\{ n \cdot \sum_{j=1}^{n} (n-k+1)(t_{n,k} - t_{n,k-1})^+ \right\}.
\]

Let \( N \geq 1 \), so that \( \beta_N \geq 0 \). Let \((b_n)\) be a decreasing sequence with non-negative entries and \( \|f\|_{p,w} = 1 \). We set \( a_1 = \cdots = a_{N-1} = 0 \) and
\[
a_{N+n} = \left( \frac{w_n}{w_{a+n}} \right)^{\frac{1}{p^*}} b_n,
\]
for all \( n \geq 1 \). We have \( \|f\|_{p,w} = \|f\|_{p,w} = 1 \), and Lemma 1.2 follows that
\[
\|f\|_{p,w} \geq \beta_N \sum_{a=1}^{\infty} w_a \left( \sum_{k=1}^{\infty} t_{n,k} a_k \right)^p
\]
\[
\geq \beta_N \sum_{a=1}^{\infty} w_a \left( \sum_{j=1}^{\infty} a_j \right)^p
\]
\[
= \beta_N \sum_{a=1}^{\infty} w_{a+n-1} \left( \sum_{j=1}^{\infty} \frac{1}{n-N-1} \right)^p
\]
\[
= \beta_N \left( \sum_{j=1}^{\infty} \frac{1}{n-N-1} \right)^p.
\]
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\[ \beta_n \sum_{n=1}^{\infty} W_{n,N} \left( \frac{1}{n+N-1} \sum_{j=1}^{n} \left( \frac{w_j}{w_j + N - 1} \right)^p b_j \right) \]

\[ \geq \beta_n \sum_{n=1}^{\infty} C_n b_n \]

Applying Proposition 1.1, we conclude that \( \| T \|_{p,w} \geq p^\star \beta_N \), and so

\[ \| T \|_{p,w} \geq p^\star m_T \]

This establishes the proof of the theorem.

In the following, we give some corollaries of Theorem 1.2. We assume \( (w_n) \) is a decreasing sequence with non-negative entries and \( \sum_{n=1}^{\infty} \frac{w_n}{n^\alpha} = \infty \).

**Corollary 1.1.** Suppose \( p > 1 \) and \( T = (t_{n,k}) \) is a lower triangular matrix with \( 0 \leq t_{n,k-1} \leq t_{n,k} \) for \( 1 < k \leq n \).

Then

\[ p^\star \left( \sup_{n \geq 1} \inf_{n \geq N} t_{n,N} \right) \leq \| T \|_{p,w} \leq p^\star \left( \sup_{n \geq 1} \left( \sum_{k=1}^{n} t_{n,k} \right) \right) \]

Moreover, if the right hand side of the above inequality is finite, then \( T \) is bounded on \( L_p(w) \).

**Proof.** We have \( M_T = \sup_{n \geq 1} \sum_{k=1}^{n} t_{n,k} \) and \( m_T = \inf_{n \geq N} \sup_{n \geq N} t_{n,N} \). This completes the proof of the statement.

**Corollary 1.2.** Assume that \( p > 1 \) and \( T = (t_{n,k}) \) is a lower triangular matrix with \( 0 \leq t_{n,k-1} \leq t_{n,k} \) for \( 1 < k \leq n \) and also \( (nt_{n,k}) \) is an increasing sequence for each \( k \).

Then

\[ \| T \|_{p,w} = p^\star \left( \sup_{n \geq 1} nt_{n,N} \right) \]

In particular, \( \| C_N \|_{p,w} = p^\star \), where \( C_N \) is the generalized Cesaro matrix defined in Lemma 1.4.

We apply the above corollary to the following two special cases.

Let \( (t_n) \) be a non-negative sequence with \( t_i \geq 0 \), and \( T_n = t_1 + \cdots + t_n \). The Norlund matrix \( N_T = (t_{n,k}) \) is defined as follows:

\[ t_{n,k} = \begin{cases} \frac{t_{n,k-1}}{T_n} & \text{for } 1 \leq k \leq n, \\ 0 & \text{for } k > n \end{cases} \]

**Corollary 1.3.** Suppose \( p > 1 \) and \( N_T = (t_{n,k}) \) is the Norlund matrix and \( (t_n) \) is a sequence decreasing with \( t_n \to \alpha \) and \( \alpha > 0 \). Then

\[ \| N_T \|_{p,w} = p^\star. \]

Let \( (t_n) \) be a non-negative sequence with \( t_i \geq 0 \). The Weighted mean matrix \( M_T = (t_{n,k}) \) is defined as follows:

\[ t_{n,k} = \begin{cases} \frac{t_k}{T_n} & \text{for } 1 \leq k \leq n, \\ 0 & \text{for } k > n \end{cases} \]

**Corollary 1.4.** Assume that \( p > 1 \) and \( M_T = (t_{n,k}) \) is the Weighted mean matrix and also \( (t_n) \) is an increasing sequence with \( t_n \to \alpha \) and \( \alpha < \infty \). Then

\[ \| M_T \|_{p,w} = p^\star. \]

**Corollary 1.5.** Suppose \( p > 1 \) and \( T = (t_{n,k}) \) is a lower triangular matrix with \( t_{n,k-1} \geq t_{n,k} \geq 0 \) for \( 1 < k \leq n \).

Then

\[ p^\star \left( \inf_{n \geq 1} \sum_{k=1}^{n} t_{n,k} \right) \leq \| T \|_{p,w} \leq p^\star \left( \sup_{n \geq 1} \left( nt_{n,1} \right) \right) \]

Moreover, if the right hand side of the above inequality is finite, then \( T \) is bounded on \( L_p(w) \).

**Proof.** We have \( M_T = \sup_{n \geq 1} \sum_{k=1}^{n} t_{n,k} \) and \( m_T \geq \inf_{n \geq N} \sup_{n \geq N} t_{n,N} \). This establishes the proof.

We apply the above corollary to the following two special cases.

**Corollary 1.6.** Assume that \( p > 1 \) and \( N_T = (t_{n,k}) \) is the Norlund matrix and \( (t_n) \) is an increasing sequence. Then

\[ p^\star \leq \| N_T \|_{p,w} \leq p^\star \left( \sup_{n \geq 1} \left( nt_{n,1} \right) \right). \]
Corollary 1.7. Suppose $p > 1$ and $M_p = (t_{n,k})$ is the Weighted mean matrix and also $(t_{n,k})$ is a decreasing sequence with $t_n \to \alpha$ and $\alpha > 0$. Then

$$p^* \leq \|M_p\|_{p,w} \leq p^\left(\frac{t_1}{\alpha}\right).$$

Example 1.1. Let $w_n = \frac{1}{\log(n+1)}$ where $0 < \gamma \leq 1$, $w_n$ and $(\frac{w_n}{s_n})$ be decreasing and also $\sum_{n=1}^\infty \frac{w_n}{s_n} = \infty$.

Therefore, if $(t_n)$ is a decreasing sequence with $t_n \to \alpha$ and $\alpha > 0$, then

$$\|N_p\|_{p,w} = p^*.$$

Also, if $(t_n)$ is an increasing sequence with $t_n \to \alpha$ and $\alpha < \infty$, then

$$\|M_p\|_{p,w} = p^*.$$

2. Matrix Operator on $d(w, 1)$ and $e(w, \infty)$

In this part of study, we consider the problem of finding the norm of matrix operator $C_N$ and $C'_N$ on $d(w, 1)$ and $e(w, \infty)$, where $d(w, 1)$ and $e(w, \infty)$ are defined as before.

If $a \in d(w, 1)$, we denote norm of $a$ with $\|a\|_{d,w}$ and if $a \in e(w, \infty)$, we denote norm of $a$ with $\|a\|_{e,w}$. We write $\left\|F T_{d,w}\right\|$ for the norm of $T$ as an operator from $d(w, 1)$ into itself, and $\left\|F T_{e,w}\right\|$ for the norm of $T$ as an operator from $e(w, \infty)$ into itself.

Suppose $T$ is a bounded matrix operator on $e(w, \infty)$. Then $T'$, the transpose matrix of $T$, is a bounded matrix operator on $d(w, 1)$ and

$$\left\|F T'\right\|_{d,w} \leq \left\|F T\right\|_{e,w}.$$

Let $N \geq 1$ and $C_N$ be defined as in Lemma 1.4, and also let $C'_N$ be the matrix transpose of $C_N$. The matrix $C'_N = (a_{n,k})$ is defined as follows:

$$a_{n,k} = \begin{cases} 
\frac{1}{k + N - 1} & \text{for } n \leq k, \\
0 & \text{for } n > k.
\end{cases}$$

If $N = 1$, $C_1$ and $C'_1$ are Cesaro and Copson matrices, respectively. $C_N$ and $C'_N$ are generalized Cesaro and Copson matrices.

The problem of finding the norm of matrix operators on $d(w, 1)$ and $e(w, \infty)$ is considered in [8]. Also in the following, we consider such problems for some matrices on weighted sequence spaces $d(w, 1)$ and $e(w, \infty)$.

Theorem 2.1. Suppose $T = (t_{n,k})$ is a matrix operator satisfying conditions (1), (2) and (3). If

$$\sup_{n} S_n < \infty,$$

where $S_n = s_1 + \cdots + s_n$, $s_n = \sum_{k=1}^\infty w_k I_{n,k}$, and $W_n = w_1 + \cdots + w_n$, then $T$ is a bounded operator from $d(w, 1)$ into itself, and also

$$\left\|F T\right\|_{d,w} = \sup_{n} \frac{S_n}{W_n}.$$

Proof. Applying Proposition 1.1, it is sufficient to consider decreasing, non-negative sequences. Let $a$ be in $d(w, 1)$ such that $a_1 \geq a_2 \geq \cdots \geq 0$ and $M = \sup_{n} \frac{S_n}{W_n}$. Then

$$\left\|F a\right\|_{d,w} = \sum_{n=1}^\infty \frac{S_n}{W_n} \left(\sum_{k=1}^\infty I_{n,k} a_k\right) = \sum_{n=1}^\infty S_n a_n = \sum_{n=1}^\infty S_n (a_n - a_{n+1}) \leq M \sum_{n=1}^\infty W_n (a_n - a_{n+1}).$$

Also, we have

$$\left\|a\right\|_{d,w} = \sum_{n=1}^\infty W_n (a_n - a_{n+1}).$$

Therefore

$$\left\|F a\right\|_{d,w} \leq M \left\|a\right\|_{d,w},$$

and hence $\left\|F T\right\|_{d,w} \leq M$.

Further, we take $a_1 = \cdots = a_k = 1$ and $a_k = 0$ for all $k \geq n + 1$, then

$$\left\|a\right\|_{d,w} = W_n, \quad \left\|F a\right\|_{d,w} = S_n.$$
Thus
\[ \|F\|_{\mu} = M. \]

This completes the proof of the theorem.

In the following statements, we consider the norm of Cesaro and Copson matrices. It is enough to consider the sequence \((\frac{w_n}{n})\) instead of \((\frac{s_n}{n})\), because of the well-known facts listed in the following lemma.

**Lemma 2.1.**

(i) If \(m \leq \frac{w_n}{n} \leq M\) for all \(n\), then \(m \leq \frac{s_n}{n} \leq M\) for all \(n\).

(ii) If \((\frac{w_n}{n})\) is increasing (or decreasing), then so is \((\frac{s_n}{n})\).

(iii) If \(\frac{w_n}{n} \to M\) as \(n \to \infty\), then \(\frac{s_n}{n} \to M\) as \(n \to \infty\).

**Proof.** It is elementary.

**Lemma 2.2.** Let \(0 < \alpha < 1\).

(i) If \(N \geq 1\) and \(X_n = \sum_{k=N}^{\infty} \frac{1}{(k + N - 1)^{\alpha}}\), then \(\frac{s_n}{n} \to \frac{1}{(N + N - 1)^{\alpha}}\) is increasing and tends to \(\frac{1}{(N + N - 1)^{\alpha}}\).

(ii) If \(X(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}\), then \(n^{-\alpha} X(\alpha)\) is decreasing.

**Proof.** It is elementary.

**Theorem 2.2.** If \(w_n = \frac{1}{(n + N - 1)^{\alpha}}\), where \(0 < \alpha < 1\), then \(C_N\) is a bounded operator on \(d(w,1)\) and also \(C_N'\) is a bounded operator on \(e(w,\infty)\). Moreover,
\[ \|C_N\|_{\mu} = \|C_N'\|_{\mu} = N\alpha \sum_{k=N}^{\infty} \frac{1}{(k + N - 1)^{1+\alpha}}. \]

In particular, \(\|C_N\|_{\mu} = \|C_N'\|_{\mu} = \xi(1+\alpha)\), where \(\xi\) is Riemann’s Zeta function.

**Proof.** Applying Theorem 2.1, we have
\[ \|C_N\|_{\mu} = \sup_{n} \frac{S_n}{W_n}. \]

Since
\[ \frac{s_n}{w_n} = (n + N - 1)^{\alpha} \sum_{k=n}^{\infty} \frac{1}{(k + N - 1)^{1+\alpha}} \]

\[ = (n + N - 1)^{\alpha} \sum_{k=n}^{\infty} \frac{1}{k^{1+\alpha}}, \]

Lemma 2.2(ii) shows that \(\frac{s_n}{w_n}\) is decreasing. Therefore applying Lemma 2.1(ii), we deduce that \(\frac{s_n}{w_n}\) is decreasing and also

**Proposition 2.1.** If
\[ r_N(w) = \sup_{n \geq 1} \frac{W_n}{(n + N - 1)^{\alpha}} < \infty, \]

then \(C_N\) maps \(d(w,1)\) into itself. Also, we have
\[ \|C_N'\|_{\mu} \leq r_N(w). \]

**Proof.** Since for all \(n\)
\[ s_n = \frac{W_n}{n + N - 1} \leq r_N(w) \cdot v_n, \]

Theorem 2.1 and Lemma 2.1(i) follow that
\[ \|C_N'\|_{\mu} \leq r_N(w), \] and this completes the proof.

**Proposition 2.2.** If
\[ \sup_{n \geq 1} \frac{1}{W_n} \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} W_k < \infty, \]

then \(C_N\) is a bounded operator on \(e(w,\infty)\) and
\[ \|C_N\|_{\mu} = \|C_N'\|_{\mu} = \xi(1+\alpha). \]

**Proof.** Applying Theorem 2.1, we have
\[ \|C_N'\|_{\mu} = \sup_{\alpha} \frac{S_n}{W_n}. \]

Since \(s_n = \frac{W_n}{\alpha + N - 2}\), and \(\|C_N'\|_{\mu} = \|C_N\|_{\mu}\), we have the statement.

**Theorem 2.3.** Suppose that \(w_n = \frac{1}{(n + N - 1)^{\alpha}}\), where \(0 < \alpha < 1\). Then \(C_N'\) maps \(d(w,1)\) into itself and also we have
\[ \|C_N\|_{\mu} = \|C_N'\|_{\mu} = \frac{1}{1-\alpha}. \]
In particular, \( \|C_1\|_\infty = \|C_1\|_p = \frac{1}{1-\alpha} \).

**Proof.** We have

\[
\frac{S_n}{w_n} = \frac{W_n}{(n+N-1)w_n} = \frac{W_n}{(n+N-1)^{1-\alpha}}.
\]

Our \( W_n \) is the \( X_\alpha \) of Lemma 2.2 (i), which tells us that \( \frac{w_n}{(n+N-1)^{1-\alpha}} \) is increasing and tends to \( \frac{1}{1-\alpha} \). Lemma 2.1 (ii) and (iii) follow the statement (Of course, this also shows that \( r_\alpha (w) = \frac{1}{1-\alpha} \)).

**References**