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Entropy Estimate for Maps on Forests

A. Golbaharan and M. Sabbaghan*

Faculty of Mathematics, Statistics and Computer Science, University College of Science, University of Tehran, Tehran, Islamic Republic of Iran

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Abstract

A 1993 result of J. Llibre, and M. Misiurewicz, (Theorem A [5]), states that if a continuous map f of a graph into itself has an s-horseshoe, then the topological entropy of f is greater than or equal to log$s$, that is $h(f) \geq \log s$. Also a 1980 result of L.S. Block, J. Guckenheimer, M. Misiurewicz and L.S. Young (Lemma 1.5 [3]) states that if G is an A-graph of f then $h(G) \leq h(f)$. In this paper we generalize Theorem A and Lemma 1.5 for continuous functions on forests. Let F be a forest and $f: F \rightarrow F$ be a continuous function. By using the adjacency matrix of a graph, we give a lower bound for the topological entropy of f.

Keywords: Entropy; Forest; Graph; Horseshoe

Introduction

A topological space homeomorphic with $[0,1]$ is called an *arc*. A topological space F is called a *forest* if it is a finite union of arcs such that any pair of them has at most one point in common, and F does not have any homeomorphic copy of the unit circle. A connected forest is called a *tree*. Clearly any connected component of a forest is a tree. A point x 0f the tree T is an *end point* if $T-\{x\}$ is connected, and it is a *branching point* if $T-\{x\}$ has at least 3 connected components. The number of all connected components of $T-\{x\}$ is called the *valence* of x and is denoted by $Val(x)$. A point x of the forest F is an *end* (resp. *branching*) point if it is an end (resp. branching) point of the connected component containing x. The set of all end points of F is denoted by $End(F)$. An end point or a branching point of a forest F is called a *node*. The set of all nodes of F is denoted by $Node(F)$. For every x and y of a connected component T of a forest F, the interval $[x,y]$ is the smallest connected subset containing x, y and $(x,y)=[x,y]-\{x,y\}$. For further information about these definitions refer to [2] and [5].

Let T be a tree and let $f: T \rightarrow T$ be continuous and let also $(I_i)_{i=1}^{s}$ be a finite sequence of pairwise disjoint (except perhaps ends) closed intervals in T such that for every $j \in \{1,2,...,s\}$ we have $\bigcup_{i=1}^{s} I_i \subseteq f(I_j)$, then the sequence $(I_i)_{i=1}^{s}$ is called an $s-$*horseshoe.* In [5] it is shown that $\log(s)$ is a lower bound for the topological entropy of f (for definition of topological entropy refer to [6], [7] and [8]).

A directed graph D is a triple $D(V(D), A(D).\psi_D)$ consisting of a nonempty set $V(D)$ of vertices, a set

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* Corresponding author, Tel.: +98(21)44671474, Fax: +98(21)66412178, E-mail: sabbagh@khayam.ut.ac.ir

$A(D)$ of arcs which is distinct from $V(D)$ and a function ψ_D which associates an ordered pair of vertices (not necessarily distinct) with each arc of D. If a is an arc and u and v are two vertices such that $\psi_D(a)=(u,v)$, we say a joins u to v, where u is the *tail* and v is the *head* of a. A directed graph D' is a subgraph of D if $V(D')\subseteq V(D), A(D')\subseteq A(D)$ and $\psi_{D'}$ is the restriction of ψ_D onto $A(D')$. A *directed walk* in D is a finite sequence $W=(v_0,a_1,v_1,...,a_k,v_k)$ such that its terms are alternately vertices and arcs, and for every $i=1,2,...,k$, the arc a_i has the tail v_{i-1} and the head v_i. Any directed walk $W=(v_0,a_1,v_1,...,a_k,v_k)$ is often represented simply by its vertices sequence $W=(v_0,v_1,...,v_k)$. In this case we say W is a walk from v_0 to v_k or W is a $(v_0,v_k)-$ walk. The vertices v_0 and v_k are called the *origin* and the *terminus* of W, respectively and $v_1,...,v_{k-1}$ are called the *internal vertices*. The integer k is the length of W. If the arcs $a_1,...,a_k$ in a directed walk W are distinct, W is called a *directed trail*. If in addition the vertices $v_0,\ldots,v_k$ are distinct, W is called a *directed path*. A cycle is a directed trail such that all vertices except the head are distinct and the head vertex is the same as the tail vertex. Let D be directed graph with m vertices, we assign an $m\times m$ matrix $M(D)=\left[m_{ij}\right]_{m\times m}$ to D called the *adjacency matrix* as follows: Assume $V(D)=\{v_1,v_2,...,v_m\}$. Then m_{ij} is the number of arcs that join v_i to v_j. Two vertices u and v in D are said to be *strongly connected* if the directed paths (u,v) and (v,u) exist. *Strongly connected* relation is an equivalence relation on the vertices set. The subgraphs $D[V_1],D[V_2],\cdots,D[V_t]$ induced by the resulting partition $V_1,V_2,...,V_t$ of $V(D)$ are called the *dicomponnents* of D. The number of arcs with the head (resp. the tail) v is the indegree (resp. outdegree) of v. For further information refer to [4]. It is clear that between any two dicomponents of D only one direction exists and any cycle in D belongs to exactly one of the dicomponents.

Let $I=[0,1]$, S^1 is the unit circle, and $f:S^1\to S^1$ or $f:I\to R$ is continuous. Furthermore, let $A=\{I_1,I_2,...,I_s\}$ be a partition of S^1 or I, into subintervals (we shall call the arc on S^1 the interval too) i.e. a family of closed interval such that $I_1\bigcup I_2\bigcup...\bigcup I_s$ is S^1 or I, respectively and for $i\neq j, I_i\bigcap I_j$ is at most a single point set. Let D be a directed graph with the set of vertices $V(D)=\{I_1,I_2,...,I_s\}$. Suppose for every $i,j\in\{1,2,\ldots,s\}$, if I_i contains exactly n closed subintevals with distinct interiors such that their images under f contain I_j (and n is the biggest integer with such property) then exactly n arcs from I_i to I_j exist. Let M be the adjacency matrix of graph D. It is shown in [4] that $h(f)\geq\lim_{n\to\infty}\frac{1}{n}\log$ (the sum of all entries of M^n).

In this paper a lower bound is given for forest maps in such a way that the two previously stated inequalities will be an special case of our result.

Materials and Methods

Let D be a directed graph and W be a directed walk in D. For every two vertices v_i and v_j in W, the number of arcs from v_i to v_j is called the *distance* from v_i to v_j and is denoted by $d(v_i,v_j)$. If the directed walk W does not have repetition vertex then W is a directed path, otherwise let u be a repletion vertex which has the least distance with itself. So a part of this walk from u to u is a cycle. Now if we replace this cycle by vertex u, we have a new directed walk in which the number of repetition vertices is less than the number of repetition vertices in W. By continuing this procedure all repetition vertices will be eliminated, and the procedure remains a directed path say P_W and we say W is generated by P_W. So in directed graph D every directed walk can be generated by a directed path in which some vertex is replaced by a cycle which can be repeated finitely many times. By induction it can be said that every directed walk is generated by a unique directed path.

Let $D[V_i]$ be a dicomponent and u and v be two vertices of $D[V_i]$. If every vertices in $D[V_i]$ has outdegree 1, $D[V_i]$ is a cycle. Assume the length of this cycle is λ. For every n belonging to $\{i\lambda+d(u,v): i=0,1,2,\ldots\}$, there exists exactly one directed walk of length n from u to v and for every other natural number m there is no directed walk from u to v of length m in the graph D.

But if $D[V_i]$ is neither a cycle nor a vertex, since it is a dicomponent, it has at least a cycle. We claim that in every arbitrary cycle in $D[V_i]$ there exist vertices x

and y (not necessarily distinct) and a path from x to y disjoint from the cycle except in origin and terminal. Let C be an arbitrary cycle in $D[V_i]$. Since $D[V_i]$ is not a cycle nor a vertex, so there exist an arc or vertex in $D[V_i]$, but not on cycle C.

Case one: If there exists a vertex z in $D[V_i]$ that is not in cycle C, then there exist two directed paths between z and two vertices of the cycle C. In other word, vertices x and y exist on cycle C such that the directed paths (x,z) and (z,y) in $D[V_i]$ have smallest length. Assume the directed walk generated by these paths is W. since (x,z) and (z,y) have been chosen with the smallest length, so the directed walk W does not have any intersection with the cycle C except in origin and terminus. Furthermore, the directed walk $W=(x,\cdots,z,\cdots,y)$ has been generated by a path from x to y, so a path exists from x to y in $D[V_i]$ disjoint from the cycle C except in origin and terminus. Therefore, there exist cycles C_1 and C_2 with length λ_1 and λ_2 in $D[V_i]$ containing vertices x and y, respectively.

Case two: There exists an arc in $D[V_i]$, that does not belong to the cycle C. If the tail and the head of this arc is belong to the cycle C, this arc is a path which is distinct from the cycle except for the tail and the head, but if neither the tail nor the head belong to the cycle C, we must return to previous case repeatedly.

Since the number of vertices and cycles in the graph is finite, it has an upper bound l. Suppose $C_1,C_2,...,C_l$ are all the cycles in the graph D with length $\lambda_1,\lambda_2,...,\lambda_l$ respectively. Let W be a walk with the length of n from u to v. This walk has been generated by a path $W_0=P_W$ such that in i^{th} step the cycle θ_i has been replaced by the vertex z_{i-1} in the walk W_{i-1}. Therefore the sentence of sequence $\{\theta_i\}_{i=1}$ is in the set $\{C_1,...C_l\}$. Let for $j\in\{1,...,l\}$ the cycle C_j appear in the sequence $\{\theta_i\}_{i=1}$ for n_j times and first time in the $i_j^{\,th}$ sentence. So in the step i_j the cycle $(z_{i_j-1},\dots,z_{i_j-1})$ that is in the same cycle C_j has been replaced the vertex z_{i_j-1} in the walk W_{i_j-1} and the walk W_{i_j} is produced. Without loss of generality suppose that $1=i_1\langle i_2\langle...\langle i_l$. If for every $j\in\{1,...,l\}$, W_j' is a walk with the length of $n_j\lambda_j$ with origin and terminus z_{i_j-1} obtained by n_j times travel the cycle C_j, then we could have a walk W' with the length of n from u to v that is constructed in $l+1$ steps. First let $W_0'=P_W$ and in $j^{\,th}$ step, the walk W_j' replace the vertex z_{i_j-1} Since the length of P_W is at most l so $\sum_{j=1}^{l}n_j\lambda_j\geq n-l$ and we have

$$n-l\leq\sum_{j=1}^{l}n_j\lambda_j\leq\sum_{j=`}^{l}n_jl=l\sum_{j=1}^{l}n_j$$

$$\leq l^2\max\{n_j\}_{j=1}^{l}\Rightarrow\frac{n-l}{l^2}\leq\max\{n_j\}_{j=1}^{l}.$$

Hence there exists at least one cycle C_{j_1} which has traveled n_{j_1} times and we have $\frac{n-l}{l^2}\leq n_{j_1}$. But by the above explanation, the cycle C_{j_1} contains two vertices as x and y which has been join by a path disjoint from the cycle except at the origin and terminus (in Figure 1 let two cycles be C_{j_1} and C_{j_2} and take the vertex $z_{i_{j_1}-1}$ on the cycle C_{j_1}). Assume $n_{j_1}-1=q\lambda_{j_2}+r:0\leq r\leq\lambda_{j_2}-1$. Since there are at least two choices for going through the walk with the length of $\lambda_{j_1}\lambda_{j_2}$ from x to x (λ_{j_1} times going through C_{j_2} or λ_{j_2} times going through C_{j_1}), for going through the walk of length $q\lambda_{j_1}\lambda_{j_2}$ from x to x there are also at least 2^q choices, and for each of these choices after r times going through the cycle C_{j_1} we obtain a walk with the length of $q\lambda_{j_1}\lambda_{j_2}+r\ \lambda_{j_1}=\lambda_{j_1}(n_{j_1}-1)$ from x to x. So there exist 2^q distinct walks of length $\lambda_j(n_j-1)$ from x to x and for each of them in the cycle $C_{j1}=(z_{i_{j1}-1},\dots,z_{i_{j1}-1})$, we can replace that walk by vertex x, in this case a walk of length $n_{j_1}\lambda_{j_1}$ from

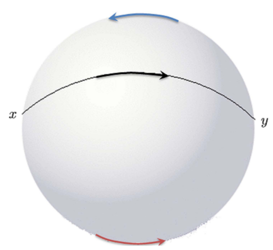

Figure 1.

$z_{i_{j_1}-1}$ to $z_{i_{j_1}-1}$ is obtained. Thus there exist at least 2^q distinct walks of length $n_{j_1}\lambda_{j_1}$ from $z_{i_{j_1}-1}$ to $z_{i_{j_1}-1}$, hence there exist at least 2^q walks of length n from u to v. However, we have

$$q \geq \frac{n_{j_1}-1}{\lambda_{j_2}} - 1 \geq \frac{\frac{n-l}{l^2}-1}{l} - 1 = \frac{n}{l^3} - \left(\frac{1}{l^2}+\frac{1}{l}+1\right)$$

$$\geq \frac{n}{l^3} - 3 \Rightarrow 2^q \geq \frac{1}{8}\left(\sqrt[l^3]{2}\right)^n .$$

Hence there exist constants α_{uv} and β_{uv} such that if the number of walk with the length of n from u to v is not zero, it is greater than or equal to $\beta_{uv}\,(\alpha_{uv})^n$. (✱)

Lemma 1: Let $D(V(D), A(D), \psi_D)$ be a directed graph of t vertices and let M be its adjacency matrix. Then the (i,j) entry of matrix M^n is the number of distinct walks of length n from the i^{th} vertex to the j^{th} vertex, so the sum of all entries of matrix M^n is equal to the number of all distinct walks of length n in graph D.

Proof: Refer to [4]. ■

We used the notation ${}^n a_{ij}$ for (i,j) entry of matrix M^n and a_n for the sum of its entries.

Now we show that for some graph D there exists a subgraph D' such that for its adjacency matrix M' and sequence $\left\{{}^n a'_{ij}\right\}_{n=1}^{\infty}$, There exists an exponential sequence such that all elements of the sequence $\left\{{}^n a'_{ij}\right\}_{n=1}^{\infty}$ except zero terms are greater than the elements exponential sequence and

$$\lim_{n\to\infty}\frac{1}{n}\log(a_n) = \lim_{n\to\infty}\frac{1}{n}\log(a'_n). \qquad (✱✱)$$

Let $D(V(D), A(D), \psi_D)$ be a directed graph of t vertices and let M be its adjacency matrix. We have $\lim_{n\to\infty}\frac{1}{n}\log(a_n) = \lim_{n\to\infty}\frac{1}{n}\log\left(\max_{1\le i,j\le t} {}^n a_{ij}\right)$, because:

$$\overline{\lim}\frac{1}{n}\log\left(\max_{1\le i,j\le t} {}^n a_{ij}\right) \le \overline{\lim}\frac{1}{n}\log\left(\sum_{i,j=1}^{t} {}^n a_{ij}\right)$$

$$\le \overline{\lim}\frac{1}{n}\log\left(t^2 \max_{1\le i,j\le t} {}^n a_{ij}\right) = \overline{\lim}\frac{1}{n}\log\left(\max_{1\le i,j\le t} {}^n a_{ij}\right)$$

$$\Rightarrow \overline{\lim}\frac{1}{n}\log\left(\max_{1\le i,j\le t}\left({}^n a_{ij}\right)\right) = \overline{\lim}\frac{1}{n}\log(a_n).$$

Similar procedure for liminf shows the equality.

For every $i,j \in V(D)$, let E_{ij} be the set of all subsequencial limits of $\left\{\frac{1}{n}\log\left({}^n a_{ij}\right)\right\}_{n=1}^{\infty}$ and $\alpha_{ij} = \overline{\lim}_{n\to\infty}\frac{1}{n}\log\left({}^n a_{ij}\right) = \sup(E_{ij})$. Since in the above equality maximum is taken over a finite number of terms (t^2 terms), so we have:

$$\overline{\lim}_{n\to\infty}\frac{1}{n}\log(a_n) = \overline{\lim}_{n\to\infty}\frac{1}{n}\log\left(\max_{1\le i,j\le t} {}^n a_{ij}\right)$$

$$= \sup\left(\bigcup_{i,j=1}^{t} E_{ij}\right) = \max_{1\le i,j\le t} \alpha_{ij}.$$

Let k, l be two vertices of a dicomponent. For every vertices k' and l' of this dicomponent, we have $\alpha_{kl} = \alpha_{k'l'}$, because $\alpha_{kl} = \overline{\lim}_{n\to\infty}\frac{1}{n}\log({}^n\alpha_{kl})$, and there exists walks with the lengths of p_1 and p_2 from k' to k and from l to l' respectively. So there exists at least ${}^n a_{kl}$ walks of length p_1+p_2+n from k' to l' and we have

$$\alpha_{k'l'} = \overline{\lim}_{n\to\infty}\frac{1}{n}\log\left({}^n a_{k'l'}\right)$$

$$\ge \overline{\lim}_{n\to\infty}\frac{1}{p_1+p_2+n}\log\left({}^n a_{kl}\right) = \alpha_{kl}$$

By using reverse inequality, we have $\alpha_{kl} = \alpha_{k'l'}$. So for each dicomponent $D[V_i]$, there exists α_i such that for every vertices l and k in $D[V_i]$ we have $\alpha_i = \overline{\lim}_{n\to\infty}\frac{1}{n}\log\left({}^n a_{kl}\right) = \sup(E_{kl})$ (for single vertex dicomponent take α_i zero).

Since for every $i,j \in \{1,2,...,t\}$, E_{ij} is closed, $\alpha_0 = \sup\left(\bigcup_{i,j=1}^{t} E_{ij}\right) \in \bigcup_{i,j=1}^{t} E_{ij}$. Hence there exist $i_0, j_0 \in \{1,2,...,t\}$ and there exist a sequence $\left\{\frac{1}{n_k}\log\left({}^{n_k} a_{i_0 j_0}\right)\right\}_{k=1}^{\infty}$ such that $\alpha_0 = \lim_{k\to\infty}\frac{1}{n_k}\log\left({}^{n_k} a_{i_0 j_0}\right)$. We show that there exist vertices i_1 and j_1 in a dicomponent such that $\alpha_0 = \sup\left(E_{i_1 j_1}\right)$. Assume there exist l paths between i_0 and j_0. For every natural number k, ${}^{n_k} a_{i_0 j_0}$ is the number of walks of length n_k from i_0 to j_0 each one has been generated by a path from i_0 to j_0. So there exists a path P_k such that at least $\frac{1}{l}\,{}^{n_k}a_{i_0 j_0}$ of the walks have been generated by the path P_k. Since the total number of paths are finite, so $\{P_k\}_{k=1}^{\infty}$ has a constant subsequence $\{P_{k_m} = P\}_{m=1}^{\infty}$. Hence if for every natural number m, the number of paths of length n_{k_m} from i_0 to j_0 generated by path P is b_m, we have $\frac{1}{l}\,{}^{n_{k_m}}a_{i_0 j_0} \le b_m \le {}^{n_{k_m}}a_{i_0 j_0}$. We conclude that $\alpha_0 = \lim_{k\to\infty}\frac{1}{n_k}\log\left({}^{n_k}a_{i_0 j_0}\right) = \lim_{m\to\infty}\frac{1}{n_{k_m}}\log(b_m)$.

Let $P = \left(i_0 = v_{t_1}, ..., u_{t_1}, v_{t_2}, ..., u_{t_2}, ..., v_{t_r}, ..., u_{t_r}\right)$ be a walk such that every subpath $\left(v_{t_i}, ..., u_{t_i}\right)$ belongs to the dicomponent $D[V_i]$. Let m be fixed, and for any non negative integers $n'_1, ..., n'_r$ such that $\sum_{i=1}^{r} n'_i = n_{k_m}$, let b'_i be the number of walks of length n'_i generated by subpath $(v_{t_i}, ..., u_{t_i})$ in the component $D[V_i]$. Since the walks are obtained by repetition of replacing a vertex by a cycle on the path, and every cycle is in one dicomponent, we have

$$b_m = \sum_{\left\{(n'_1,...,n'_r):\sum_{i=1}^{r} n'_i = n_{k_m}\right\}} \prod_{i=1}^{r} b'_i .$$

Let $\hat{\alpha} = \max\{\alpha_i : 1 \le i \le r\}$ and let $\varepsilon \rangle 0$. There exists a natural number N such that for every vertices k and l in a dicomponent, if $n \ge N$ then ${}^{n}a_{kl} \langle \left(e^{\hat{\alpha}+\varepsilon}\right)^n$. Let

$$A = \left\{\left(n'_1, n'_2, ..., n'_r\right) : \sum_{i=1}^{r} n'_i = n_{k_m}, n'_i \rangle N ; i = 1, 2, ..., r\right\}.$$

If $m \rangle rN$ then $\sum_{i=1}^{r} n'_i = n_{k_m} \ge m \rangle rN$. So for $\left(n'_1, n'_2, ..., n'_r\right) \in A^c$ there exists i_1 such that $n'_{i_1} \ge N$, so $b'_{i_1} \ge \left(e^{\hat{\alpha}+\varepsilon}\right)^{n'_{i_1}}$. However, for the other entries, since the length of walk is less than N, the number of obtained walks for a constant K is less K. Therefore, we have

$$b_m = \sum_{\left\{(n'_1,...,n'_r):\sum_{i=1}^{r} n'_i = n_{k_m}\right\}} \prod_{i=1}^{r} b'_i = \sum_{A}\prod_{i=1}^{r} b'_i + \sum_{A^c}\prod_{i=1}^{r} b'_i$$

$$\le \sum_{A}\prod_{i=1}^{r}\left(e^{\hat{\alpha}+\varepsilon}\right)^{n'_i} + \sum_{A^c} K \prod_{j}\left(e^{\hat{\alpha}+\varepsilon}\right)^{n'_{i_j}}$$

$$= \sum_{A}\left(e^{\hat{\alpha}+\varepsilon}\right)^{\sum_{i=1}^{r} n'_i} + \sum_{A^c} K e^{\sum_j n'_{i_j}(\hat{\alpha}+\varepsilon)} \le \sum_{A} e^{n_{k_m}(\hat{\alpha}+\varepsilon)}$$

$$+\sum_{A^c} K e^{n_{k_m}(\hat{\alpha}+\varepsilon)} \le \sum_{\left\{(n'_1,n'_2,...,n'_r):\sum_{i=1}^{r} n'_i = n_{k_m}\right\}} K e^{n_{k_m}(\hat{\alpha}+\varepsilon)}$$

$$= \binom{n_{k_m} + r - 1}{r-1} K e^{n_{k_m}(\hat{\alpha}+\varepsilon)}.$$

Hence

$$\alpha_0 = \lim_{n\to\infty}\frac{1}{n}\log(a_n) = \lim_{k\to\infty}\frac{1}{n_k}\log\left(a_{n_k}\right)$$

$$= \lim_{m\to\infty}\frac{1}{n_{k_m}}\log(b_m)$$

$$\le \lim_{m\to\infty}\frac{1}{n_{k_m}}\log\left(\binom{n_{k_m} + r - 1}{r-1} K e^{n_{k_m}(\hat{\alpha}+\varepsilon)}\right) = \hat{\alpha} + \varepsilon.$$

That is

$\alpha_0 \le \hat{\alpha} = \max\{\alpha_i : 1 \le i \le r\} \le \alpha_0$. So $\hat{\alpha} = \alpha_0$.

Therefore, there exists a dicomponent in graph D such that for every vertices u and v in this dicomponent we have $\lim_{n\to\infty}\frac{1}{n}\log(a_n) = \alpha_0 = \lim_{n\to\infty}\frac{1}{n}\log\left({}^{n}a_{uv}\right)$. Let M' be the matrix of the subgraph of this component and let a_n' be the sum of its entries, we have $\lim_{n\to\infty}\frac{1}{n}\log(a_n) = \lim_{n\to\infty}\frac{1}{n}\log\left(a_n'\right)$.

If this component is either a cycle or a single vertex, the sequence $\{a'_n\}$ is bounded, hence $\lim_{n\to\infty}\frac{1}{n}\log\left(a_n'\right) = 0$ which is a trivial bounded for topological entropy. However if this component is neither a single vertex nor a cycle, by (✱) for every vertices u and v in this dicomponent, there exist constants α_{uv} and β_{uv} such that for every natural number n, the number of walks with the length of n from u to v, if it is not zero, are greater than or equal to $\beta_{uv}(\alpha_{uv})^n$. So we may assume for every (i, j) entry in M'^n, the elements of $\left\{{}^{n}a_{ij}'\right\}_{n=1}^{\infty}$, except zero terms, are greater than or equal to the element of an exponential sequence.

Theorem 1: Let X be a compact topological space and

$f: X \to X$ be continuous. Let also $(A_r)_{r=1}^{t}$ be a collection of nonempty pairwise disjoint closure subsets of X. Assume for every $r \in \{1,2,...,t\}$ we have $A_r = \bigcup_{s=1}^{t_r} I_{rs} : t_r \in N$, in such a way that I_{rs}'s are nonempty and their closures are pairwise disjoint. Take the directed graph $D(V(D), A(D), \psi_D)$ as follows: $V(D) = \{A_1, A_2, ..., A_t\}$ and if there exist $r, r' \in \{1,2,...,t\}$ and $s \in \{1,2,...,t_r\}$ such that $f(I_{rs}) \supseteq A_{r'}$, then there exists an arc from A_r to $A_{r'}$. Let M be the adjacency matrix of D. Then $h(f) \geq \lim_{n\to\infty} \frac{1}{n} \log$ (the sum of all entries of M^n).

Proof: Since X is compact, $Y \subseteq f(Z)$ implies $\overline{Y} \subseteq f(\overline{Z})$. Hence without lose of generality we may assume all I_{rs}'s are closed. For every $n \in N$ and every walk $\tau = (j_0, ..., j_n)$ in D and every $i \in \{1,2,...,n\}, (j_{i-1}, j_i)$ is an arc in D. So there exists a subset $I_{j_{i-1}k_{i-1}} \subseteq A_{j_{i-1}}$ such that $f(I_{j_{i-1}k_{i-1}}) \supseteq A_{j_i}$. For every arbitrary subset $I_{j_n k_n}$ of A_{j_n} we show that $\bigcap_{i=0}^{n} f^{-i}(I_{j_i k_i})$ is nonempty. Induction is used for every walk $\tau = (j_0, j_1)$ with the length of1 there exist $I_{j_0 k_0} \subseteq A_{j_0}$ such that $f(I_{j_0 k_0}) \supseteq A_{j_1} = \bigcup_{k_1=1}^{t_{j_1}} I_{j_1 k_1}$, hence for each subset $I_{j_1 k_1} \subseteq A_{j_1}$ we have $f(I_{j_0 k_0}) \supseteq I_{j_1 k_1}$. Since $I_{j_1 k_1}$ is nonempty there exists $y \in I_{j_1 k_1} \subseteq f(I_{j_0 k_0})$, hence there exists $x \in I_{j_0 k_0}$ such that $y = f(x)$. Therefore $x \in I_{j_0 k_0} \cap f^{-1}(I_{j_1 s})$ that is $I_{j_0 k_0} \cap f^{-1}(I_{j_1 s}) \neq \varphi$. Now assume for $n \in N$ we have $\bigcap_{i=0}^{n} f^{-i}(I_{j_i k_i}) \neq \varphi$. Let $\tau = (j_0, ..., j_{n+1})$. The walk $\tau' = (j_1, ..., j_{n+1})$ satisfies the induction hypothesis and $\bigcap_{i=1}^{n+1} f^{-i+1}(I_{j_i k_i})$ is a nonempty subset of $I_{j_1 k_1}$ and we have $f(I_{j_0 k_0}) \supseteq I_{j_1 k_1} \supseteq \bigcap_{i=1}^{n+1} f^{-i+1}(I_{j_i k_i})$. So similar to the previous argument we have $I_{j_0 k_0} \cap f^{-1}\left(\bigcap_{i=1}^{n+1} f^{-i+1}(I_{j_i k_i})\right) \neq \varphi$ and we conclude that $\bigcap_{i=0}^{n+1} f^{-i}(I_{j_i k_i}) \neq \varphi$, which completes the induction.

Take an open cover of X as

$$\rho = \left\{ U_{kl} = X - \bigcup_{\substack{1\leq r\leq t \\ 1\leq s\leq t_r \\ (r,s)\neq(k,l)}} I_{rs} : k = 1,2,...,t, l = 1,2,...,t_r \right\}.$$

For every $n \in N$ and every walk $\tau = (j_0, ..., j_n)$, the nonempty subset $\bigcap_{i=0}^{n} f^{-i}(I_{j_i k_i})$ implies that every subcover of $\vee_{i=0}^{n} f^{-i}(\rho)$ must contain $\bigcap_{i=0}^{n} f^{-i}(U_{j_i k_i})$ as one of its elements. To see this for every $x \in \bigcap_{i=0}^{n} f^{-i}(I_{j_i k_i})$ there exists an element $\bigcap_{i=0}^{n} f^{-i}(U_{p_i q_i}) \in \vee_{i=0}^{n} f^{-i}(\rho)$ containing x. Hence for every $i \in \{0,1,2,...,n\}, f^{i}(x) \in U_{p_i q_i}$ and for every $i \in \{0,1,2,...,n\}, f^{i}(x) \in I_{j_i k_i}$. We conclude that for every $i \in \{0,1,2,...,n\}, f^{i}(x) \in I_{j_i k_i} \cap U_{p_i q_i}$, so $I_{j_i k_i} \subseteq U_{p_i q_i}$ and $\bigcap_{i=0}^{n} f^{-i}(U_{p_i q_i}) = \bigcap_{i=0}^{n} f^{-i}(U_{j_i k_i})$. Therefore, the result follows and every subcover of $\vee_{i=0}^{n} f^{-i}(\rho)$ must contain the element $\bigcap_{i=0}^{n} f^{-i}(U_{j_i k_i})$ corresponding to walk $\tau = (j_0, ..., j_n)$ in the graph D. As a result, the number of every finite subcover is greater than or equal to the number of walks with the length of n in the graph D. By considering Lemma 1, $N\left(\vee_{i=0}^{n} f^{-i}(\rho)\right)$ is greater than or equal to the sum of entries of M^n and we have:

$$h(f) = \lim_{n\to\infty} \frac{1}{n+1} \log\left(N\left(\vee_{i=0}^{n} f^{-i}(\rho)\right)\right) \geq \lim_{n\to\infty} \frac{1}{n} \log$$

(the sum of all entries of M^n). ■

A part of the following lemma has been presented in [1]. It will be presented completely with proof.

Lemma 2: Let f be a real function. If I, J are two closed intervals such that $f(I) \supseteq J$, then there exists a closed subinterval K of I such that $f(K) = J$ and $f(K^\circ) = J^\circ$.

Proof: Let $J=[a,b]$. Since $a,b\in J$, there exist $a',b'\in I$ such that $a=f(a')$ and $b=f(b')$. Without lose of generality assume $a'\langle b'$. Take $A=f^{-1}(\{a\})\cap[a',b']$. Since A is a closed nonempty subset of $[a',b']$, So $\alpha=\sup(A)\in A$. Similarly by taking $B=f^{-1}(\{b\})\cap[\alpha,b']$, we have $\beta=\inf(B)\in B$. Hence none of the elements of (α,β) images to a or b. Let $K=[\alpha,\beta]$. Since $f(K)$ is connected and compact, it is a closed interval containing $a=f(\alpha)$ and $b=f(\beta)$ and containing $J=[a,b]$. If $f(K)\not\subset[a,b]$, there exists $c\in K$ such that $f(c)\notin[a,b]$. So we have $b\langle f(c)$ or $f(c)\langle a$. By the main value theorem, from $f(\alpha)=a\langle b\langle f(c)$, there exists $\alpha\langle x\langle c$ such that $f(x)=b$. So there exists $x\in(\alpha,\beta)$ images to b which is a contradiction. Similarly from $f(c)\langle a$ we have a contradiction too. Thus we have $[a,b]=f(K)$. Since non of the points in (α,β) images to a or b and we have $[a,b]=f(K)$, hence $f(\alpha,\beta)\subseteq(a,b)$. But:

$$f(K^\circ)=f(\alpha,\beta)=f(K-\{\alpha,\beta\})\supseteq f(K)-\{f(\alpha),f(\beta)\}=[a,b]-\{a,b\}=(a,b)=J^\circ.$$

So the result follows. ■

Theorem 2: Let F be a forest and $f:F\to F$ be continuous and let I and J be two subforests of F such that $f(I)\supseteq J$. Then there exists a subforest K of I such that $f(K)=J$ and the end points of K are either an end point of I or image to a node of J.

Proof: Since J is a forest, we have $J=\bigcup_{j=1}^{m}V_j$ where V_j's are arcs without any branching points in their interior and their end points are nodes in J and every pair of them have at most one point in common which is a branching point of J. Take the closed interval A in forest I in which its end points are the end points of I. Since f is continuous and A is connected and compact, $f(A)$ is a tree. Hence $B=f(A)\cap J=\bigcup_{j=1}^{m}(f(A)\cap V_j)$ is a forest.

We claim that for every $j\in\{1,2,...,m\}$ if $f(A)\cap V_j$ is nonempty, it is a closed interval. It is enough to show that $f(A)\cap V_j$ is a connected and closed subset of V_j. Suppose there exists a j such that $f(A)\cap V_j$ is not connected; so it has at least two components. Choose x and y in different components. The subinterval $[x,y]\subseteq V_j$ is a path in F. Since $f(A)$ is a tree containing x and y, it contains the path between x and y. By uniquely arcwise connectivity, we have $[x,y]\subseteq f(A)$. Hence $[x,y]\subseteq f(A)\cap V_j=B_j$ which is a contradiction. So B_j is a connected subset of V_j and it is an interval. On the other hand, since $f(A)$ is compact, $f(A)\cap V_j=B_j$ is closed, hence it is an arc without any branching points in its interior, and all nonempty B_j's forms B as a forest and since for every $j\in\{1,2,...,m\}$, $B_j\subseteq f(A)$, by Lemma 2 there exists a subinterval $K_j\subseteq A$ such that $f(K_j)=B_j$ and $f(K_j^\circ)=B_j^\circ$. Assume $A=[a,b]$, $K_j=[a_j,b_j]$, $B_j=[c_j,d_j]$, and $V_j=[a_j,y_j]$. Let U_j be a connected component of J containing V_j. Then $U_j-V_j^\circ$ has two connected components D_{x_j} and D_{y_j} containing x_j and y_j respectively. Considering homeomorphism between closed intervals and $[0,1]$, three cases for B_j is possible. In each case we want to have K_j in such a way that $f(K_j)=B_j$ and either the end points of K_j are the end points of I or are those points of K_j which are imaged to a node merely.

Case 1: If $B_j=V_j$, no change is necessary. From $f(K_j)=B_j$ and $f(K_j^\circ)=V_j^\circ$ we conclude that the end points of K_j are those points of this interval which image to a node merely.

Case 2: $B_j\subseteq V_j^\circ$, that is $x_j\langle c_j\langle d_j\langle y_j$. In this case we have $f(A)\subseteq V_j^\circ$, otherwise there exists $z\in f(A)$ which is not in V_j°. So z belongs to either D_{x_j} or D_{y_j}, suppose $z\in D_{y_j}$, since $f(A)$ is a connected set containing z and d_j and $[z,d_j]$ is the smallest connected set containing these two points, $y_j\in[z,d_j]\subseteq f(A)$, that is $y_j\in f(A)\cap V_j=$

$B_j = [c_j, d_j]$ which is a contradiction with our assumption (and similar procedure for $z \in D_{x_j}$). In this case we replace K_j by A and the end points of K_j are the end points of I .

Case 3: Either $x_j = c_j \langle d_j \langle y_j$ or $x_j \langle c_j \langle d_j = y_j$. We will investigate the first one, the second one can be investigated in a similar way. We know $f(a_j, b_j) = (c_j, d_j)$ and $f(\{a_j, b_j\}) = \{c_j, d_j\}$. Without lose of generality assume $f(a_j) = c_j$ and $f(b_j) = d_j$. If $f(b_j, b) \subseteq V_j^{\circ}$ we replace K_j by $[a_j, b]$ and if $f(b_j, b) \not\subset V_j^{\circ}$ there exists $z \in (b_j, b)$ such that $f(z)$ does not belong to V_j° . So it belongs to either D_{x_j} or D_{y_j} .If $f(z) \in D_{y_j}$ similar to the previous case we conclude that $y_j \in f(B_j)$, which is a contradiction with our assumption. So we have $f(z) \in D_{x_j}$. Hence $f(b_j, b)$ contains x_j . Notice that $L_j = f^{-1}(x_j) \cap [b_j, b]$ is a closed nonempty subset of $[b_j, b]$, so $\alpha_j = \inf(L_j) \in L_j$. We will replace K_j with $[a_j, \alpha_j]$. Therefore, we have $f(a_j) = f(\alpha_j) = x_j$ and $f(a_j, \alpha_j) \subseteq V_j^{\circ}$.

Now for every $a, b \in End(I)$, set the interval K_j obtained from this procedure by K_{jab} and let $K = \bigcup K_{jab}$. So the end points of K are either the end points of I or image to a node in J and thus, we have $f(K) = \bigcup f(K_{jab}) = \bigcup (f(A) \cap J) = f(\bigcup A) \cap J = f(I) \cap J = J$. ■

Results

Now we will state the most important theorem of the paper. The proof is rather long and we will prove it in three steps. The difference between this theorem and theorem 1 is that the topological space X is a forest and the collection of nonempty pairwise disjoint closed subsets are replaced by a collection of nonempty pairwise disjoint interior subforests.

Theorem 3: Let F be a forest and $f : F \to F$ be continuous and let $(F_r)_{r=1}^{t}$ be a collection of nonempty subset of F with pairwise disjoint interior and each one has finite connected components. Assume for every $r \in \{1, 2, ..., t\}$ we have $F_r = \bigcup_{s=1}^{t_r} I_{rs}, t_r \in N$ such that I_{rs}'s are nonempty subset of F_r with finite connected components and with pairwise disjoint interior. Let $D(V(D), A(D), \psi_D)$ be the directed graph and M be its adjacency matrix similar to Theorem 1. Then $h(f) \geq \lim_{n\to\infty} \frac{1}{n} \log$ (the sum of all entries of M^n).

Proof: We know for every $r \in \{1, 2, ..., t\}$, $F_r^{\circ} = \overline{F_r}^{\circ}$. Also $B \subseteq f(A)$ implies $\overline{B} \subseteq f(\overline{A})$, so without loss of generality we may assume that F_r's are closed. Hence all F_r's and I_{rs}'s are forest.

Step one: Let $\tau = (j_0, j_1, ..., j_n)$ be an arbitrary walk with the length of n in D . For every $i \in \{1, 2, ..., n\}$, (j_{i-1}, j_i) is an arc in D , so there exists subset $I_{j_{i-1}k_{i-1}}$ of $F_{j_{i-1}}$ such that $f(I_{j_{i-1}k_{i-1}}) \supseteq F_{j_i}$. We claim that corresponding to walk τ , and by taking $F_{j_n} = I_{j_n k_n}$, there exists a subforest K_τ of $I_{j_0 k_0}$ such that $f^i(K_\tau) \subseteq I_{j_i k_i}, i = 0, 1, ..., n-1$ and $f^n(K_\tau) = I_{j_n k_n}$ and every end point of K_τ is either an end point of $I_{j_0 k_0}$ or images to a node of $I_{j_i k_i}$ by f^i for some $i \in \{1, 2, ..., n\}$. Using induction, based on theorem 2, the assumption is true for walks with the length of 1. Suppose that the assumption is true for every walk with the length of n . Let $\tau = (j_0, j_1, ..., j_{n+1})$ be a walk with the length of $n+1$ in D . since $\tau' = (j_1, ..., j_{n+1})$ is a walk with the length of n , by taking $F_{j_{n+1}} = I_{j_{n+1}k_{n+1}}$ and using the induction hypothesis, there exists a subforest $K_{\tau'}$ of $I_{j_1 k_1}$ such that $f^i(K_{\tau'}) \subseteq I_{j_{i+1}k_{i+1}}$, $i = 0, 1, ..., n-1$ and $f^n(K_{\tau'}) = I_{j_{n+1}k_{n+1}}$ and every end point of $K_{\tau'}$ is either an end point of $I_{j_1 k_1}$ or images to a node in $I_{j_{i+1}k_{i+1}}$ by f^i for some $i \in \{1, 2, ..., n\}$. We have $K_{\tau'} \subseteq I_{j_1 k_1} \subseteq f(I_{j_0 k_0})$, so based on theorem 2 there exists a subforest K_τ of $I_{j_0 k_0}$ such that $K_{\tau'} = f(K_\tau)$ and every end point of K_τ is either an end point of $I_{j_0 k_0}$ or images to a node of $I_{j_1 k_1}$ by f . Hence for every $i \in \{1, 2, ..., n-1\}$ we have:

$$f^{i+1}(K_\tau) = f^i(K_{\tau'}) \subseteq I_{j_{i+1}k_{i+1}} ,$$

$$f^{n+1}(K_\tau) = f^n(K_{\tau'}) = I_{j_{n+1}k_{n+1}} .$$

We can conclude that for every $i \in \{0,1,...,n\}$, $f^{i}(K_{\tau}) \subseteq I_{j_i k_i}$ and $f^{n+1}(K_{\tau}) = I_{j_{n+1}k_{n+1}}$. For any arbitrary end point x of K_{τ} either x is an end point of $I_{j_0 k_0}$ or $f(x)$ is a node of $K_{\tau'}$, if $f(x)$ is a branching point of $K_{\tau'}$ then it is a node of $I_{j_1 k_1}$ and if $f(x)$ is an end point of $K_{\tau'}$, based on the induction hypothesis there exists $i \in \{0,1,...,n\}$ such that $f^{i+1}(x)$ is a node of $I_{j_{i+1}k_{i+1}}$. Thus we can conclude that every end point of K_{τ} is either an end point of $I_{j_0 k_0}$ or for some $i \in \{1,2,...,n+1\}$, $f^{i}(x)$ is a node of $I_{j_i k_i}$.

Step two: We claim that for any distinct walks τ_1 and τ_2 with the length of n, $K_{\tau_1}^{\circ} \cap K_{\tau_2}^{\circ} = \varphi$,. We use induction for the length of walks. For $n=1$ take $\tau_1 = (j_0, j_1)$ and $\tau_2 = (j'_0, j'_1)$. So there exist subforests $I_{j_0 k_0}$ and $I_{j'_0 k'_0}$ of F_{j_0} and $F_{j'_0}$ respectively, such that $F_{j_1} \subseteq f(I_{j_0 k_0})$ and $F_{j'_1} \subseteq f(I_{j'_0 k'_0})$. Since τ_1 and τ_2 are distinct, either $I_{j_0 k_0} \neq I_{j'_0 k'_0}$ or $F_{j_0} \neq F_{j'_0}$. If $I_{j_0 k_0} \neq I_{j'_0 k'_0}$, we have $K_{\tau_1}^{\circ} \cap K_{\tau_2}^{\circ} \subseteq I_{j_0 k_0}^{\circ} \cap I_{j'_0 k'_0}^{\circ} = \varphi$. Assume $I_{j_0 k_0} = I_{j'_0 k'_0}$ and $F_{j_0} \neq F_{j'_0}$, we have K_{τ_1}, $K_{\tau_2} \subseteq I_{j_0 k_0}$, $f(K_{\tau_1}) = F_{j_1}$, $f(K_{\tau_2}) = F_{j'_1}$ and $F_{j_1}^{\circ} \cap F_{j'_1}^{\circ} = \varphi$ so $f(K_{\tau_1}^{\circ} \cap K_{\tau_2}^{\circ}) \subseteq f(K_{\tau_1}) \cap f(K_{\tau_2}) = F_{j_1} \cap F_{j'_1} \subseteq End(F_{j_1}) \cup End(F_{j'_1})$. Now, if $K_{\tau_1}^{\circ} \cap K_{\tau_2}^{\circ} \neq \varphi$, it is an open subset of $I_{j_0 k_0}$, so it contains an open interval say (a,b), and $f(a,b)$ is a connected subset of $End(F_{j_1}) \cup End(F_{j'_1})$, so it is a single point set. In other words, the interval (a,b), that is a subset of $K_{\tau_1}^{\circ} \cap K_{\tau_2}^{\circ}$, images to an end point, which contradicts theorem 2. Thus, $K_{\tau_1}^{\circ} \cap K_{\tau_2}^{\circ} = \varphi$.

Now assume the induction hypothesis is true for any two distinct walks with the length of n. Let $\tau_3 = (j_0, j_1, ..., j_{n+1})$ and $\tau_4 = (j'_0, j'_1, ..., j'_{n+1})$ be two walks in graph D. If $I_{j_0 k_0} \neq I_{j'_0 k'_0}$, based on the previous argument we have $K^{\circ}_{\tau_3} \cap K^{\circ}_{\tau_4} = \varphi$. Otherwise, since $\tau_3 \neq \tau_4$ there exists $i \in \{1,2,...,n+1\}$ such that $I_{j_i k_i} \neq I_{j'_i k'_i}$. Hence $\tau'_3 = (j_1, ..., j_{n+1})$ and $\tau'_4 = (j'_1, ..., j'_{n+1})$ are two distinct walks with the length of n in graph D. Based on the induction hypothesis we have $K_{\tau'_3}^{\circ} \cap K_{\tau'_4}^{\circ} = \varphi$ so $K_{\tau'_3} \cap K_{\tau'_4} \subseteq End(K_{\tau'_3}) \cup End(K_{\tau'_4})$. Also we have $K_{\tau'_3} = f(K_{\tau_3})$ and $K_{\tau'_4} = f(K_{\tau_4})$, and

$$f(K^{\circ}_{\tau_3} \cap K^{\circ}_{\tau_4}) \subseteq f(K_{\tau_3}) \cap f(K_{\tau_4})$$

$$= K_{\tau'_3} \cap K_{\tau'_4} \subseteq End(K_{\tau'_3}) \cup End(K_{\tau'_4}).$$

and similar to the previous argument, we conclude that $K^{\circ}_{\tau_3} \cap K^{\circ}_{\tau_4} = \varphi$. This completes the induction.

Step three: We show that $h(f) \geq \lim_{n\to\infty} \frac{1}{n} \log$ (the sum of all entries of M^{n}).

Let a_n be the sum of all M^{n}'s entries. Fixed n and let A_n be the set of all walks with the length of n in D and let

$$C_n = \left\{ f^{i}(x) : x \in \bigcup_{r=1}^{t} \bigcup_{s=1}^{t_r} Node(I_{rs}), i = 0,1,...,n \right\},$$

$$B_n = \{K_{\tau} : \tau \in A_n, K_{\tau} \cap C_n = \varphi\},$$

and for every z in C_n let $B_n^{z} = \{K_{\tau} : \tau \in A_n, z \in K_{\tau}\}$.

That is $\{K_{\tau} : \tau \in A_n\} = B_n \cup \left(\bigcup_{z \in C_n} B_n^{z} \right)$. Hence $a_n = |A_n| \leq |B_n| + \sum_{z \in C_n} |B_n^{z}|$.

Let $N = \left| \bigcup_{r=1}^{t} \bigcup_{s=1}^{t_r} Node(I_{rs}) \right|$ and $\Delta = \max\left\{ val(y) : y \in \bigcup_{r=1}^{t} \bigcup_{s=1}^{t_r} Node(I_{rs}) \right\}$. Since for every $z \in C_n$ we have $z \in \bigcap_{\tau \in B_n^{z}} K_{\tau}$, $|B_n^{z}| \leq val(z) \leq \Delta$ and $|C_n| \leq N(n+1)$, so $b_n = \sum_{z \in C_n} |B_n^{z}| \leq \Delta N (n+1)$.

We know for every $\tau = (j_0, j_1, ..., j_n) \in A_n$ and for every end point x of K_{τ}, there exists an $i \in \{0,1,2,...,n\}$ such that $f^{i}(x) \in \bigcup_{r=1}^{t} \bigcup_{s=1}^{t_r} Node(I_{rs})$, so $f^{n}(x) \in C_n$ and we have:

$$f^{n}(K_{\tau}^{\circ}) = f^{n}(K_{\tau} - End(K_{\tau}))$$

$$\supseteq f^{n}\left(K_{\tau}\right)-f^{n}\left(End\left(K_{\tau}\right)\right)\supseteq F_{j_n}-C_{n}.$$

Hence $f^{n}\left(K_{\tau}^{\circ}\right)$ contains the union of those elements in B_n which are subset of F_{j_n}. For every $r\in\{1,2,...,t\}$, let F_r' be the union of all elements of B_n contained in F_r. We have $f^{n}\left(K_{\tau}^{\circ}\right)\supseteq F_{j_n}'$. Since endpoints of K_τ image to some point of C_n by f^n and based on theorem 2, there exists a forest H_τ contained in K_τ° such that $f^{n}\left(H_{\tau}\right)=F_{j_n}'$. For every $r\in\{1,2,...,t\}$, let $G_r=\bigcup_{\{\tau:K_\tau\subseteq F_r''\}}H_\tau=\bigcup_{\{\tau\in B_n:K_\tau\subseteq F_r\}}H_\tau$. Since ${}^{n}a_{ij}$ is the number of all distinct walks with the length of n from i to j, there are ${}^{n}a_{ij}$ forest H_τ contained in F_i such that $f^{n}\left(H_{\tau}\right)\supseteq F_j'$. Also there are at least ${}^{n}a_{ij}-b_n$ forest H_τ contained in G_i such that $f^{n}\left(H_{\tau}\right)\supseteq G_j$. Since K_τ°'s are pairwise disjoint, so are H_τ's. Thus all conditions of theorem 1 for f^n, the family $\left(G_r\right)_{r=1}^{t}$ with condition $G_r=\bigcup_{\{\tau\in B_n:K_\tau\subseteq F_r\}}H_\tau$ and the matrix $M_n'=\left[m_{ij}'\right]$ with condition $m_{ij}'={}^{n}a_{ij}-b_n$ are established. Hence we have $h\left(f^{n}\right)\geq\lim_{m\to\infty}\frac{1}{m}\log$ (the sum of all $M_n'^{m}$ entries). As it was explained in (✱✱), without lose of generality we may assume that all sequence $\left\{{}^{n}a_{ij}\right\}_{n=1}^{\infty}$ are exponential. Meanwhile $\left\{b_n\right\}_{n=1}^{\infty}$ is a small linear sequence, so $\lim_{n\to\infty}\frac{b_n}{{}^{n}a_{ij}}=0$, for every $i,j\in\{1,2,...,t\}$. Let $\varepsilon\rangle 0$. There exists a natural number N_0 such that for every $n\rangle N_0$ and every $i,j\in\{1,2,...,t\}$, $\left|\frac{b_n}{{}^{n}a_{ij}}\right|\leq\varepsilon$. By taking $M_n''=\left[m_{ij}''={}^{n}a_{ij}\left(1-\varepsilon\right)\right]$, we have $M_n''^{m}=\left[{}^{nm}a_{ij}\left(1-\varepsilon\right)^{m}\right]$ and

$$\lim_{m\to\infty}\frac{1}{m}\log\text{ (the sum of all } M_n''^{m}\text{ entries)}=$$

$$=\lim_{m\to\infty}\frac{1}{m}\log\left(\sum_{i,j=1}^{t}{}^{nm}a_{ij}\left(1-\varepsilon\right)^{m}\right)$$

$$=\lim_{m\to\infty}\frac{1}{m}\log\left(a_{nm}\right)+\log\left(1-\varepsilon\right)$$

$$=n\lim_{m\to\infty}\frac{1}{nm}\log\left(a_{nm}\right)+\log\left(1-\varepsilon\right)$$

$$=n\lim_{i\to\infty}\frac{1}{i}\log\left(a_{i}\right)+\log\left(1-\varepsilon\right).$$

But for n large enough, the entries of $M_n''^{m}$ are smaller than the entries of $M_n'^{m}$. So

$h\left(f\right)=\frac{1}{n}h\left(f^{n}\right)\geq\frac{1}{n}\lim_{m\to\infty}\frac{1}{m}\log$ (the sum of all $M_n'^{m}$ entries) $\geq\frac{1}{n}\lim_{m\to\infty}\frac{1}{m}\log$ (the sum of all $M_n''^{m}$ entries) $=\lim_{i\to\infty}\frac{1}{i}\log$ (the sum of all M^{i} entries) $+\frac{1}{n}\log(1-\varepsilon)$. Therefore the results follows. ■

References

1. Alseda L., Libre J., and Misiurewicz M., *Combinatorial dynamics and entropy in dimension one,* Advanced Studies in Nonlinear Dynamics*,* Vol. **5** World Science, Singapore, (1993).
2. Baldwin S., Entropy estimates for transitive maps on trees*, Topology* **40:** 551-569 (2001).
3. Block L.S., Guckenheimer J., Misiurewicz M., and Young L.S., *Periodic Points and topological entropy of one dimensional maps*, Global theory of dynamical systems, *Lecture notes in mathematics,* Vol. **819**, Springer, Berlin, 18-43 (1980).
4. Bondy J.A., Murty U.S.R., *Graph theory with its applications*, Elsevier North Holland, (1967).
5. Llibre J., and Misiurewicz M., Horseshoes, entropy and periods for graph maps, *Topology,* **32:**649-666 (1993).
6. Mai j.,and shao s., The structure of graph maps without periodic points. *Topology and its Application,***154**:2714-2728 (2007).
7. Stefankova M., On topological entropy of transitive triangular maps. *Topology and its Applications,***153**:2673-2679 (2006).
8. Walter P., *Ergodic theory: introductory lecture*, Springer-Verlag (1975).