Journal of Sciences, Islamic Republic of Iran 21(3): 259-264 (2010) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Fractional Probability Measure and Its Properties

H. Mostafaei* and P. Ahmadi Ghotbi

Department of Statistics, Faculty of Basic Sciences, the Islamic Azad University
North Tehran Branch; Islamic Republic of Iran

Received: 29 January 2009 / Revised: 18 May 2010 / Accepted: 30 August 2010

Abstract

Based on recent studies by Guy Jumarie [1] which defines probability density of fractional order and fractional moments by using fractional calculus (fractional derivatives and fractional integration), this study expands the concept of probability density of fractional order by defining the fractional probability measure, which leads to a fractional probability theory parallel to the classical one. According to the probability principles in classical probability theory and the definition of probability density of fractional order by Guy Jumarie, at first, the fractional probability principles are discussed. Then the fractional probability space (Ω, F, P_α) is introduced. Consequently, the fractional probability measure $P_\alpha : F \to [\circ,1]$, $\circ < \alpha < 1$ is explained. Moreover, validity of the classical "probability measure continuity" theorem ($P(\lim_{n\to\infty} X_n(\omega)) = \lim_{n\to\infty} P(X_n(\omega))$) for the fractional probability measure P_α is verified, which results in "Fatou Lemma" and some theorems in convergence concept.

Keywords: Probability density of fractional order α; fractional probability measure; fractional probability space; probability measure continuity

Introduction

The probability density of fractional order $P_\alpha(x)$, $P_\alpha(x) \geq \circ$ $P\{x < X \leq x'\} =: F(x, x') = \int_x^{x'} P_\alpha(\xi)(d\xi)^\alpha$, has been defined by using fractional calculus (fractional derivatives and fractional integration) [2-8], by Guy Jumarie in 2007[1] which can be considered as the first step in expanding a fractional probability theory. Two classical probability principles ($p(\Omega) = 1$ and for any $i \neq j$, $A_i \bigcap A_j = \varnothing$, $P(\bigcup_i A_i) = \sum_i P(A_i)$) [11, 12] are validated for a fractional probability measure, for instance, the uniform fractional probability density function, which can be explained by the probability density of fractional order [1]. Furthermore, the fractional probability principles are evaluated. These evaluations also result in the fractional probability space (Ω, F, P_α).

Having defined the fractional probability measure, its properties in regards to the corresponding properties of classical probability measure [11, 12] are studied. One important theorem in the classical probability theory is the probability measure continuity

* Corresponding author, Tel.: :+98(21)22946018, Fax: :+98(21)22949556, E-mail: h_mostafaei@iau-tnb.ac.ir

($P(\lim_{n\to\infty} X_n(\omega)) = \lim_{n\to\infty} P(X_n(\omega))$) [11, 12], which plays a great role in proving other theorems in the probability theory. The problem here is to verify whether this theorem is satisfied in the fractional probability space (Ω, F, P_α) or not. Is it possible to set $P_\alpha(\lim_{n\to\infty} X_n(\omega))$ $= \lim_{n\to\infty} P_\alpha(X_n(\omega))$?

Furthermore, definition of fractional moments by Guy Jumarie is mentioned and properties of mathematical expectation of fractional order are evaluated.

In following section, dominated, bounded, and monotone convergence theorems in fractional probability space are stated. Subsequently "Fatou Lemma" is proved by the means of the results of probability measure continuity in the fractional probability space (Ω, F, P_α).

Probability Principles

Definition 1 (Classical probability principles) [11, 12]. Given a sample space Ω and an associated σ -field F , a probability measure is a set function $P: F \to [\circ, 1]$ that satisfies

1. $p(A) \geq \circ$ for all $A \in F$
2. $p(\Omega) = 1$
3. for all $A_i \in F$, if A_is are pairwise disjoint, then $P(\bigcup_{i=1} A_i) = \sum_{i=1} P(A_i)$

Probability Density of Fractional Order

Definition 2 [1] Let X denote a real-valued random variable with the probability density $P_\alpha(x), P_\alpha(x) \geq \circ$. X is referred to as a random variable with fractional probability density of order α, $\circ < \alpha < 1$. Whenever one has

$$P\{x < X \leq x'\} =: F(x, x') = \int_x^{x'} P_\alpha(\xi)(d\xi)^\alpha,$$

with normalizing condition

$$\int_{-\infty}^{+\infty} P_\alpha(x)(dx)^\alpha = 1 .$$

Example 1 (Uniform probability density function of fractional order α) [1]

According to Definition 2 and normalizing condition $\int_{-\infty}^{+\infty} P_\alpha(x)(dx)^\alpha = 1$, for a uniform random α -variable X on the interval $[a,b]$, one has

$$P_\alpha(x) = \frac{1}{(b-a)^\alpha} \quad , \quad a \leq x \leq b$$

Fractional Probability Measure

If it is assumed that the fractional probability density $P_\alpha(x)$ is a fractional probability measure, according to the corresponding classical probability principles ($p(A) \geq \circ$ *for all* $A \in F$ and $p(\Omega) = 1$), in Definition 2 the expressions $P_\alpha(x) \geq \circ$ and $\int_{-\infty}^{+\infty} P_\alpha(x)(dx)^\alpha = 1$ can be considered as two initial principles of fractional probability measure. Therefore, the first principle of fractional probability measure can be defined as $P_\alpha(x) \geq \circ$ and the second principle can be defined as $p_\alpha(\Omega) = \int_{-\infty}^{+\infty} P_\alpha(x)(dx)^\alpha = 1$.

The third principle of classical probability measure, *for all* $A_i \in F$ *, if* A_i*s are pairwise disjoint, then* $P(\bigcup_{i=1} A_i) = \sum_{i=1} P(A_i)$, is verified for the uniform probability density of fractional order α

$$P_\alpha(x) = \frac{1}{(b-a)^\alpha} \quad , \quad a \leq x \leq b$$

Here two disjoint events are defined as $A_1 = [a,c)$, $A_2 = [c,b]$, then

$$P_\alpha(a < x < c) = \int_a^c \frac{1}{(b-a)^\alpha}(dx)^\alpha$$

$$= \frac{(c-a)^\alpha}{(b-a)^\alpha}$$

$$P_\alpha(c < x < b) = \int_c^b \frac{1}{(b-a)^\alpha}(dx)^\alpha$$

$$= \frac{(b-c)^\alpha}{(b-a)^\alpha}$$

$$P_\alpha(a < x < b) = 1$$

In fact it should be proved that

$$1 \leq \frac{(c-a)^\alpha}{(b-a)^\alpha} + \frac{(b-c)^\alpha}{(b-a)^\alpha}$$

or

$$(b-a)^\alpha \leq (c-a)^\alpha + (b-c)^\alpha$$

which is clear by referring to Minkovski's inequality [9, 11, 12]. Therefore,

$$P_\alpha(a<x<b)<P_\alpha(a<x<c)+P_\alpha(c<x<b)$$

It is concluded that the third principle of classical probability measure is not satisfied in fractional probability space. So, the principles of fractional probability measure can be described as follows:

Principles of Fractional Probability Measure

Given a sample space Ω and an associated σ -field F , a fractional probability measure of order α , $\circ<\alpha<1$, is a set function $P_\alpha : F \to [\circ,1]$, $\circ<\alpha<1$ that satisfies

1. $P_\alpha(A)\geq\circ$ for all $A\in F$
2. $P_\alpha(\Omega)=1$
3. for all $A_i \in F$, even if $A_i s$ are pairewise disjoint, then $P_\alpha(\bigcup_{i=1} A_i)\leq\sum_{i=1} P_\alpha(A_i)$

Definition 3 (fractional probability space). A fractional probability space is a triple (Ω, F, P_α) where;

- Ω is the sample space corresponding to outcomes of some experiment.
- F is the σ -algebra of subsets of Ω . These subsets are called events.
- $P_\alpha : F \to [\circ,1]$, $\circ<\alpha<1$ is a fractional probability measure.

Fractional probability space indicates three points;

First, events are considered as subsets of Ω .

Second, it is clarified that each particular subset is considered as an event only if it is a member of F .

Third, fractional probability of events is counted by the means of the fractional probability measure $P_\alpha : F \to [\circ,1]$, $\circ<\alpha<1$.

Theorem 1 (fractional probability measure properties). Let (Ω, F, P_α) be a fractional probability space, then one has

a) $p_\alpha(\varnothing)=\circ$

b) If A, B are two events that $A\subset B$, then $P_\alpha(A)\leq P_\alpha(B)$

c) $1-P_\alpha(A^c)\leq P_\alpha(A)\leq 1$

Proof a) since $\Omega=\Omega\bigcup\varnothing$ and $\Omega\bigcap\varnothing=\varnothing$, Assuming part b is true, one has

$$P_\alpha(\Omega)\leq P_\alpha(\Omega\bigcup\varnothing)\leq P_\alpha(\Omega)+P_\alpha(\varnothing)$$

$$1\leq P_\alpha(\Omega\bigcup\varnothing)\leq 1+P_\alpha(\varnothing)$$

$$1\leq P_\alpha(\Omega)\leq 1+P_\alpha(\varnothing)$$

According to the press theorem, it shall be

$$1+P_\alpha(\varnothing)=1$$

So,

$$P_\alpha(\varnothing)=\circ .$$

b) According to the assumption $A\subset B$,

$$B=A\bigcup(B-A)$$

$$P_\alpha(B)=P_\alpha(A\bigcup(B-A))$$

And according to the third principle of fractional probability measure, one has

$$P_\alpha(B)-P_\alpha(B-A)\leq P_\alpha(A) .$$

Now there would be two forms;

$$P_\alpha(B)-P_\alpha(B-A)\leq P_\alpha(B)\leq P_\alpha(A)$$

or

$$P_\alpha(B)-P_\alpha(B-A)\leq P_\alpha(A)\leq P_\alpha(B)$$

An example is set to refute the first one

$$P_\alpha(B)-P_\alpha(B-A)\leq P_\alpha(B)\leq P_\alpha(A)$$

As it is derived from a uniform random α -variable X on the interval $[a,b]$, one has

$$P_\alpha(x)=\frac{1}{(b-a)^\alpha} \quad , \quad a\leq x\leq b$$

Here, we set a sample space as $[\circ,1]$ $(\Omega=[\circ,1])$ and two events as

$A=[\circ,\frac{1}{3})$, $B=[\circ,1]$ that $A\subset B$, then

$$P_\alpha(x)=1 \quad , \quad \circ\leq x\leq 1$$

$$P_\alpha(A)=P_\alpha(\circ\leq x<\frac{1}{3})=\int_\circ^{\frac{1}{3}}(dx)^\alpha=(\frac{1}{3})^\alpha$$

$$P_\alpha(B)=P_\alpha(\Omega)=1$$

As it is clearly observed

$$1 \geq (\frac{1}{3})^{\alpha} \Rightarrow P_{\alpha}(B) \geq P_{\alpha}(A)$$

So the first form $\leq P_{\alpha}(B)P_{\alpha}(B) - P_{\alpha}(B-A) \leq P_{\alpha}(A)$ is refuted. Therefore, one only has

$$P_{\alpha}(B) - P_{\alpha}(B-A) \leq P_{\alpha}(A) \leq P_{\alpha}(B)$$

And it is proved that

$$A \subset B \Rightarrow P_{\alpha}(A) \leq P_{\alpha}(B).$$

c) $\Omega = A \bigcup A^{c}$, according to the third principle of fractional probability measure,

$$1 = P_{\alpha}(\Omega) = P_{\alpha}(A \bigcup A^{c}) \leq P_{\alpha}(A) + P_{\alpha}(A^{c}).$$

And according to the part (b),

$$A \subset \Omega \Rightarrow P_{\alpha}(\Omega) - P_{\alpha}(A^{c}) \leq P_{\alpha}(A) \leq P_{\alpha}(\Omega)$$

Since $P_{\alpha}(\Omega) = 1$, we have

$$1 - P_{\alpha}(A^{c}) \leq P_{\alpha}(A) \leq 1.$$

Continuity of Fractional Probability Measure

Theorem 2 Let (Ω, F, P) be a probability space, then one has

$$P(\lim_{n\to\infty} X_n(\omega)) = \lim_{n\to\infty} P(X_n(\omega)). \text{ [11, 12]}$$

According to fractional probability principles, it is declared, by giving an example, that the theorem of continuity of probability measure ($P(\lim_{n\to\infty} X_n(\omega)) = \lim_{n\to\infty} P(X_n(\omega))$) is not satisfied for fractional probability.

Example 2 let $([\circ, a], \mathrm{B}([\circ, a]), P_{\alpha})$ be a fractional probability space. So that

$$P_{\alpha} = (l([\circ, a]))^{-\alpha} = a^{-\alpha}, a > 1, \circ < \alpha < 1$$

The sequence of functions X_n on $[\circ, a]$ is set as

$$X_n(\omega) = \frac{\omega}{n}, \omega \in [\circ, a]$$

For any n, X_n is a fractional random variable or random α-variable. As follows

$$\lim_{n\to\infty} X_n(\omega) = \circ \qquad \circ \leq \omega \leq a$$

$$P_{\alpha}(\lim_{n\to\infty} X_n(\omega)) = P_{\alpha}(\circ) = P_{\alpha}(\circ \leq \omega \leq a) = a^{-\alpha}$$

$$P_{\alpha}(X_n(\omega)) = P_{\alpha}(\frac{\omega}{n})$$

Based on the fractional probability density of ω, $P_{\alpha}(\omega)$ and using transformation of fractional probability density, fractional probability function (fractional probability density) of the random α-variable X_n, $P_{\alpha}(\frac{\omega}{n})$, is calculated as bellow

$$P_{\alpha}(\omega) = a^{-\alpha}, \ \circ \leq \omega \leq a$$

$$\int_{\circ}^{a} a^{-\alpha}(d\omega)^{\alpha} = 1$$

$$X_n(\omega) = \frac{\omega}{n}$$

$$d(X_n(\omega)) = \frac{d\omega}{n}, \ d\omega = nd(X_n)$$

$$\int_{\circ}^{\frac{a}{n}} P_{\alpha}(nX_n)(ndX_n)^{\alpha} = 1, \ \circ \leq X_n(\omega) \leq \frac{a}{n}$$

Since $P_{\alpha}(nX_n) = a^{-\alpha}$, one has

$$\int_{\circ}^{\frac{a}{n}} a^{-\alpha} n^{\alpha} (dX_n)^{\alpha} = 1, \ \circ \leq X_n(\omega) \leq \frac{a}{n}$$

So,

$$P_{\alpha}(X_n(\omega)) = n^{\alpha} a^{-\alpha}.$$

$$\lim_{n\to\infty} P_{\alpha}(X_n(\omega)) = \lim_{n\to\infty} \frac{n^{\alpha}}{a^{\alpha}} = \infty$$

So, it is observed that $P_{\alpha}(\lim_{n\to\infty} X_n(\omega)) \neq \lim_{n\to\infty} P_{\alpha}(X_n(\omega))$; and it is concluded that the continuity of probability measure is not satisfied in a fractional probability space.

Properties of Mathematical Expectation of Fractional Order

Definition 4 [1] For any k positive integer, k[th] moment of fractional order α, $\circ < \alpha < 1$, of random variable X is defined by the expression

$$m_{k\alpha} := E\left\{X^{k\alpha}\right\} = \int_{R} x^{k\alpha} P_{\alpha}(x)(dx)^{\alpha}$$

First moment of fractional order α,

$$m_\alpha := E\{X^\alpha\} = \int_R x^\alpha P_\alpha(x)(dx)^\alpha$$

Which is the expected value of fractional order α and fractional variance of order α results from the expression

$$\sigma_\alpha^2 := m_{2\alpha} - (m_\alpha)^2$$

Theorem 3

a) $E_\alpha(aX) = a^\alpha E_\alpha(X)$, $E_\alpha(b) = b^\alpha$.

b) If $g_1(x) \geq \circ$, for all x, then $E_\alpha(g_1(X)) \geq \circ$.

c) If $g_1(x) \geq g_2(x)$, for all x, then $E_\alpha(g_1(X)) \geq E_\alpha(g_2(X))$.

d) If $a \leq g_1(x) \leq b$, for all x, then $a^\alpha \leq E_\alpha(g_1(x)) \leq b^\alpha$.

e) $E_\alpha(aX+bY) \leq a^\alpha E_\alpha(X) + b^\alpha E_\alpha(Y)$

Proof e) First, it is proved that

$$E_\alpha(X+Y) \leq E_\alpha(X) + E_\alpha(Y)$$

According to Definition 4 one has

$$E_\alpha(X+Y) = \int (x+y)^\alpha d^\alpha p$$

So it is sufficient to prove

$$(x+y)^\alpha \leq x^\alpha + y^\alpha$$

or equally

$$x+y \leq (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}$$

Since $\circ < \alpha \leq 1$, therefore, $\frac{1}{\alpha} > 1$. Now if we set $\frac{1}{\alpha} = n$ and $x^\alpha = z, y^\alpha = l$, binominal approximation $(z+l)^n = \sum_{k=\circ}^{n} \binom{n}{k} z^k l^{n-k}$ can be used.

$$(x^\alpha + y^\alpha)^{\frac{1}{\alpha}} = \sum_{k=\circ}^{\frac{1}{\alpha}} \binom{\frac{1}{\alpha}}{k} (x^\alpha)^k (y^\alpha)^{\frac{1}{\alpha}-k}$$
$$= y + (*) + x$$

(*) is a positive expression. So,

$$(x^\alpha + y^\alpha)^{\frac{1}{\alpha}} \geq x + y$$

or

$$x^\alpha + y^\alpha \geq (x+y)^\alpha$$

Then by integrating of both sides of inequality with respect to $d^\alpha p$, one has

$$\int (x^\alpha + y^\alpha) d^\alpha p \geq \int (x+y)^\alpha d^\alpha p$$
$$\int x^\alpha d^\alpha p + \int y^\alpha d^\alpha p \geq \int (x+y)^\alpha d^\alpha p$$
$$E_\alpha(X) + E_\alpha(Y) \geq E_\alpha(X+Y)$$

So,

$$E_\alpha(aX+bY) \leq a^\alpha E_\alpha(X) + b^\alpha E_\alpha(Y).$$

Fatou's Lemma in Fractional Probability Space

Theorem 4 (Dominated convergence theorem) If $\lim_{n\to\infty} X_n = X$ *a.e.* or merely in fractional probability measure on Ω and $\forall n : |X_n| \leq Y$ *a.e. on* Ω, with $\int_\Omega Y d^\alpha P < \infty$, then by assuming $P_\alpha(\lim_{n\to\infty} X_n) \leq \lim_{n\to\infty} P_\alpha(X_n)$, one has

$$\int_\Omega (\lim_{n\to\infty} X_n) P_\alpha(\lim_{n\to\infty} X_n) d^\alpha x \leq \lim_{n\to\infty} \int_\Omega X_n P_\alpha(X_n) d^\alpha x$$

$$\int_\Omega X P_\alpha(X) d^\alpha x \leq \lim_{n\to\infty} \int_\Omega X_n P_\alpha(X_n) d^\alpha x \,. \qquad (**)$$

As a result,

$$\int_\Omega X^\alpha P_\alpha(X) d^\alpha x \leq \lim_{n\to\infty} \int_\Omega (X_n)^\alpha P_\alpha(X_n) d^\alpha x \,.$$

So,

$$E_\alpha(X) \leq \lim_{n\to\infty} E_\alpha(X_n)$$

or

$$E_\alpha(\lim_{n\to\infty} X_n) \leq \lim_{n\to\infty} E_\alpha(X_n).$$

Theorem 5 (Bounded convergence theorem) If $\lim_{n\to\infty} X_n = X$ *a.e.* or merely in fractional probability measure on Ω and there exists a constant M such that $\forall n : |X_n| \leq M$ *a.e. on* Ω, by assuming $P_\alpha(\lim_{n\to\infty} X_n) \leq \lim_{n\to\infty} P_\alpha(X_n)$, then (**) is true.

Theorem 6 (Monotone convergence theorem) If $X_n \geq \circ$ and $X_n \uparrow X$ *a.e. on* Ω or merely in fractional probability measure on Ω, by assuming $P_\alpha(\lim_{n\to\infty} X_n) \leq \lim_{n\to\infty} P_\alpha(X_n)$, then (**) is again true provided that $+\infty$ is allowed as a value for either member.

Theorem 7 (Fatou's lemma) If $X_n \geq \circ \, a.e. on \, \Omega$ or merely in fractional probability measure on Ω, then by assuming $P_\alpha(\lim_{n\to\infty} X_n) \leq \lim_{n\to\infty} P_\alpha(X_n)$, one has

$$\int_\Omega (\underline{\lim} X_n)^\alpha P_\alpha(\underline{\lim} X_n) d^\alpha x \leq \underline{\lim} \int_\Omega (X_n)^\alpha P_\alpha(X_n) d^\alpha x$$

or

$$E_\alpha(\underline{\lim} X_n) \leq \underline{\lim} E_\alpha(X_n)$$

Proof. If $Y_n = \inf_{k\geq n} X_k$ then $\underline{\lim} X_n = \lim_{n\to\infty} Y_n$, $X_n \geq Y_n$, $Y_n \geq \circ$.

So, $Y_n \uparrow \underline{\lim} X_n$.

According to Monotone convergence theorem 7.3, one has

$$\lim_{n\to\infty} E_\alpha(Y_n) \geq E_\alpha(\lim_{n\to\infty} Y_n)$$

So,

$$\lim_{n\to\infty} E_\alpha(Y_n) \geq E_\alpha(\underline{\lim} X_n)$$

Since $E_\alpha(X_n) \geq E_\alpha(Y_n)$, then

$$\underline{\lim} E_\alpha(X_n) \geq \lim_{n\to\infty} E_\alpha(Y_n) \geq E_\alpha(\underline{\lim} X_n).$$

Results and Discussion

In this study, some basic definitions such as "fractional probability space (Ω, F, P_α)" and "fractional probability measure P_α" were proposed, in order to expand a probability theory of fractional order completely parallel to the classical probability theory. Also validities of some theorems such as "continuity of fractional probability measure" and "fractional probability measure properties" were discussed.

In future it would be of interest to study some other theorems of Convergence concept.

References

1. Jumarie G. Probability calculus of fractional order and fractional Taylor's series application to Fokker-Planck equation and information of non-random functions. Chaos, Solitons and Fractals (2007).
2. Kober H. On fractional integrals and derivatives. Quart J Math Oxford, **11**: 193-215 (1940).
3. Ross B. Fractional calculus and its applications. Lecture notes in mathematics, Vol. 457. Berlin, Springer, (1974).
4. Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. New York, Wily, (1973).
5. Nishimoto K. Fractional calculus. Koroyama, Descartes Press (1989).
6. Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions Further results. Comput Math Appl, **51**: 1367-76 (2006).
7. Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives. Theory and applications. London, Gordon and Breach Science Publishers (1987).
8. Oldham KB, Spanier J. The fractional calculus. Theory and application of differentiation and integration to arbitrary order. New York, Acadenic Press (1974).
9. Casella G, L.Berger R. Statistical Inference, Duxbery Press (1990).
10. Parsian A. Basic Concepts of Mathematical statistics, Persian Language. Publisher Isfahan University of Technology Iran, (2009).
11. Pasha E. Probability Theory, Persian Language, Publisher Teacher Training University Iran (2007).
12. Sidney I Resnik. A Probability Path. Birkhauser, Boston (1998).