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# A Pure First Order Representation for Undamped Second Order Descriptor Systems

A. Aasaraai and K. Nasiri*

*Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Islamic Republic of Iran*

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## Abstract

The aim of this paper is to analyze the undamped second order descriptor systems to show the possibility of transform an undamped second order descriptor system to a pure first order descriptor system ,on different ways, while stating some of its benefits. Meanwhile we will introduce pure first order systems and undamped second order systems and indicate that under some assumptions every undamped second order descriptor system can be changed to a pure first order one.

**Keywords:** Descriptor systems; Undamped second order systems; Pure representation

## Introduction

At first we review briefly first and second order descriptor systems and some properties of them that we will need later.

A first order descriptor system with constant coefficients is of the form

$$\begin{cases} E\dot{z}(t) = Fz(t) + Gu(t) \\ y(t) = Hz(t) \end{cases} \tag{1}$$

where $E$ and $F$ are $n \times n$ matrices, $G$ and $H$ are $n \times m$ and $p \times n$ matrices respectively. $z$ is the state, $u$ the input and $y$ the output of the system.

The order of the system is $n$, the dimension of state. The system (1) is called regular if there exists a $\lambda \in \mathbb{C}$ such that $\det(\lambda E - F) \neq 0$, otherwise, it is called *singular*. The descriptor system (1) is called pure if $F$ is nonsingular and $F^{-1}E$ is nilpotent.It is easy to show that every pure system is a regular system.

A second order descriptor system with constant coefficients is of the form

$$\begin{cases} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2}$$

where $M, D, K$ are $n \times n$ matrices and $B, C$ are $n \times m$ and $p \times n$ matrices respectively. $x$ is the state, $u$ the input and $y$ the output of the system.

If $D = 0$ the system is called undamped second order descriptor system.

Second order descriptor systems arise in various areas, for example, the mechanical and electrical oscillation, the control of constrained mechanical systems and the control of electrical systems; see [1, 2, 3, 4, and 5].

The system (2) can be transformed into a first order descriptor system in different ways. For instance, the system (2) can be transformed into a first order descriptor system as,

* Corresponding author, Tel.: +98(131)3233901, Fax: +98(131)3233509, E-mail: k-nasiri@guilan.ac.ir

$$\begin{cases}\begin{pmatrix}A & 0\\ 0 & M\end{pmatrix}\begin{pmatrix}\dot{x}(t)\\ \ddot{x}(t)\end{pmatrix}=\begin{pmatrix}0 & A\\ -K & -D\end{pmatrix}\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}+\begin{pmatrix}0\\ B\end{pmatrix}u(t)\\ y(t)=\begin{pmatrix}C & 0\end{pmatrix}\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\end{cases} \quad (3)$$

The system (2) also can be written as

$$\begin{cases}\begin{pmatrix}0 & A\\ M & D\end{pmatrix}\begin{pmatrix}\ddot{x}(t)\\ \dot{x}(t)\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & -K\end{pmatrix}\begin{pmatrix}\dot{x}(t\\ x(t)\end{pmatrix}+\begin{pmatrix}0\\ B\end{pmatrix}u(t)\\ y(t)=\begin{pmatrix}0 & C\end{pmatrix}\begin{pmatrix}\dot{x}(t\\ x(t)\end{pmatrix}\end{cases} \quad (4)$$

at systems (3) and (4) $A$ is an arbitrary $n\times n$ nonsingular matrix, and usually is assumed that $A=I$.

Other representation for system (2) is

$$\begin{cases}\begin{pmatrix}-K & 0\\ 0 & M\end{pmatrix}\begin{pmatrix}\dot{x}(t)\\ \ddot{x}(t)\end{pmatrix}=\begin{pmatrix}0 & -K\\ -K & -D\end{pmatrix}\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}+\begin{pmatrix}0\\ B\end{pmatrix}u(t)\\ y(t)=\begin{pmatrix}C & 0\end{pmatrix}\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\end{cases} \quad (5)$$

This representation has this advantage that some properties of system (2) transfer to this system. For instance, if $M$ and $K$ are symmetric and positive definite and $D$ is symmetric then $F$ is symmetric and $E$ is symmetric and positive definite.

## 1. Undamped Second Order Descriptor System and a Pure First Order Representation

In this section we consider an undamped second order descriptor system of the form

$$\begin{cases}M\ddot{x}(t)+Kx(t)=Bu(t)\\ y(t)=Cx(t)\end{cases} \quad (6)$$

and show under some assumptions it can be transformed to a pure first order descriptor system.

We assume that matrices $M$ and $K$ are nonsingular. So we can transform the system (6) into a first order descriptor system of the form (4) by choosing $A=M$,

$$\begin{cases}\begin{pmatrix}0 & M\\ M & 0\end{pmatrix}\begin{pmatrix}\ddot{x}(t)\\ \dot{x}(t)\end{pmatrix}=\begin{pmatrix}M & 0\\ 0 & -K\end{pmatrix}\begin{pmatrix}\dot{x}(t\\ x(t)\end{pmatrix}+\begin{pmatrix}0\\ B\end{pmatrix}u(t)\\ y(t)=\begin{pmatrix}0 & C\end{pmatrix}\begin{pmatrix}\dot{x}(t)\\ x(t)\end{pmatrix}\end{cases} \quad (7)$$

It is clear that the systems (3), (4), (5), (7) are first order descriptor systems of the form (1).

**Lemma 1:**

If $M$ and $K$ are nonsingular matrices and

$$E=\begin{bmatrix}0 & M\\ M & 0\end{bmatrix} \qquad F=\begin{bmatrix}M & 0\\ 0 & -K\end{bmatrix}$$

then

$$(F^{-1}E)^{2n+1}=\begin{bmatrix}0 & (-K^{-1}M)^{n}\\ (-K^{-1}M)^{n+1} & 0\end{bmatrix} \quad n=0,1,2,\ldots$$

$$(F^{-1}E)^{2n}=\begin{bmatrix}(-K^{-1}M)^{n} & 0\\ 0 & (-K^{-1}M)^{n}\end{bmatrix} \quad n=1,2,\ldots$$

**Proof:**

Proof by induction:

By computing $F^{-1}E$ directly, we have

$$F^{-1}E=\begin{bmatrix}M^{-1} & 0\\ 0 & -K^{-1}\end{bmatrix}\begin{bmatrix}0 & M\\ M & -0\end{bmatrix}=\begin{bmatrix}0 & I\\ -K^{-1}M & 0\end{bmatrix}$$

that is equal to $(F^{-1}E)^{2n+1}$ when $n=0$.

Also

$$(F^{-1}E)^{2}=\begin{bmatrix}0 & I\\ -K^{-1}M & 0\end{bmatrix}\begin{bmatrix}0 & I\\ -K^{-1}M & 0\end{bmatrix}=\begin{bmatrix}-K^{-1}M & 0\\ 0 & -K^{-1}M\end{bmatrix}$$

that is equal to $(F^{-1}E)^{2n}$ when $n=1$.

Now we suppose that

$$(F^{-1}E)^{2k+1}=\begin{bmatrix}0 & (-K^{-1}M)^{k}\\ (-K^{-1}M)^{k+1} & 0\end{bmatrix}$$

so

$$(F^{-1}E)^{2k+3}=(F^{-1}E)^{2k+1}(F^{-1}E)^{2}$$

$$=\begin{bmatrix}0 & (-K^{-1}M)^{k}\\ (-K^{-1}M)^{k+1} & 0\end{bmatrix}\begin{bmatrix}-K^{-1}M & 0\\ 0 & -K^{-1}M\end{bmatrix}$$

$$=\begin{bmatrix}0 & (-K^{-1}M)^{k+1}\\ (-K^{-1}M)^{k+2} & 0\end{bmatrix}$$

so, the first equality proved. The proof of the second equality is similar.

**Lemma 2:**

Under assumptions of lemma1, if $K^{-1}M$ is nilpotent matrix with the nilpotence index $\upsilon$, $F^{-1}E$ is nilpotent matrix with the nilpotence index $2\upsilon$.

**Proof:**

Since $K^{-1}M$ is nilpotent matrix with the nilpotence index $\upsilon$,

$$(K^{-1}M)^{\upsilon} = 0\,, \qquad (K^{-1}M)^{\upsilon-1} \neq 0$$

we must show

$$(F^{-1}E)^{2\upsilon} = 0\,, \qquad (F^{-1}E)^{2\upsilon-1} \neq 0$$

According to lemma 1

$$(F^{-1}E)^{2\upsilon} = \begin{bmatrix} (-K^{-1}M)^{\upsilon} & 0 \\ 0 & (-K^{-1}M)^{\upsilon} \end{bmatrix} = 0$$

$$(F^{-1}E)^{2\upsilon-1} = \begin{bmatrix} 0 & (-K^{-1}M)^{\upsilon} \\ (-K^{-1}M)^{\upsilon-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ (-K^{-1}M)^{\upsilon-1} & 0 \end{bmatrix} \neq 0$$

since $(K^{-1}M)^{\upsilon-1} \neq 0$.

**Corollary 1:**

If $M$ and $K$ are nonsingular matrices and $K^{-1}M$ is nilpotent, then undamped second order descriptor system (6) can be transformed to a pure first order descriptor system.

**Proof:**

Define

$$z(t) = \begin{pmatrix} \dot{x}(t \\ x(t) \end{pmatrix} \quad \text{E=}\begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix} \quad \text{F=}\begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}$$

$$G = \begin{pmatrix} 0 \\ B \end{pmatrix} \qquad H = \begin{pmatrix} C & 0 \end{pmatrix}$$

Then, the system (6) is equivalent to the system

$$\begin{cases} E\dot{z}(t) = Fz(t) + Gu(t) \\ \quad y(t) = Hz(t) \end{cases}$$

$F$ is nonsingular since $M$ and $K$ are nonsingular, $F^{-1}E$ is nilpotent according to lemma 2, then the last system is pure and proof is complete.

## Results

In this paper, under some assumptions, we transformed an undamped second order descriptor system to a first order descriptor system. The pure representation is important since we can find a normal form for a SISO pure first order descriptor system, which is suitable to analyze some properties of system, like impulse controllability and impulse observability. see [6].

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