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Isotropic Lagrangian Submanifolds in Complex Space Forms

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Abstract

In this paper we study isotropic Lagrangian submanifolds M^n, in complex space forms $\tilde{M}^n(4c)$. It is shown that they are either totally geodesic or minimal in the complex projective space $\mathbb{C}P^n$, if $n \geq 3$. When $n = 2$, they are either totally geodesic or minimal in $\tilde{M}^2(4c)$. We also give a classification of semi-parallel Lagrangian H-umbilical submanifolds.

Keywords: Lagrangian; Isotropic; Semi-parallel submanifold; H-umbilical; Complex space form

Introduction

The notion of an isotropic submanifold of a Riemannian manifold was introduced by B. O'Neill [14]. These submanifolds which can be considered as generalized totally geodesic submanifolds usually have been studied under some additional hypothesis, [12,13]. Here, we assume that these submanifolds are semi-parallel.

On the other hand, Lagrangian submanifolds of complex space forms have been deeply studied since the decade 1970's. A survey of the main results about Lagrangian submanifolds can be found in [7]. Since there is no complete classification of Lagrangian submanifolds, it is natural to study these submanifolds with some additional constraint.

Recall that, an n–dimensional Riemannian submanifold M of an m–dimensional Riemannian manifold $\tilde{M}$ is called parallel if its second fundamental form $\mathbb{I}$, satisfies

$$\left(\overline{\nabla}\mathbb{I}\right)(X,Y,Z) = \nabla^{\perp}_{Z}\mathbb{I}(X,Y) - \mathbb{I}(\nabla_Z X,Y) - \mathbb{I}(X,\nabla_Z Y) = 0,$$

for all vectors X, Y, Z, tangent to M where $\overline{\nabla}$ is the Van der Waerden–Bortolotti connection [11]. By their definition, semi-parallel submanifolds are generalized parallel submanifolds. The classification of semi-parallel submanifolds in real space forms is still an open problem, although several authors have obtained many important results. We can refer the reader to [10] for a survey. Also a recent good refernce for th whole subject of Lagrangian and symplectic manifolds is [1].

Since 2009, it is known that all isotropic Lagrangian submanifolds are parallel, hence semi-parallel [2]. In [9], semi-parallel isotropic Lagrangian submanifolds have been studied. In this paper we follows [2,12,13] to continue the study of isotropic Lagrangian submanifolds in complex space forms.

Our main results are Proposition 1 and Theorems 2,5.

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Preliminaries

We recall some prerequisites from [2,6,12,14]. Let $(\tilde{M}^m, \langle , \rangle)$ be an m-dimensional Riemannian complex manifold and M^n be an n-dimensional real submanifold of $\tilde{M}$. We denote by $X, Y, W, Z, ...$ vectors tangent to M and by $U, V, ...$ generic vectors tangent to $\tilde{M}$. ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M and $\tilde{M}$, respectively. $\mathbb{I}$ is the second fundamental form of M, and A_ξ is the shape operator of M in the direction of the normal vector field $\xi \in \mathcal{X}^\perp(M)$.

The curvature $\tilde{R}$ of $\tilde{\nabla}$ is defined by:

$$\tilde{R}(U_1, U_2)V = [\tilde{\nabla}_{U_1}, \tilde{\nabla}_{U_2}]V - \tilde{\nabla}_{[U_1, U_2]}V,$$

and the sectional curvature of a plane spanned by $\{U, V\}$ is given by

$$\langle R(U,V)U, V\rangle / (\|U\|^2 \|V\|^2 - \langle U, V\rangle^2).$$

If R denote the Riemannian curvature tensor of ∇, then the Gauss equation is,

$$\langle \tilde{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle + \langle \mathbb{I}(X,Z), \mathbb{I}(Y,W)\rangle - \langle \mathbb{I}(X,W), \mathbb{I}(Y,Z)\rangle.$$

$\bar{\nabla} = \nabla \oplus \nabla^\perp$ is the Van der Waerden–Bortolotti connection, where $\nabla^\perp$ is normal curvature. The curvature operator $\bar{R}(X,Y)$ of $\bar{\nabla}$, can be extended as derivation of tensor fields in the usual way. Its action on $\mathbb{I}$ is as follows,

$$(\bar{R}(X,Y)\cdot\mathbb{I})(Z,W) = R^\perp(X,Y)(\mathbb{I}(Z,W)) - \mathbb{I}(R(X,Y)Z, W) - \mathbb{I}(Z, R(X,Y)W), \tag{1.1}$$

where $R^\perp$ denote the Riemannian curvature operator of $\nabla^\perp$. The submanifold M of $\tilde{M}$ is called semi-parallel if its second fundamental form $\mathbb{I}$ satisfies

$$\bar{R}(X,Y)\cdot\mathbb{I} = 0 \tag{1.2}$$

An almost complex structure on $\tilde{M}^m$ is a tensor field J of type (1,1) on $\tilde{M}$, such that $J^2 = -\mathrm{Id}_{T\tilde{M}}$. If J is an isometry, i.e. $\langle U, V\rangle = \langle JU, JV\rangle$, $\tilde{M}^m$ is called an almost Hermitian manifold. The most interesting Hermitian manifolds are, the Kähler manifolds $(\tilde{M}, J, \langle , \rangle)$ defined by the condition $\tilde{\nabla}J = 0$, where $(\tilde{\nabla}J)(U,V) = \tilde{\nabla}_U JV - J\tilde{\nabla}_U V$.

The holomorphic sectional curvature of an almost Hermitian manifold is the restriction of the sectional curvature to holomorphic planes (the planes that are spanned by U and JU) in the tangent spaces. The curvature tensor of a space of constant holomorphic sectional curvature $4c$, $\tilde{M}(4c)$, is given by:

$$\tilde{R}(U_1, U_2)V = c((U_1 \wedge U_2)V + (JU_1 \wedge JU_2)V + 2\langle JU_1 \wedge U_2\rangle JV) \tag{1.3}$$

where $(U_1 \wedge U_2)V = \langle U_1, V\rangle U_2 - \langle U_2, V\rangle U_1$.

A complex space form is a complete, simply connected, Kähler manifold with constant holomorphic sectional curvature. So, a complex space form is isometric to either the complex projective space $\mathbb{C}P^n(4c)$, if $c > 0$, or the complex Euclidean space $\mathbb{C}^n$, if $c = 0$, or the complex projective hyperbolic space $\mathbb{H}P^n(4c)$, if $c < 0$.

An n-dimensional submanifold M^n of an almost Hermitian complex manifold $\tilde{M}^m$ is said to be totally real if $J(T_pM) \subset (T_pM)^\perp$ for all $p \in M$. A totally real submanifold M^n of $\tilde{M}^m$ is said to be Lagrangian when $m = n$. For Lagrangian submanifolds of a Kähler manifold the following relations hold, [2]

$$JA_{JX}Y = \mathbb{I}(X,Y) = JA_{JY}X,$$
$$\langle \mathbb{I}(X,Y), JZ\rangle = \langle \mathbb{I}(Y,Z), JX\rangle = \langle \mathbb{I}(Z,X), JY\rangle, \tag{1.4}$$
$$R^\perp(X,Y)JZ = JR(X,Y)Z.$$

Moreover, from the Gauss equation one has

$$R(X,Y) = \tilde{R}(X,Y) + A_{JX}A_{JY} - A_{JY}A_{JX}. \tag{1.5}$$

In [4], it is proved that there exists no totally umbilic Lagrangian submanifold in a complex space form $\tilde{M}^n(4c)$ with $n \geq 2$ except the totally geodesic ones. The Lagrangian H-umbilical submanifolds are the simplest Lagrangian submanifolds next to the totally geodesic submanifolds in a complex space form. A Lagrangian H-umbilical submanifold of a Kähler manifold $\tilde{M}^n(4c)$ is a Lagrangian submanifold whose second fundamental form takes the following simple form, [6].

$$A_{Je_1}e_1 = \lambda e_1, \quad A_{Je_2}e_2 = \ldots = A_{Je_n}e_n = \mu e_1,$$
$$A_{Je_1}e_j = \mu e_j, \quad A_{Je_j}e_k = 0, \quad 2 \leq j \neq k \leq n, \tag{1.6}$$

with respect to some suitable orthonormal local frame field, and for some suitable functions λ and μ.

A Lagrangian submanifold M of $\tilde{M}$ is said to be λ-isotropic if there exists an smooth function $\lambda : M \to \mathbb{R}$ such that $\|\mathbb{I}(X,X)\|^2 = \lambda^2(p)$, for any unit vector $X \in T_pM$ and for all $p \in M$, [14]. In particular, if λ is constant then M is called constant isotropic.

From [12], it is known that, if M^n ($n \geq 3$) is a minimal totally real and isotropic submanifold of a Kähler manifold, then either M is totally geodesic or $n = 5, 8, 14, 26$. Also, if M^n is a complete, constant isotropic totally real submanifold of $\mathbb{C}P^n(4c)$, then either M is totally geodesic or M is locally isometric to $S^1 \times S^{n-1}$ ($n \geq 2$); $SU(3)/SO(3)$, $n = 5$; $SU(3)$, $n = 8$; $SU(6)/Sp(3)$, $n = 14$; E_6, $n = 26$.

Results

In [2], P. M Chacon and G. A. Lobos give some properties of semi-parallel Lagrangian H-umbilical submanifold. Here, we give the classification of such submanifolds, by using their result.

Proposition 1: If $n \geq 3$ and M^n is a semi-parallel Lagrangian H-umbilical submanifold of $\tilde{M}^n(4c)$, then M^n is one of the following submanifolds.

a) *A totally geodesic one,*

b) *A flat submanifold of* $\mathbb{C}^n$ *,*

c) *A non-flat and non-totally geodesic minimal submanifold of* $\mathbb{C}P^n(4c)$.

Proof: suppose that $\{e_1, \ldots, e_n\}$ is a suitable orthonormal local frame field, such that with respect to it the shape operators of M have the form (1.6). From the Gauss equation for $i, j = 2, \ldots, n$ and $i \neq j$ we have,

$$R(e_i, e_j)e_i = (c - \mu^2)e_j, \qquad R(e_i, e_1)e_i = (c + \mu^2 - \mu\lambda)e_1. \tag{2.1}$$

From (1.6), for Lagrangian H-umbilical submanifolds, $H = J\frac{1}{n}\sum_{i=1}^{n} A_{Je_i}e_i = \frac{\lambda + (n-1)\mu}{n} Je_1$. If $\mu \neq 0$ from [2], we have $\lambda = (1-n)\mu$ and $c = n\mu^2 > 0$, so $H = 0$, i.e. M is minimal. Also, from (2.1) we have $\langle R(e_i, e_j)e_i, e_j\rangle = (n-1)\mu^2 > 0$,0 so M is a non-flat, non-totally geodesic minimal submanifold of $\mathbb{C}P^n(4c)$.

A semi-parallel Lagrangian submanifold M^n of constant sectional curvature c_1 in $\tilde{M}^n(4c)$ is flat or totally geodesic [2]. If $\mu = 0$, from (2.1) we get that $R(e_i, e_j)e_i = ce_j$, and $R(e_i, e_1)e_i = ce_1$. So M has constant sectional curvature c, Hence, M is either totally geodesic or flat. If M is flat, we get that $c = 0$, i.e. M^n is a flat submanifold of $\mathbb{C}^n$. □

One should see [3] for new results about Lagrangian H-umbilical submanifolds of para-Kahler manifolds.

It is known that for $n \geq 3$ any λ-isotropic Lagrangian submanifold M^n of $\tilde{M}^n(4c)$ is constant isotropic and M^n is parallel in $\tilde{M}^n(4c)$, [2]. So, any isotropic Lagrangian submanifold of $\tilde{M}^n(4c)$ is semi-parallel. In [12] minimal isotropic Lagrangian submanifolds have been studied. Now, we use the fact that isotropic Lagrangian submanifolds are semi-parallel, and give the classification of such submanifolds.

Theorem 2: Let M^n ($n \geq 3$) be a λ-isotropic Lagrangian submanifold of $\tilde{M}^n(4c)$. Then M is either totally geodesic or minimal in $\mathbb{C}P^n(4c)$.

Proof: From [9], a semi-parallel isotropic Lagrangian submanifold of dimension $n \geq 3$ is either totally geodesic or $c = 2\lambda^2 > 0$. So, every isotropic Lagrangian submanifold of $\tilde{M}^n(4c)$ with $c \leq 0$ is totally geodesic. It follows that non-totally geodesic isotropic Lagrangian submanifolds can only exist in $\mathbb{C}P^n(4c)$.

If $X = \sin\theta e_i + \cos\theta e_j$, we have,

$$\begin{aligned} \lambda^2 = \|\mathbb{I}(X,X)\|^2 &= \lambda^2\cos^4\theta + \lambda^2\sin^4\theta \\ &+ (\|\mathbb{I}(e_i, e_j)\|^2 + \tfrac{1}{2}\langle\mathbb{I}(e_i, e_i), \mathbb{I}(e_j, e_j)\rangle)\sin^2 2\theta \\ &+ 4\langle\mathbb{I}(e_j, e_j), \mathbb{I}(e_i, e_j)\rangle\sin^3\theta\cos\theta \\ &+ 4\langle\mathbb{I}(e_i, e_i), \mathbb{I}(e_i, e_j)\rangle\sin\theta\cos^3\theta, \end{aligned} \tag{2.2}$$

Since λ is independent of θ, we obtain from (2.2) that,

$$\begin{aligned} 0 = \tfrac{d}{d\theta}\lambda^2 &= -2\lambda^2\sin 2\theta\cos 2\theta + 2(\|\mathbb{I}(e_i, e_j)\|^2 \\ &+ \tfrac{1}{2}\langle\mathbb{I}(e_i, e_i), \mathbb{I}(e_j, e_j)\rangle)\sin 4\theta \\ &+ 4\langle\mathbb{I}(e_j, e_j), \mathbb{I}(e_i, e_j)\rangle(3\sin^2\theta\cos^2\theta - \sin^4\theta) \\ &+ 4\langle\mathbb{I}(e_i, e_i), \mathbb{I}(e_i, e_j)\rangle(\cos^4\theta - 3\sin^2\theta\cos^2\theta), \end{aligned} \tag{2.3}$$

Choose $\theta = 0$ in (2.3) to get

$$\langle \mathbb{I}(e_i, e_i), \mathbb{I}(e_i, e_j)\rangle = 0 \tag{2.4}$$

Choose $\theta = \frac{\pi}{8}$ to obtain

$$2\left\|\mathbb{I}(e_i, e_j)\right\|^2 + \langle \mathbb{I}(e_i, e_i), \mathbb{I}(e_j, e_j)\rangle = \lambda^2 \tag{2.5}$$

From [12], $\forall p \in M$ there exists an orthonormal basis $\{e_1, ..., e_n\}$ of T_pM satisfying $A_{Je_1}e_i = \lambda_i e_i$, and $\lambda_1 = \lambda$, and $i = 2, ..., n$, λ_i is either $-\lambda$ or $\frac{1}{2}\lambda$. Let V_1 and V_2 be the eigenspaces of A_{Je_1} corresponding to the eigenvalues $-\lambda$ and $\frac{1}{2}\lambda$ respectively. Then, $\mathbb{I}(x, y) = -\langle x, y\rangle \lambda Je_1$, $\forall x, y \in V_1$, and $\langle \mathbb{I}(v, w), Jz\rangle = 0$ for $v, w, z \in V_2$, hence $A_{Jv}w$ belongs to $V_1 \cup \text{span}_{\mathbb{R}}\{e_1\}$. So, $\sum_{k=1}^{n} A_{Je_k} e_k$ belongs to $V_1 \cup \text{span}_{\mathbb{R}}\{e_1\}$.

Now we consider four possible cases for V_1 and V_2.

case i : If $V_1 = \varnothing$, we have $\mathbb{I}(e_i, e_i) = \frac{1}{2}\lambda Je_1$ for $e_i \in V_2$, so $\left\|\mathbb{I}(e_i, e_i)\right\|^2 = \frac{1}{4}\lambda^2$. Since M is λ-isotropic, hence $\frac{1}{4}\lambda^2 = \lambda^2$, so $\lambda = 0$.

case ii : If $V_2 = \varnothing$, since M is an H-umbilical Lagrangian submanifold. From Proposition 2, M is either totally geodesic or minimal. We have $H = (1 - \dim V_1)\lambda Je_1$ and $\dim V_1 > 1$, so if M is minimal, hence $\lambda = 0$ and M is totally geodesic.

case iii : If $\dim V_1 = 1$ and $\dim V_2 = n - 2$, from (2.4) and (2.5) we obtain that,

$$\begin{aligned} &A_{Je_1}e_1 = \lambda e_1, \quad A_{Je_2}e_2 = -\lambda e_1, \\ &A_{Je_1}e_2 = -\lambda e_2, \quad A_{Je_1}e_i = \tfrac{1}{2}\lambda e_i, \\ &A_{Je_i}e_i = \tfrac{1}{2}\lambda e_1 + \varepsilon_i \tfrac{\sqrt{3}}{2}\lambda e_2, \\ &A_{Je_2}e_i = \varepsilon_i \tfrac{\sqrt{3}}{2}\lambda e_i, \quad A_{Je_i}e_j = 0, \end{aligned} \tag{2.6}$$

where $e_2 \in V_1$ and $e_i, e_j \in V_2$, and $\varepsilon_i = \pm 1$. We have from Gauss equation and (2.6) that,

$$\begin{aligned} R(e_1, e_i)e_1 &= c(e_1 \wedge e_i)e_1 + A_{Je_1}A_{Je_i}e_1 - A_{Je_i}A_{Je_1}e_1 \\ &= ce_i + \tfrac{1}{2}\lambda A_{Je_1}e_i - \lambda A_{Je_i}e_1 \\ &= ce_i + \tfrac{1}{4}\lambda^2 e_i - \tfrac{1}{2}\lambda^2 e_i = (c - \tfrac{1}{4}\lambda^2)e_i, \end{aligned}$$

$$\begin{aligned} R(e_1, e_i)e_2 &= c(e_1 \wedge e_i)e_2 + A_{Je_1}A_{Je_i}e_2 - A_{Je_i}A_{Je_1}e_2 \\ &= \varepsilon_i \tfrac{\sqrt{3}}{2}\lambda A_{Je_1}e_i + \lambda A_{Je_i}e_2 \\ &= \varepsilon_i \tfrac{\sqrt{3}}{4}\lambda^2 e_i + \varepsilon_i \tfrac{\sqrt{3}}{2}\lambda^2 e_i = \tfrac{3\sqrt{3}}{4}\varepsilon_i \lambda^2 e_i, \end{aligned}$$

$$\begin{aligned} R(e_2, e_i)e_1 &= c(e_2 \wedge e_i)e_1 + A_{Je_2}A_{Je_i}e_1 - A_{Je_i}A_{Je_2}e_1 \\ &= \tfrac{1}{2}\lambda A_{Je_2}e_i + \lambda A_{Je_i}e_2 = \varepsilon_i \tfrac{3\sqrt{3}}{4}\lambda^2 e_i, \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(e_2, e_i)e_2 &= c(e_2 \wedge e_i)e_2 + A_{Je_2}A_{Je_i}e_2 - A_{Je_i}A_{Je_2}e_2 \\ &= ce_i + \varepsilon_i \tfrac{\sqrt{3}}{2}\lambda A_{Je_2}e_i + \lambda A_{Je_i}e_1 \\ &= ce_i + \tfrac{3}{4}\lambda^2 e_i + \tfrac{1}{2}\lambda^2 e_i = (c + \lambda^2)e_i. \end{aligned}$$

From (2.6) we obtain that

$$\sum_{k=1}^{n} A_{Je_k}e_k = \tfrac{n-2}{2}\lambda e_1 + \tfrac{\sqrt{3}}{2}\lambda \sum_{k=3}^{n} \varepsilon_k e_2. \tag{2.8}$$

For semi-parallel Lagrangian submanifold M^n of $\tilde{M}^n(4c)$ we have, [2]

$$R(X, Y)\sum_{k=1}^{n} A_{Je_k}e_k = 0 \tag{2.9}$$

Then from (2.7), (2.8) and (2.9) one obtains that,

$$\begin{aligned} 0 = R(e_2, e_i)\sum_{k=1}^{n} A_{Je_k}e_k &= \tfrac{n-2}{2}\lambda R(e_2, e_i)e_1 \\ &+ \tfrac{\sqrt{3}}{2}\lambda \sum_{k=3}^{n} \varepsilon_k \; R(e_2, e_i)e_2 \\ &= \varepsilon_i \tfrac{3\sqrt{3}}{8}(n-2)\lambda^3 e_i + \tfrac{3\sqrt{3}}{2}\lambda^3 \sum_{k=3}^{n} \varepsilon_k \; e_i, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} 0 = R(e_1, e_i)\sum_{k=1}^{n} A_{Je_k}e_k &= \tfrac{n-2}{2}\lambda R(e_1, e_i)e_1 \\ &+ \tfrac{\sqrt{3}}{2}\lambda \sum_{i=3}^{n} \varepsilon_i \; R(e_1, e_i)e_2 \\ &= \tfrac{n-2}{2}\lambda(c - \tfrac{1}{4}\lambda^2)e_i + \tfrac{9}{8}\varepsilon_i \lambda^3 \sum_{k=3}^{n} \varepsilon_k \; e_i, \end{aligned} \tag{2.11}$$

If $\lambda \neq 0$, from (2.10), $\sum_{k=3}^{n} \varepsilon_k = -\varepsilon_i \frac{1}{4}(n-2)$, then from (2.11), $n = 2$, this is in contrast with the assumption $n \geq 3$. So, $\lambda = 0$.

case iv : If $V_2 \neq \varnothing$ and $\dim V_1 \geq 2$, from Gauss equation we have,

$$R(e_i,e_j)e_k = c(e_i \wedge e_j)e_k + A_{Je_j}A_{Je_i}e_k - A_{Je_i}A_{Je_j}e_k = 0,$$
$$R(e_i,e_j)e_1 = c(e_i \wedge e_j)e_1 + A_{Je_j}A_{Je_i}e_1 - A_{Je_i}A_{Je_j}e_1 = -\lambda A_{Je_j}e_i + \lambda A_{Je_i}e_j = 0, \tag{2.12}$$
$$R(e_i,e_j)e_i = c(e_i \wedge e_j)e_i + A_{Je_j}A_{Je_i}e_i - A_{Je_i}A_{Je_j}e_i = ce_j + \lambda^2 e_j,$$

for $e_i, e_j, e_k \in V_1$. Then, the restriction of $R(e_i,e_j)$ to $V_1 \cup \mathrm{span}_{\mathbb{R}}\{e_1\}$ is equal to $R(e_i,e_j) = (c+\lambda^2)(e_i \wedge e_j)$. Using (2.9) gives,

$$\begin{aligned} 0 = R(e_i,e_j)\sum_{k=1}^{n} A_{Je_k}e_k &= \sum_{k=1}^{n} R(e_i,e_j)A_{Je_k}e_k \\ &= (c+\lambda^2)\sum_{k=1}^{n}(e_i \wedge e_j)A_{Je_k}e_k \\ &= (c+\lambda^2)\Big(\sum_{k=1}^{n}\langle e_i, A_{Je_k}e_k\rangle e_j - \sum_{k=1}^{n}\langle e_j, A_{Je_k}e_k\rangle e_i\Big) \\ &= (c+\lambda^2)\Big(\langle e_i, \sum_{k=1}^{n}A_{Je_k}e_k\rangle e_j - \langle e_j, \sum_{k=1}^{n}A_{Je_k}e_k\rangle e_i\Big), \end{aligned}$$

If M is not totally geodesic, $c+\lambda^2 = 3\lambda^2 \neq 0$, so for each $e_k \in V_1$, $\langle e_i, \sum_{k=1}^{n}A_{Je_k}e_k\rangle = 0$, therefore $\sum_{k=1}^{n}A_{Je_k}e_k$ is in the direction of e_1. Then,

$$\sum_{k=1}^{n}A_{Je_k}e_k = \sum_{k=1}^{n}\langle A_{Je_k}e_k, e_1\rangle e_1 = \lambda(1-\dim V_1 + \tfrac{1}{2}\dim V_2)e_1 \tag{2.13}$$

Let $e_k \in V_1$ and $e_l \in V_2$, from Gauss equation it is seen that

$$R(e_k,e_l)e_1 = (e_k \wedge e_l)e_1 + A_{Je_l}A_{Je_k}e_1 - A_{Je_k}A_{Je_l}e_1 = -\lambda A_{Je_l}e_k - \tfrac{1}{2}\lambda A_{Je_k}e_l = -\tfrac{3}{2}\lambda A_{Je_l}e_k \tag{2.14}$$

If $A_{Je_l}e_k = 0$, (2.5) yields $\langle \mathbb{I}_{ll}, \mathbb{I}_{kk}\rangle = \lambda^2$, but since $\langle \mathbb{I}_{ll}, \mathbb{I}_{kk}\rangle = \frac{1}{2}\lambda^2$, so $\lambda = 0$. From (2.9), (2.13) and (2.14) we have

$$\begin{aligned} 0 = R(e_k,e_l)\sum_{m=1}^{n}A_{Je_m}e_m &= \lambda(1-\dim V_1 + \tfrac{1}{2}\dim V_2)R(e_k,e_l)e_1 \\ &= -\tfrac{3}{2}(1-\dim V_1 + \tfrac{1}{2}\dim V_2)\lambda^2 A_{Je_l}e_k, \end{aligned} \tag{2.15}$$

If $A_{Je_l}e_k \neq 0$ from (2.15) we get that either $\lambda = 0$ or $(1-\dim V_1 + \frac{1}{2}\dim V_2) = 0$. Then from (2.13), $H = 0$.

So, isotropic Lagrangian submanifolds of $\tilde{M}^n(4c)$ are either totally geodesic or minimal in $\mathbb{C}\mathrm{P}^n(4c)$. □

From the classification of minimal isotropic Lagrangian submanifolds of $\tilde{M}^n(4c)$ and constant isotropic Lagrangian submanifolds of $\tilde{M}^n(4c)$ given in [12], we obtain the following corollaries:

Corollary 3: Let $n \geq 3$, non-totally geodesic isotropic Lagrangian submanifolds M^n in complex space form $\tilde{M}^n(4c)$ are minimal and $n = 5, 8, 14$ or 26.

Corollary 4: Let M^n ($n \geq 3$) be a complete isotropic Lagrangian submanifold of $\tilde{M}^n(4c)$. Then M^n is a totally geodesic or minimal submanifold in $\mathbb{C}\mathrm{P}^n(4c)$, and locally isometric to one of the following spaces:

a) $n = 5$; $SU(3)/SO(3)$,

b) $n = 8$; $SU(3)$,

c) $n = 14$; $SU(6)/Sp(3)$,

d) $n = 26$; E_6/F_4, *where* E_6, F_4 *are exceptional Lie groups.*

From [7], we know that all submanifolds in Corollary 4, are minimal.

We also know that isotropic Lagrangian submanifolds, are semi-parallel, for $n \geq 3$, [2]. In Theorem 2, we proved that, for $n \geq 3$, isotropic Lagrangian submanifolds are either totally geodesic or minimal. To prove the theorem we used the condition that M is semi-parallel. Now we prove the same result for isotropic Lagrangian surfaces.

Theorem 5: If M^2 is a λ-isotropic Lagrangian surface in the complex space form $\tilde{M}^2(4c)$, then M is either totally geodesic or minimal. Moreover, all constant isotropic Lagrangian surfaces in $\tilde{M}^2(4c)$, are either totally geodesic or flat and minimal surfaces in $\mathbb{C}\mathrm{P}^2(4c)$.

Proof: If $\lambda = 0$ from (2.5) M is totally geodesic. If

$\lambda \neq 0$, for Lagrangian surfaces, from [12], we have that $\forall p \in M$ there exist an orthonormal basis $\{e_1, e_2\}$ of T_pM satisfying $A_{Je_1}e_1 = \lambda_1 e_1$ and $A_{Je_1}e_2 = \lambda_2 e_2$, $\lambda_1 \geq 2\lambda_2$. Since, M is λ-isotropic, $\lambda_1^2 = \lambda^2$. From (1.4) we get that $\langle A_{Je_1}e_2, e_2\rangle = \langle A_{Je_2}e_2, e_1\rangle = \lambda_2$, and from (2.4),

$$0 = \langle A_{Je2}e_2, A_{Je_1}e_2\rangle = \lambda_2 \langle A_{Je_2}e_2, e_2\rangle,$$

so, $A_{Je_2}e_2 = \lambda_2 e_1$. Using the fact that M is λ-isotropic gives, $\lambda_2^2 = \lambda^2$. Also, from $\lambda_1 \geq 2\lambda_2$, one gets that $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$. So, $H = JA_{Je_1}e_1 + JA_{Je_2}e_2 = 0$, i.e. M is minimal.

Now, suppose that M^2 is a constant isotropic Lagrangian surface, so M^2 is either totally geodesic or minimal. From Gauss equation, we obtain that M has constant Gaussian curvature $c - 2\lambda^2$.

In [8], it has been shown that, minimal Lagrangian submanifolds of constant sectional curvature in complex space forms are either totally geodesic or flat. So, if M^2 is not totally geodesic, we have $c - 2\lambda^2 = 0$, so $c = 2\lambda^2 > 0$, i.e. M^2 is a flat, minimal Lagrangian surface in $\mathbb{C}P^2(4c)$. □

From [12], we know that every minimal Lagrangian surface M^2 in $\tilde{M}^2(4c)$ is isotropic. From Gauss equation we know that the Gaussian curvature of each λ–isotropic Lagrangian surface is $c - 2\lambda^2$. So, from Theorem 5 we obtain the following corollary.

Corollary 6: A minimal Lagrangian surface M^2 of constant Gaussian curvature c_1 in $\tilde{M}^2(4c)$ is either totally geodesic or flat, constant λ-isotropic surface in $\mathbb{C}P^2(4c)$.

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