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On *R*-Right (*L*-Left) Cancellative and Weakly *R*(*L*)-Cancellative Semigroups

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Abstract

In this paper we introduce *R*-right (left), *L*-left (right) cancellative and weakly *R*(*L*)-cancellative semigroups and will give some equivalent conditions for completely simple semigroups, (completely) regular right (left) cancellative semigroups, right (left) groups, rectangular groups, rectangular bands, groups and right (left) zero semigroups according to *R*-right (left), *L*-left (right) and weak *R*(*L*)-cancellativity.

Keywords: *R*-right (left) cancellative; *L*-left (right) cancellative; Weakly *R*(*L*)-cancellative

1. Introduction and Preliminaries

Throughout this paper S will denote a semigroup. We refer the reader to [1] and [2] for basic results, definitions and terminology relating to semigroups. If S is a semigroup, then $E(S)$ is the set of all idempotent elements of S. If $e, f \in E(S)$, then $e \leq f$ if and only if $ef = fe = e$. If $ef = fe = f \neq 0$ implies that $e = f$, then e is called a primitive idempotent. A semigroup without zero is called simple if it has no proper ideals, It is called completely simple if it is simple and has a primitive idempotent. A semigroup S is called left (uniquely) solvable if for all $a, b \in S$ there exists (a unique) $s \in S$ such that $sa = b$. Analogously right (uniquely) solvable semigroups are defined. An element a of a semigroup S is called regular if there exists x in S such that $axa = a$. The semigroup S is called regular if all of its elements are regular. If every element of S lies in a subgroup of S, then S is called a completely regular semigroup. S is called a rectangular band if for $a,b \in S$, $aba = a$. If $S \cong G \times E$ where G is a group and E is a rectangular band, then S is called a *rectangular group*.

2. Results

Definition 2.1. A semigroup S is called R-right cancellative if for all $a,b,c \in S$, $ac = bc$ implies that aRb and S is called L-left cancellative if $ca = cb$ implies that aLb. R-left cancellative and L-right cancellative are defined similarly (R and L are Green's equivalences).

A semigroup S is called weakly R-cancellative (L-cancellative) if for all $a,b,c \in S$, $ac = bc$ and $ca = cb$ imply that aRb (aLb). It is obvious that R-right (left) cancellativity (L-left (right) cancellativity) implies weak R-cancellativity (L-cancellativity).

Note that every weakly R-cancellative non trivial semigroup has no zero element, otherwise for a non zero element $s \in S$, $s0 = 00$ and $0s = 00$, imply that $sR0$ which is a contradiction. Similarly, it can be shown that every weakly L-cancellative non trivial semigroup has no zero element, too.

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Lemma 2.2. Let S be a semigroup. If S is weakly R - cancellative, then every idempotent is primitive.

Proof. Suppose that S is weakly R - cancellative and let $e, f \in E(S)$, with $e \leq f$. Then $e = ef = fe$, and so $ee = ef$ and $ee = fe$. Thus eRf, and so there exist $x, y \in S$ such that $ex = f$, and $fy = e$. Consequently, $ex = eex = ef$, which implies that $f = ef$, and so $e = f$.

Note that Lemma 2.2 is also valid for weak L - cancellativity. By Theorems 3.3.3 and 4.1.2, Howie in [1], gave equivalent conditions of a completely simple semigroup. Now we give eight more equivalent conditions.

Theorem 2.3. For any semigroup S the following statements are equivalent:

(1) S is completely simple.

(2) S is completely regular and R - right cancellative.

(3) S is completely regular and L - left cancellative.

(4) S is regular and R - right cancellative.

(5) S is regular and L - left cancellative.

(6) S is completely regular and weakly R - cancellative.

(7) S is completely regular and weakly L - cancellative.

(8) S is regular and weakly R - cancellative.

(9) S is regular and weakly L - cancellative.

Proof. $(1) \Rightarrow (2)$.

By ([1, Theorem 4.1.2]), every completely simple semigroup is completely regular. By ([1, Theorem 3.3.1]), every completely simple semigroup is isomorphic to a Rees matrix semigroup $M[G; I, \Lambda; P]$. Thus it suffices to show that $M[G; I, \Lambda; P]$ is R - right cancellative. Suppose that,

$$(i_1, g_1, \lambda_1)(i, g, \lambda) = (i_2, g_2, \lambda_2)(i, g, \lambda),$$

For $(i_1, g_1, \lambda_1), (i_2, g_2, \lambda_2), (i, g, \lambda) \in M[G; I, \Lambda; P]$. Then $(i_1, g_1 p_{\lambda_1 i} g, \lambda) = (i_2, g_2 p_{\lambda_2 i} g, \lambda)$ and so $i_1 = i_2$, $g_1 p_{\lambda_1 i} = g_2 p_{\lambda_2 i}$. If $x = p^{-1}{}_{\lambda_1 i_1} g_1^{-1} g_2$ and $y = p^{-1}{}_{\lambda_2 i_2} g_2^{-1} g_1$, then $(i_1, g_1, \lambda_1)\ (i_1, x, \lambda_2) = (i_1, g_2, \lambda_2)$, and $(i_1, g_2, \lambda_2)\ (i_1, y, \lambda_1) = (i_1, g_1, \lambda_1)$ Thus $(i_1, g_1, \lambda_1) R (i_2, g_2, \lambda_2)$ and so S is R - right cancellative.

$(2) \Rightarrow (1)$.

Since S is completely regular, then S is regular and that S is R - right cancellative, then S has no zero. Also by Lemma 2.2, every idempotent is primitive. Thus by ([1, Theorem 3.3.3]), S is completely simple.

$(1) \Leftrightarrow (3)$.

It is similar to $(1) \Leftrightarrow (2)$.

$(1) \Rightarrow (4)$.

Since (1) implies (2) and that every completely regular semigroup is regular, then we are done.

$(4) \Rightarrow (1)$.

Since S is R - right cancellative, then S has no zero. Also by Lemma 2.2, every idempotent element is primitive. Hence by ([1, Theorem 3.3.3]), S is completely simple.

$(1) \Leftrightarrow (5)$.

It is similar to $(1) \Leftrightarrow (4)$.

$(1) \Rightarrow (6)$.

By ([1, Theorem 4.1.2]), and ([1,Theorem 3.3.3]), every completely simple semigroup is completely regular and weakly cancellative. As every weakly cancellative semigroup is weakly R - cancellative, then we are done.

$(6) \Rightarrow (1)$.

Since by Lmma 2.2, every idempotent element is primitive and that every completely regular semigroup is regular, then by ([1, Theorem 3.3.3]), S is a completely simple semigroup.

$(1) \Leftrightarrow (7)$.

It is similar to $(1) \Leftrightarrow (6)$.

$(1) \Rightarrow (8)$.

Since every completely regular semigroup is regular, and that $(1) \Rightarrow (6)$, then we are done.

$(8) \Rightarrow (1)$.

By Lemma 2.2, every idempotent element is primitive and so by ([1,Theorem 3.3.3]), S is a completely simple semigroup.

$(1) \Leftrightarrow (9)$.

It is similar to $(1) \Leftrightarrow (8)$.

In Theorem 2.3, if we substitute R - left cancellative and L - right cancellative, for R - right cancellative and L - left cancellative, respectively, then we will have the following theorems:

Theorem 2.4. For any semigroup S the following statements are equivalent:

(1) S is completely regular and left cancellative.

(2) S is completely regular and R-left cancellative.

(3) S is regular and left cancellative.

(4) S is regular and R-left cancellative.

(5) S is isomorphic to a Rees matrix semigroup $M[G;I,\Lambda;P]$, with |I| = 1.

Proof. Implications (1)$\Rightarrow$(2), (2)$\Rightarrow$(4) and (3)$\Rightarrow$(4) are obvious.

(4)$\Rightarrow$(5).

By Lemma 2.2, every idempotent is primitive. Thus by ([1, Theorem 3.3.3]), S is completely simple. Thus by ([1, Theorem 3.3.1]), S is isomorphic to a Rees matrix semigroup $M[G;I,\Lambda;P]$. We claim that $|I|=1$. Suppose that $x=p^{-1}_{\lambda i_2}p_{\lambda i_1}$, for $i_1,i_2\in I$ and $\lambda\in\Lambda$. Then,

$$(i_1,e,\lambda)(i_1,e,\lambda)=(i_1,e,\lambda)(i_2,x,\lambda).$$

Since S is R-left cancellative, then $(i_1,e,\lambda)R(i_2,x,\lambda)$, and so there exists $(i_0,y,\lambda_0)\in M[G;I,\Lambda;P]$ such that

$$(i_1,e,\lambda)(i_0,y,\lambda_0)=(i_2,x,\lambda).$$

Thus $(i_1,p_{\lambda i_0}y,\lambda_0)=(i_2,x,\lambda)$, which implies that $i_1=i_2$, and so $|I|=1$ as required.

(5)$\Rightarrow$(3).

By ([1, Theorems 3.3.1, 3.3.3]), S is regular. We show that S is left cancellative. Since S is isomorphic to a Rees matrix semigroup $M[G;I,\Lambda;P]$, then it suffices to show that $M[G;I,\Lambda;P]$ is left cancellative. Thus we suppose that

$$(i,g,\lambda)(i,g_1,\lambda_1)=(i,g,\lambda)(i,g_2,\lambda_2),$$

for $(i,g,\lambda),(i,g_1,\lambda_1)$ and $(i,g_2,\lambda_2)\in M[G;I,\Lambda;P]$. Then, $(i,gp_{\lambda i}g_1,\lambda_1)=(i,gp_{\lambda i}g_2,\lambda_2)$, which implies that $\lambda_1=\lambda_2$, $gp_{\lambda i}g_1=gp_{\lambda i}g_2$. Thus $g_1=g_2$ and so $(i,g_1,\lambda_1)=(i,g_2,\lambda_2)$.

(5)$\Rightarrow$(1).

Since by ([1,Theorems 3.3.1,4.1.2]), S is completely regular, then by (5)$\Rightarrow$(3), it is obvious.

Similarly we have,

Theorem 2.5. For any semigroup S the following statements are equivalent:

(1) S is completely regular and right cancellative.

(2) S is completely regular and L-right cancellative.

(3) S is regular and right cancellative.

(4) S is regular and L-right cancellative.

(5) S is isomorphic to a Rees matrix semigroup $M[G;I,\Lambda;P]$ with $|\Lambda|=1$.

By ([1, page 62, Exercise 11]), every cancellative regular semigroup is a group. Thus by Theorems 2.4, and 2.5 we have,

Corollary 2.6. For any (completely) regular semigroup S the following statements are equivalent:

(1) S is L-right and R-left cancellative.

(2) S is isomorphic to a Rees matrix semigroup $M[G;I,\Lambda;P]$ with $|\Lambda|=|I|=1$.

(3) *S is a group.*

Definition 2.7. A semigroup S is called a right group if it is right simple $(R=S\times S)$ and left cancellative.

By ([1, page 61, exercise 6]), and ([2, I, 3.19, 3.22]), it can easily be seen that:

Theorem 2.8. For any semigroup S the following statements are equivalent:

(1) S is a right group.

(2) S is left cancellative and has no proper right ideal.

(3) $S\cong G\times E$ where G is a group and E is a right zero semigroup.

(4) S is right uniquely solvable.

Now see four more equivalent conditions for right groups in the following theorem:

Theorem 2.9. For any semigroup S the following statements are equivalent:

(1) *S is a right group.*

(2) S is right inverse and R-right cancellative.

(3) S is right inverse and L-left cancellative.

(4) S is right inverse and weakly R-cancellative.

(5) S is right inverse and weakly L-cancellative.

Proof. (1)$\Rightarrow$(2).

Suppose that S is a right group. Then by Theorem 2.8, $S\cong G\times E$, where G is a group and E is a right zero sem-igroup. It is obvious that S is regular and also $E(S)=1_G\times E$. If $e=(1_G,e')$ and $f=(1_G,f')$ are idempotents, then

$$efe = (1_G, e')(1_G, f')(1_G, e') = (1_G, e'f'e') = (1_G, e') = e.$$

Also

$$fe = (1_G, f')(1_G, e') = (1_G, f'e') = (1_G, e') = e.$$

Thus $efe = fe$, and so by ([2, I, 3.39]), S is right inverse. Suppose that $ac = bc$, for $a = (g_1, e_1)$, $b = (g_2, e_2)$ and $c = (g, e)$. Then $(g_1, e_1)(g, e) = (g_2, e_2)(g, e)$, and so $g_1 g = g_2 g$, which implies that $g_1 = g_2$. Now we have $(g_1, e_1)(1_G, e_2) = (g_1, e_2) = (g_2, e_2)$ and $(g_2, e_2)(1_G, e_1) = (g_2, e_1) = (g_1, e_1)$. Hence aRb and so S is R- right cancellative.

(2) $\Rightarrow$ (1).

Since S is right inverse, then by ([2, I, 3.38]), S is orthodox. Also by Theorem 2.3, S is completely simple. Thus by ([1, page 139, Exercise 10]), $S \cong G \times E$, where G is a group and E is a rectangular band. Now we show that E is right zero. Suppose that $e = (1_G, e')$, $f = (1_G, f') \in E(S) = 1_G \times E$. Then $efe \in E(S)$ and $efe \leq e$. Hence by Le-mma 2.2, $efe = e$, that is, $(1_G, e')(1_G, f')(1_G, e') = (1_G, e')$ and so $(1_G, e'f'e') = (1_G, e')$, which implies that $e'f'e' = e'$. Since S is right inverse, then $efe = fe$, and so $e'f'e' = f'e'$. Thus $f'e' = e'$ and hence E is right zero which implies by Theorem 2.8, that S is a right group.

(1) $\Leftrightarrow$ (3).

It is similar to (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (4).

Since every R- right cancellative is weakly R- cancellative, then by (1) $\Rightarrow$ (2) we are done.

(4) $\Rightarrow$ (1).

Since S is right inverse, then S is regular and so by Theorem 2.3, S is completely simple. Hence by Theorem 2.3, S is R- right cancellative. Now by (2) $\Rightarrow$ (1) S is a right group.

(1) $\Leftrightarrow$ (5).

It is similar to (1) $\Leftrightarrow$ (4).

Note that in Theorem 2.9, if we substitute left inverse for right inverse, then we have a similar theorem for left groups.

By ([1, page 139, Exercise 10]), and ([1, page 236, Exercise 10]), it can be shown that

Theorem 2.10. For any semigroup S the following statements are equivalent:

(1) S *is rectangular group.*

(2) S is completely simple and orthodox.

(3) S is completely regular and satisfies the law $x^{-1}yy^{-1}x = x^{-1}x$.

(4) S is orthodox and $E(S)$ is rectangular band.

Now see four more equivalent conditions for rectangular groups in the following theorem:

Theorem 2.11. For any semigroup S the following statements are equivalent:

(1) S *is a rectangular group.*

(2) S is orthodox and R- right cancellative.

(3) S is orthodox and L- left cancellative.

(4) S is orthodox and weakly R- cancellative.

(5) S is orthodox and weakly L- cancellative.

Proof. (1) $\Rightarrow$ (2).

By Theorem 2.10, S is orthodox and completely simple. Thus by Theorem 2.3, S is R- right cancellative.

(2) $\Rightarrow$ (1).

Since S is orthodox, then S is regular. Also by Theorem 2.3, S is completely simple. Thus by Theorem 2.10, S is a rectangular group.

(1) $\Leftrightarrow$ (3).

It is similar to (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (4).

Since every R- right cancellative is weakly R- cancellative, then by (1) $\Rightarrow$ (2) we are done.

(4) $\Rightarrow$ (1).

The same argument as (2) $\Rightarrow$ (1) can be used.

(1) $\Leftrightarrow$ (5).

It is similar to (1) $\Leftrightarrow$ (4).

By ([1, Theorem 1.1.3]), and ([1, page 38, Exercise 4]), we have

Theorem 2.12. For any semigroup S the following statements are equivalent:

(1) S is a rectangular band.

(2) S is idempotent semigroup and for all $a,b,c \in S$, $abc = ac$.

(3) There exist a left zero semigroup L and a right zero semigroup R such that $S \cong L \times R$.

(4) S is isomorphic to a semigroup of the form $A \times B$, where A and B are non-empty sets and where

multiplication is given by $(a_1,b_1)(a_2,b_2)=(a_1,b_2)$.

(5) For all $a,b \in S$, $ab = ba$ implies that $a = b$.

Now see four more equivalent conditions for rectangular band in the following theorem:

Theorem 2.13. For any semigroup S the following statements are equivalent:

(1) S *is a rectangular band.*
(2) S is a band and R - right cancellative.
(3) S is a band and L - left cancellative.
(4) S is a band and weakly R - cancellative.
(5) S is a band and weakly L - cancellative.

Proof. (1)$\Rightarrow$(2).

By Theorem 2.12, every rectangular band is a band. Now we suppose that $e_1 f = e_2 f$, for $e_1, e_2, f \in S$. Since S is a rectangular band, then $e_1 f e_1 = e_1$, and $e_2 f e_2 = e_2$. Thus $e_1 = e_2(fe_1)$, $e_2 = e_1(fe_2)$, and so $e_1 R e_2$. Hence, S is R - right cancellative.

(1)$\Rightarrow$(4).

Since every R - right cancellative is weakly R - cancellative, then by (1)$\Rightarrow$(2), it is obvious.

(4)$\Rightarrow$(1).

Suppose that $e, f \in S$. Since S is a band, then we have, $(ef)^2 = ef$ and $(fe)^2 = fe$, that is, $efe.f = e.f$ and $f.efe = f.e$. Since S is weakly R - cancellative, then from the above equalities we have $efeRe$. Thus there exists $x \in S$ such that $efex = e$, which implies that

$$efe = ef(efex) = (ef)^2 ex = efex = e.$$

Hence S is a rectangular band.

(1)$\Leftrightarrow$(5).

It is similar to (1)$\Leftrightarrow$(4).

(2)$\Rightarrow$(1).

Since every R - right cancellative is weakly R - cancellative, then by (4)$\Rightarrow$(1), it is obvious.

(1)$\Leftrightarrow$(3).

It is similar to (1)$\Leftrightarrow$(2).

Theorem 2.14. For any semigroup S the following statements are equivalent:

(1) S *is a group.*
(2) S is clifford and R - right cancellative.
(3) S is clifford and L - left cancellative.
(4) S is an inverse semigroup and R - right cancellative.
(5) S is an inverse semigroup and L - left cancellative.
(6) S is clifford and weakly R - cancellative.
(7) S is clifford and weakly L - cancellative.
(8) S is an inverse semigroup and weakly R - cancellative.
(9) S is an inverse semigroup and weakly L - cancellative.

Proof. (1)$\Rightarrow$(6).

It is obvious.

(6)$\Rightarrow$(1).

Since S is Clifford, then S is regular and so $E(S) \neq \emptyset$. Now suppose that $e, f \in E(S)$. Then $ef = fe$ and so $E(S)$ is a subsemigroup of S. Thus $(ef)^2 = ef$, $(fe)^2 = fe$, that is, $efef = ef$ and $fefe = fe$. Since S is weakly R - cancellative, then $fef\ R\ f$. Thus there exists $x \in S$ such that $fefx = f$, which implies that

$$fef = fe(fefx) = (fe)^2 fx = fefx = f.$$

But $ef = fe$ and so $f = fe$. Similarly $e = ef$, and so $e = f$. Thus $|E(S)| = 1$ and hence by ([1, page 62, Exercise 11]), S is a group.

(1)$\Rightarrow$(2).

It is obvious.

(2)$\Rightarrow$(1).

Since every R - right cancellative semigroup is weakly R - cancellative, then by (6)$\Rightarrow$(1) it is obvious.

(1)$\Leftrightarrow$(3).

It is similar to (1)$\Leftrightarrow$(2).

(1)$\Rightarrow$(4).

It is obvious.

(4)$\Rightarrow$(1).

Since every R - right cancellative semigroup is weakly R - cancellative and for an inverse semigroup S and all $e, f \in E(S)$, $ef = fe$, then the argument is the same as in (6)$\Rightarrow$(1).

(1)$\Leftrightarrow$(5).

It is similar to (1)$\Leftrightarrow$(4).

(1) $\Leftrightarrow$ (7).

It is similar to (1) $\Leftrightarrow$ (6).

(1) $\Rightarrow$ (8).

It is obvious.

(8) $\Rightarrow$ (1).

Since for an inverse semigroup S and for every $e, f \in E(S)$, $ef = fe$, then the same argument as in (6) $\Rightarrow$ (1) can be used.

(1) $\Leftrightarrow$ (9).

It is similar to (1) $\Leftrightarrow$ (8).

Note that in Theorem 2.14, we can substitute R - left and L - right cancellative for R - right and L - left cance- llative, respectively.

Theorem 2.15. For any semigroup S the following statements are equivalent:

(1) S *is right zero.*
(2) S is rectangular band and R - left cancellative.
(3) S is a band, R - right and R - left cancellative.
(4) S is a band, L - left and R - left cancellative.
(5) S is a band and R - left cancellative.
(6) S is a band, weakly L - cancellative and R - left cancellative.

Proof.

(1) $\Rightarrow$ (2). It is obvious that S is a rectangular band. Let $fe_1 = fe_2$, for $e_1, e_2, f \in S$. Since S is right zero, then $e_1 = e_2$, and so $e_1 R e_2$. Thus S is R - left cancellative.

(2) $\Rightarrow$ (1).

Let $e, f \in S$. Since S is rectangular band, then $efe = e$, that is, $efe = ee$. Since S is R - left cancell- ative, then $feRe$ and so there exists $x \in S$ such that $fex = e$. Thus $fe = e$ and hence S is right zero.

(2) $\Leftrightarrow$ (3)

and (2) $\Leftrightarrow$ (4) are obvious by Theorem 2.13.

(1) $\Rightarrow$ (5).

It follows by (1) $\Rightarrow$ (2).

(5) $\Rightarrow$ (1).

Let $e, f \in S$. Since S is R - left cancellative, then $eef = ef$, implies that $ef \; R \; f$ and so there exists $x \in S$ such that $efx = f$. Thus $ef = f$ and hence S is right zero.

Implication (1) $\Leftrightarrow$ (6) is obvious by (1) $\Leftrightarrow$ (2) and Theorem 2.13.

Note that Theorem 2.15, is also valid for left zero semigroups if we substitute L - right cancellative for R - left cancellative.

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