**Journal of Sciences, Islamic Republic of Iran 18(1): 41-48 (2007)** http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

# The Topological Center of the Banach Algebra $UC_l(K)^*$

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## Abstract

Let $K$ be a (commutative) locally compact hypergroup with a left Haar measure. Let $L^1(K)$ be the hypergroup algebra of $K$ and $UC_l(K)$ be the Banach space of bounded left uniformly continuous complex-valued functions on $K$. In this paper we show, among other things, that the topological (algebraic) center of the Banach algebra $UC_l(K)^*$ is $M(K)$, the measure algebra of $K$.

**Keywords:** Hypergroup; Hypergroup algebra; Measure algebra; Second conjugate algebra; Algebraic center

## 1. Introduction

The theory of hypergroups was initiated by Dunkl [4], Jewett [8] and Spector [21] in the early 1970's and has received a good deal of attention from harmonic analysts (note that Jewett calls hypergroups "convos" in his paper [8]). In [16], Pym also considers convolution structures which are close to hypergroups. A fairly complete history is given in Ross's survey article [17,18]. Hypergroups arise in a natural way as a double coset space, and the space of conjugacy classes of a compact group [17,1]. In particular, locally compact groups are hypergroups. Here we follow the method of Jewett [8]. It is still unknown if an arbitrary hypergroup admits a left Haar measure but all the known examples do [8, §5]. In particular, discrete, compact and commutative hypergroups possess Haar measures [10].

Throughout, $K$ will denote a hypergroup with a left Haar measure $\lambda$. Let $L^1(K)$ denote the hypergroup algebra of $K$, *i.e.* all Borel measurable functions $\phi$ on $K$ with $\|\phi\|_1 = \int_K |\phi(x)|d\lambda(x) < \infty$ (with functions equal almost everywhere identified), and the multiplication defined by

$$\phi * \psi(x) = \int_K \phi(x * y)\psi(y)d\lambda(y) \qquad \text{(see [8, §5.5])}.$$

Let the second dual $L^1(K)^{**}$ $(= L^\infty(K)^*)$ of $L^1(K)$ be equipped with the first Arens product [3]. Then $L^1(K)^{**}$ is a Banach algebra with this product. The *topological center* of $L^1(K)^{**}$ is defined by

$Z(L^1(K)^{**}) = \{m \in L^1(K)^{**}$ : the mapping $n \mapsto mn$ is $w^*$-continuous on $L^1(K)^{**}\}$. We have shown [9] that the topological center of $L^1(K)^{**}$ is $L^1(K)$. This fact has been shown by Lau and Losert in [13] for locally compact groups (see also [14] and [2]).

Let $UC_l(K)$ be the Banach space of all bounded left uniformly continuous complex-valued functions on $K$ (see Section 2 for definition) and $UC_l(K)^*$ be its dual Banach space. Then there is a natural multiplication on $UC_l(K)^*$ under which it is a Banach algebra. More

*2000 Mathematics Subject Classification.* Primary 43A20, 43A10, 43A22 Secondary 46H05, 46H10

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specifically, for $m,n \in UC_l(K)^*$, $f \in UC_l(K)$, and $x \in K$,

$$\langle mn, f\rangle = \langle m, nf\rangle \text{ where } nf(x) = \langle n, {}_x f\rangle .$$

This product is, in fact, the restriction of the first Arens product on $L^1(K)^{**}$ to $UC_l(K)^*$, which will be proved in Lemma 3.1. The *topological center* of $UC_l(K)^*$ is defined by

$Z(UC_l(K)^*) = \{m \in UC_l(K)^*$: the mapping $n \mapsto mn$ is $w^*$-continuous on $UC_l(K)^*\}$. Note that when $K$ is commutative, then $Z(UC_l(K)^*)$ is precisely the algebraic center of $UC_l(K)^*$. For a locally compact group $G$, Lau in [12] has shown that $Z(UC_l(G)^*)$ is $M(G)$, the algebra of bounded regular Borel measures on $G$. However the method of his proof cannot be applied to hypergroups in general. The purpose of this paper is to establish these results for hypergroups. Our proof also provides a new proof of Lau's result [12, Theorem 1] in the group case.

This paper is organized as follows:

Section 2 consists of some notations and preliminary results that we need in the sequel. The technical Lemma 2.7 in this section plays a key role in proving our main result (Theorem 3.11). In Section 3, we shall prove that the topological center of $UC_l(K)^*$ is $M(K)$. The results of this section generalize the corresponding ones for locally compact groups [12].

## 2. Preliminaries and Some Technical Lemmas

The notations used in this paper are those of [8] with the following exceptions:

The mapping $x \to \breve{x}$ denotes the involution on the hypergroup $K$, $\delta_x$ the Dirac measure concentrated at $x$ ($x \in K$), and $1_X$ the characteristic function of the non-empty set $X \subseteq K$. For $C \subseteq K$ and $y \in K$, $C * y$ denotes the subset $C * \{y\}$ of $K$.

**Lemma 2.1.** *Let K be a locally compact non-compact hypergroup. Then there exists a family* $\{C_i : i \in I\}$ *of compact subsets of K, and* $y_i, z_i \in K$*, for each* $i \in I$ *such that* $C_i^\circ$ *(the interior of* $C_i$*) is non-empty,* $\cup_{i\in I} C_i^\circ = K$, $\{C_i : i \in I\}$ *is closed under finite unions, and*

*(a) the families* $\{C_i * y_i : i \in I\}$ *and* $\{C_i * z_i : i \in I\}$ *are pairwise disjoint.*

*(b)* $C_i * y_i * \breve{y}_j \cap C_p * z_p * \breve{z}_q = \varnothing$, $i \neq j$ *and* $p \neq q$, $i, j, p, q \in I$.

**Proof.** See [9, Lemma 2.1]. □

For a Borel function $f$ on $K$ and $x \in K$, ${}_x f$ denotes the left translation

$${}_x f(y) = f(x * y) = \int_K f(t)\, d(\delta_x * \delta_y)(t),$$

and $f_x$ is the right translation

$$f_x(y) = f(y * x) = \int_K f(t)\, d(\delta_y * \delta_x)(t),$$

if the integrals exist. We write ${}_{x*y} f$ and $f_{x*y}$ for ${}_y({}_x f)$ and $(f_y)_x$, respectively.

The function $\breve{f}$ is given by $\breve{f}(x) = f(\breve{x})$. The integral $\int \ldots d\lambda(x)$ is often denoted by $\int \ldots dx$.

Let $(L^p(K), \|.\|_p)$, $1 \le p \le \infty$, denote the usual $L^p$ spaces on $K$ [8, §6.2]. Then $L^\infty(K)$ is a commutative Banach algebra with pointwise multiplication and the essential supremum norm $\|.\|_\infty$, and moreover, $L^\infty(K) = L^1(K)^*$ [8, §6.2]. We say that $X \subseteq L^\infty(K)$ is *translation invariant* if ${}_x f \in X$ and $f_x \in X$ for all $f \in X$, $x \in K$; also $X$ is *topologically translation invariant* if $\phi * f \in X$ and $f * \breve{\phi} \in X$ for all $f \in X$, $\phi \in P^1(K) = \{\phi \in L^1(K) : \phi \ge 0, \|\phi\|_1 = 1\}$.

In addition, we use the following notations:

$C_{00}(K)$: the set of continuous functions with compact supports on $K$.

$C(K)$: the set of bounded continuous functions on $K$.

$UC_l(K) = \{f \in C(K) : x \mapsto {}_x f$ is continuous from K into $(C(K), \|.\|_\infty)\}$.

$UC_r(K) = \{f \in C(K) : x \mapsto f_x$ is continuous from K into $(C(K), \|.\|_\infty)\}$.

It is known that $UC_l(K) = \{f \in C(K) : x \mapsto {}_x f$ is continuous from $K$ into $C(K)$ with the weak-topology$\}$ [20, Theorem 4.2.2, p. 88].

Each of the spaces $UC_l(K)$ and $UC_r(K)$ is a normed closed, conjugate closed, translation invariant and topologically translation invariant subspace of $C(K)$ containing the constant functions and $C_0(K)$

[19, Lemma 2.2]. Furthermore

(i) $UC_l(K) = L^1(K) * UC_l(K) = L^1(K) * L^\infty(K)$ [19, Lemma 2.2]

(ii) $UC_r(K) = UC_l(K) * L^1(K)\breve{} = L^\infty(K) * L^1(K)\breve{}$ [19, Lemma 2.2].

Note that $UC_l(K)$ is not an algebra in general [19, Remark 2.3(b)].

For $\phi \in L^1(K)$, we write $\tilde{\phi}(x) = \Delta(\breve{x})\phi(\breve{x})$ where $\Delta$ is the modular function on $K$; then $\|\tilde{\phi}\| = \|\phi\|_1$. If $f \in L^p(K)$, $1 \le p \le \infty$, $x \in K$, then $\|{}_x f\|_p \le \|f\|$, and this is in general not an isometry [8, §3.3]. The mapping $x \mapsto {}_x f$ is continuous from $K$ to $(L^p(K), \|.\|_p)$, $1 \le p < \infty$, [8, 2.2B and 5.4H].

It is easy to show that $L^1(K)$ has a bounded approximate identity (B.A.I) $\{e_i : i \in I\} \subseteq C_{00}^+(K)$ such that $\|e_i\| = 1$ (see [19, Lemma 2.1]).

For any Banach space $X$, we denote its first and second dual by $X^*$ and $X^{**}$. Let $A$ be a Banach algebra. For any $f \in A^*$ and $a \in A$, we may define a linear functional $fa$ on $A$ by $\langle fa, b\rangle = \langle f, ab\rangle, (b \in A)$. One can check that $fa \in A^*$ and $\|fa\| \le \|f\| \|a\|$. Now for $n \in A^{**}$, we may define $nf \in A^*$ by $\langle nf, a\rangle = \langle n, fa\rangle$; clearly we have $\|nf\| \le \|n\| \|f\|$. Next for $m \in A^{**}$, define $mn \in A^{**}$ by $\langle mn, f\rangle = \langle m, nf\rangle$. We have $\|mn\| \le \|m\| \|n\|$, and $A^{**}$ becomes a Banach algebra with the multiplication $mn$, just defined, referred to as the first *Arens product*. There is another multiplication on $A^{**}$, called the second Arens product, which is denoted by $m \circ n$ and defined successively as follows:

$\langle m \circ n, f\rangle = \langle n, fm\rangle$, where $\langle fm, a\rangle = \langle m, af\rangle$, $\langle af, b\rangle = \langle f, ba\rangle$, and $m, n, f, a, b$ are taken as above.

From now on $A^{**}$ will always be regarded as a Banach algebra with the first Arens product.

Let $Z(A^{**})$ denote the set of all $m \in A^{**}$ such that $mn = m \circ n$ for all $n \in A^{**}$. We call $Z(A^{**})$ the *topological center* of $A^{**}$.

**Lemma 2.2.** *$Z(A^{**})$ is a closed subalgebra of $A^{**}$ containing $A$.*

**Proof.** [3, p. 310] or [13, Lemma 1]. □

**Lemma 2.3** *For any $m \in A^{**}$, the following are equivalent:*

*(a) $m \in Z(A^{**})$;*

*(b) the map $n \to mn$ from $A^{**}$ into $A^{**}$ is $w^*$-$w^*$ continuous;*

*(c) the map $n \to mn$ from $A^{**}$ into $A^{**}$ is $w^*$-$w^*$ continuous on norm bounded subsets of $A^{**}$.*

**Proof.** [3, p. 313]. □

Note that for $n$ fixed in $A^{**}$, the mapping $m \mapsto mn$ is always $w^*$-$w^*$ continuous.

We collect here some facts about the Arens product on $L^1(K)^{**}$ that we shall need later.

**Lemma 2.4.** *Let $\phi, \psi \in L^1(K)$, $f \in L^\infty(K)$. Then*

*(i) $\langle \psi f, \phi\rangle = \langle f\phi, \psi\rangle$.*

*(ii) $\psi f = f * \breve{\psi} \in UC_r(K)$, $f\phi = \tilde{\phi} * f \in UC_l(K)$.*

*(iii) ${}_a(\psi f) = \psi({}_a f)$, $(f\phi)_a = (f_a)\phi$ for $a \in K$.*

**Proof.** immediate. □

**Lemma 2.5.** *Let $0 \ne m \in L^\infty(K)^*$. Then there is a net $\{u_\alpha\}$ in $L^1(K)$ such that $\|u_\alpha\| \le \|m\|$, all $u_\alpha$ have compact support and $u_\alpha \to m$ in the $w^*$-topology of $L^\infty(K)^*$.*

**Proof.** This follows from Goldstine's theorem and the density of $C_{00}(K)$ in $L^1(K)$. □

**Lemma 2.6.** *If $m \in Z(L^1(K)^{**})$ and $f \in L^\infty(K)$, then $fm \in UC_l(K)$ and $(fm)(x * y) = \langle m, f_{x*y}\rangle$.*

**Proof.** See [9, Lemma 2.6]. □

**Lemma 2.7.** *If $n \in Z(L^1(K)^{**})$ and $u \in L^1(K)$ are such that $(n - u)(f) = 0$ for all $f \in C_0(K)$, then $n = u$.*

**Proof.** See [9, Lemma 2.7]. □

## 3. Topological Center of $UC_l(K)^*$

In this section we show that the topological center of $UC_l(K)^*$ is $M(K)$. Let $f \in UC_l(K)$ and $m \in UC_l(K)^*$. Define the function $mf$ on $K$ by $mf(x)$

$= \langle m, {}_x f \rangle$. Then $mf \in UC_l(K)$. Indeed, it is easy to see that $mf \in C(K)$. Also

$$\begin{aligned} {}_x(mf)(y) &= mf(x*y) \\ &= \int_{x*y} mf(t)d(\delta_x * \delta_y)(t) \\ &= \int_{x*y} \langle m, {}_t f\rangle d(\delta_x * \delta_y)(t) \\ &= \langle m, \int_{x*y} {}_t f\, d(\delta_x * \delta_y)(t)\rangle \qquad (*) \end{aligned}$$

But the Bochner integral $\int_{x*y} {}_t f\, d(\delta_x * \delta_y)(t)$ is ${}_y({}_x f)$ since

$$\begin{aligned} \int_{x*y} {}_t f\, d(\delta_x * \delta_y)(t)(\xi) &= \langle \delta_\xi, \int_{x*y} {}_t f\, d(\delta_x * \delta_y)(t)\rangle \\ &= \int_{x*y} \langle \delta_\xi, {}_t f\rangle d(\delta_x * \delta_y)(t) \\ &= \int_{x*y} {}_t f(\xi) d(\delta_x * \delta_y)(t) \\ &= \int_{x*y} f_\xi(t) d(\delta_x * \delta_y)(t) \\ &= f_\xi(x*y) = {}_y({}_x f)(\xi). \end{aligned}$$

So (*) implies that

$${}_x(mf)(y) = \langle m, {}_y({}_x f)\rangle = m({}_x f)(y),$$

that is,

$${}_x(mf) = m({}_x f). \qquad (1)$$

Hence

$$\begin{aligned} \| {}_x(mf) - {}_y(mf) \| &\le \| m({}_x f) - m({}_y f) \| \\ &\le \| m \| \| {}_x f - {}_y f \|. \end{aligned}$$

Note that if $m = \delta_a$ for some $a \in K$, then $\delta_a f = f_a$.

Now we may define a product on $UC_l(K)^*$ by $\langle nm, f\rangle = \langle n, mf\rangle$ for $m, n \in UC_l(K)^*$ and $f \in UC_l(K)$. With this product, one can see that $UC_l(K)^*$ is a Banach algebra. **Lemma 3.1** *The product on $UC_l(K)^*$ is the restriction of the first Arens product on $L^\infty(K)^*$ to $UC_l(K)^*$.*

**Proof.** See [15, Theorem 7]. □

Note that we can even identify $UC_l(K)^*$ as a closed right ideal of the Banach algebra $L^\infty(K)^*$ with the first Arens product (see [14, p. 13]).

**Lemma 3.2.** *If we take $C_0(K)^\perp = \{m \in UC_l(K)^* : m|_{C_0(K)} = 0\}$, then $UC_l(K)^* = C_0(K)^\perp \oplus M(K)$. If $m \in UC_l(K)^*$ and $m = m_1 + \mu$ for $m_1 \in C_0(K)^\perp$ and $\mu \in M(K)$, then $\|m\| = \|m_1\| + \|\mu\|$ and $C_0(K)^\perp$ is a closed ideal in $UC_l(K)^*$.*

**Proof.** See [15, Theorem 4]. □

**Remark 3.3.** For $m \in UC_l(K)^*$ and $f \in UC_l(K)$, we may define a bounded complex function $fm$ on $K$ by $fm(x) = \langle m, f_x\rangle$. Generally, $fm$ is not in $UC_l(K)$ but for $m = \delta_a$ $(a \in K)$ $fm = f\delta_a = {}_a f \in UC_l(K)$. If $n \in UC_l(K)^*$ and $fm \in UC_l(K)$, for all $f \in UC_l(K)$, then we may define another product on $UC_l(K)^*$ by $\langle m \circ n, f\rangle = \langle n, fm\rangle$.

Let $Z(UC_l(K)^*)$ denote the set of all $m \in UC_l(K)^*$ such that $fm \in UC_l(K)$ for all $f \in UC_l(K)$ and $mn = m \circ n$ for all $n \in UC_l(K)^*$. One can check that $Z(UC_l(K)^*)$ contains all point evaluation functionals $\delta_x$, $x \in K$.

**Note 3.4.** For $m \in UC_l(K)^*$, define the linear operator $L_m$ from $UC_l(K)^*$ into itself by

$$L_m(n) = mn, \quad n \in UC_l(K)^*.$$

Put

$C = \{m \in UC_l(K)^* : L_m$ is $w^*$-$w^*$ continuous on norm bounded subset of $UC_l(K)^*\}$.

**Lemma 3.5.** $M(K) \subseteq C$.

**Proof.** For $\mu \in M(K)$, we need to show that the map $m \to \mu m$ is $w^*$-$w^*$ continuous on any norm bounded subset of $UC_l(K)^*$. Let $\{m_\alpha\}$ be a net in $UC_l(K)^*$ with $\|m_\alpha\| \le c$, for some constant $c$, converging to $m \in UC_l(K)^*$ in the $w^*$-topology of $UC_l(K)^*$. Then for any $f \in UC_l(K)$ and $s, t \in K$, we have

$|m_\alpha f(s) - m_\alpha f(t)| = |\langle m_\alpha, {}_s f - {}_t f\rangle| \le c\,\|{}_s f - {}_t f\|$. Hence by [11, p. 232] the family $\{m_\alpha f\}$ in $UC_l(K)$ is equicontinuous. Since $m_\alpha f \to mf$ pointwise on $K$, the convergence is uniform on every compact set in $K$ [11, Theorem 7.15]. Let $\mu \in M(K)$ be with compact support, then $\langle \mu m_\alpha - \mu m, f\rangle = \langle \mu, m_\alpha f - mf\rangle = \int_K (m_\alpha f - mf)(x)\,d\mu(x) \to 0$. Since measures with compact supports are norm dense in $M(K)$ and $\|m_\alpha f\| \le c\|f\|$, it follows that $\mu m_\alpha \to \mu m$ in the $w^*$-topology of $UC_l(K)^*$ and we are done. □

**Lemma 3.6.** *If* $m \in C$ *and* $f \in UC_l(K)$, *then* $fm \in C(K)$ *and* $fm(x*y) = \langle m, f_{x*y}\rangle$ *for all* $x, y \in K$.

**Proof.** If $\{x_\alpha\}$ is a net in $K$ converging to $x$, then the net $\{\delta_{x_\alpha}\}$ converges to $\delta_x$ in the $w^*$-topology of $UC_l(K)^*$ (see [8, Lemma 2.2B] and Lemma 3.2). Hence

$$\begin{aligned} fm(x_\alpha) &= \langle m, f_{x_\alpha}\rangle = \langle m, \delta_{x_\alpha} f\rangle \\ &= \langle m\delta_{x_\alpha}, f\rangle \to \langle m\delta_x, f\rangle \\ &= \langle m, \delta_x f\rangle = \langle m, f_x\rangle = fm(x), \end{aligned}$$

since $m \in C$ and $\{\delta_{x_\alpha}\}$ is bounded. Furthermore, we know that $fm$ is also bounded. Consequently $fm \in C(K)$. Note that for every $x, y \in K$, the Bochner's integral $\int_{x*y} f_t\, d(\delta_x * \delta_y)$ exists. Indeed, the map $t \to f_t$ from the compact subset $x*y$ of $K$ into $UC_l(K)$ is continuous in the topology $\sigma(UC_l(K), C)$ of $UC_l(K)$, and $C$ separates the points of $UC_l(K)$ ($C$ contains the point evaluations). Hence for any $m \in C$

$$\begin{aligned} \langle m, \int_{x*y} f_t\, d(\delta_x * \delta_y)(t)\rangle &= \int_{x*y} \langle m, f_t\rangle d(\delta_x * \delta_y)(t) \\ &= \int_{x*y} fm(t) d(\delta_x * \delta_y)(t) \quad (*) \\ &= fm(x*y). \end{aligned}$$

On the other hand, the Bochner's integral $\int_{x*y} f_t\, d(\delta_x * \delta_y)(t)$ is equal to $f_{x*y}$. By using Lemma 2.4(iii), for every $\phi \in L^1(K) \subseteq C$ (Lemma 3.5), (*) implies that

$$\begin{aligned} \langle \phi, \int_{x*y} f_t\, d\delta_x * \delta_y(t)\rangle &= f\phi(x*y) = (f\phi)_y(x) \\ &= ((f_y)\phi)(x) = ((f_y)\phi)_x(e) \\ &= (f_y)_x\,\phi(e) = \langle \phi, (f_y)_x\rangle\rangle. \end{aligned}$$

Hence from (*) we have $\langle m, f_{x*y}\rangle\rangle = fm(x*y)$. □

**Lemma 3.7.** *For each* $m \in UC_l(K)^*$ *the following are equivalent:*

*(a)* $m \in Z(UC_l(K)^*)$,

*(b) The operator* $L_m$ *is* $w^*$-$w^*$ *continuous,*

*(c)* $m \in C$.

**Proof.** First we show that (a) implies (b). Let $\{n_\alpha\}$ be a net in $UC_l(K)^*$ converging to $n \in UC_l(K)^*$ in the $w^*$-topology of $UC_l(K)^*$. Then for every $f \in UC_l(K)$,

$$\begin{aligned} \lim_\alpha mn_\alpha(f) &= \lim_\alpha \langle mn_\alpha, f\rangle \\ &= \lim_\alpha \langle m \circ n_\alpha, f\rangle = \lim_\alpha \langle n_\alpha, fm\rangle \\ &= \langle m \circ n, f\rangle = mn(f). \end{aligned}$$

(b) clearly implies (c).

To show that (c) implies (a), let $m \in C$ and $f \in UC_l(K)$, then by Lemma 3.6, $fm \in C(K)$. To see that $fm \in UC_l(K)$, we first show that if $\theta \in C(K)^*$ and $a \in K$, then

$$\langle \theta, {}_a(fm)\rangle = \langle m\delta_a\theta, f\rangle. \qquad (**)$$

Indeed, for $\theta = \delta_x$ ($x \in K$), by Lemma 3.6, we have

$$\begin{aligned} \langle \delta_x, {}_a(fm)\rangle &= {}_a(fm)(x) = fm(a*x) \\ &= \langle m, f_{a*x}\rangle = \langle m, (f_x)_a\rangle \\ &= \langle m, \delta_a(f_x)\rangle = \langle m\delta_a, f_x\rangle \\ &= \langle m\delta_a, \delta_x f\rangle = \langle m\delta_a\delta_x, f\rangle. \end{aligned}$$

If $\theta$ is a mean on $C(K)$, then there is $\theta_\beta = \Sigma_{i=1}^{n_\beta} \lambda_i \delta_{x_i}$, a convex combinations of point evaluations, such that $\theta_\beta \to \theta$ in the $w^*$-topology of

$C(K)^*$. Hence

$$\langle\theta, {}_a(fm)\rangle = \lim_\beta \langle\theta_\beta, {}_a(fm)\rangle$$
$$= \lim_\beta \langle m\delta_a\theta_\beta, f\rangle = \langle m\delta_a\theta, f\rangle.$$

by $w^*$-$w^*$ continuity of $L_m$ on norm bounded subsets of $UC_l(K)^*$. Consequently (**) holds.

Now to see that $fm \in UC_l(K)$, by [20, Theorem 4.2.2, p. 88], it is enough to show that the map $x \to_x (fm)$ from $K$ to $C(K)$ is weakly continuous. Let $\{x_\alpha\}$ be a net in $K$ converging to $x$ and $\theta \in C(K)^*$, then by (**),

$$\lim_\alpha \langle\theta, {}_{x_\alpha}(fm)\rangle = \lim_\alpha \langle m_{x_\alpha}\theta, f\rangle$$
$$= \langle m\delta_x\theta, f\rangle = \langle\theta, {}_x(fm)\rangle,$$

by $w^*$-$w^*$ continuity of $L_m$ on norm bounded subsets of $UC_l(K)^*$. Hence, $fm \in UC_l(K)$.

If $n$ is a mean on $UC_l(K)$, there exists a net $n_\alpha = \Sigma_{i=1}^{l_\alpha}\lambda_i\delta_{x_i}$ in $Z(UC_l(K)^*)$ (see Remark 3.3) where $\lambda_i > 0$ and $\Sigma_{i=1}^{l_\alpha}\lambda_i = 1$ such that $n_\alpha \to n$ in the $w^*$-topology of $UC_l(K)^*$. Hence for each $f \in UC_l(K)$, considering Remark 3.3, we have

$$\langle m\circ n, f\rangle = \langle n, fm\rangle$$
$$= \lim_\alpha \langle n_\alpha, fm\rangle = \lim_\alpha \langle m\circ n_\alpha, f\rangle$$
$$= \lim_\alpha \langle mn_\alpha, f\rangle = \langle mn, f\rangle$$

by the continuity of $L_m$. Now by linearity, we have $m\circ n = mn$ for all $n \in UC_l(K)^*$, *i.e.* $m \in Z(UC_l(K)^*)$. □

**Remark 3.8.** For $\phi \in L^1(K)$ and $m \in UC_l(K)^*$, the product $\phi m$ makes sense both as an element of $UC_l(K)^*$ and as an element of $L^\infty(K)^*$ (see [14, §3, p. 13]).

**Lemma 3.9.** *Let* $\pi : L^\infty(K)^* \to UC_l(K)^*$ *be the adjoint of the inclusion map of* $UC_l(K)$ *into* $L^\infty(K)$. *Then* $\pi$ *is* $w^*$-$w^*$ *continuous and* $mn = m\pi(n)$ *for each* $m, n \in L^\infty(K)^*$.

**Proof.** It is easy to check that $\pi$ is $w^*$-$w^*$ continuous. For the second part, we first define a continuous map $f \mapsto Ff$ of $L^\infty(K)$ into itself for each $F \in UC_l(K)^*$. Note that for $f \in L^\infty(K), \phi \in L^1(K)$, we know that $f\phi \in UC_l(K)$ (Lemma 2.4(ii)), so $\phi \mapsto \langle F, f\phi\rangle$ is a continuous linear functional on $L^1(K)$ and therefore corresponds to an element $Ff$ of $L^\infty(K)$. The adjoint of $\phi \mapsto fF$ is a continuous and $w^*$-continuous map $m \mapsto mF$ of $L^\infty(K)^*$ into itself. Thus for $\phi \in L^1(K)$, $f \in L^\infty(K)$, $F \in UC_l(K)^*$, and $m \in L^\infty(K)^*$,

$$\langle Ff, \phi\rangle = \langle F, f\phi\rangle \quad , \quad \langle mF, f\rangle = \langle m, Ff\rangle \quad (*).$$

Let $\{\phi_i\} \subseteq L^1(K)$ be a net converging to $m$ in the $w^*$-topology of $L^\infty(K)^*$ then for each $f \in L^\infty(K)$, by (*),

$$\langle mn, f\rangle = \lim_i \langle\phi_i n, f\rangle = \lim_i \langle\phi_i \circ n, f\rangle$$
$$= \lim_i \langle n, f\phi_i\rangle = \lim_i \langle\pi(n), f\phi_i\rangle$$
$$= \lim_i \langle\pi(n)f, \phi_i\rangle = \lim_i \langle\phi_i, \pi(n)f\rangle$$
$$= \langle m, \pi(n)f\rangle = \langle m\pi(n), f\rangle. \qquad \square$$

**Lemma 3.10.** $Z(UC_l(K)^*) = \{ m \in UC_l(K)^* : \phi m \in Z(L^\infty(K)^*)$ for each $\phi \in L^1(K)\}$.

**Proof.** Let $\phi \in L^1(K)$ and $m \in Z(UC_l(K)^*)$. By Remark 3.8, we may consider $\phi m$ in $L^1(K)^{**}$. To prove that $\phi m \in Z(L^\infty(K)^*)$, by Lemma 2.3, it is enough to show that $n \to \phi mn$ is $w^*$-$w^*$ continuous. If $n_\alpha \to n$ in the $w^*$-topology of $L^\infty(K)^*$, then $\pi(n_\alpha) \mapsto \pi(n)$ (since $\pi$ is $w^*$-$w^*$ continuous) in the $w^*$-topology of $UC_l(K)^*$. Hence, by Lemma 3.7, for any $f \in L^\infty(K)$,

$$\langle\phi mn_\alpha, f\rangle = \langle\phi\circ(mn_\alpha), f\rangle$$
$$= \langle mn_\alpha, f\phi\rangle$$
$$= \langle m\pi(n_\alpha), f\phi\rangle \to \langle m\pi(n), f\phi\rangle$$
$$= \langle\phi\circ(m\pi(n)), f\rangle$$
$$= \langle\phi m\pi(n), f\rangle = \langle\phi mn, f\rangle,$$

so by Lemma 2.3, $\phi m \in Z(L^\infty(K)^*)$.

Conversely, let $m \in UC_l(K)^*$, and $n_\alpha \to n$ in the $w^*$-topology of $UC_l(K)^*$, then for each $f \in UC_l(K)$, there exists $g \in UC_l(K)$ and $\phi \in L^1(K)$ such that $f = g\phi$ ([19, Lemma 2.2] and Lemma 2.4(ii)). Hence

$$\langle mn_\alpha, f\rangle = \langle mn_\alpha, g\phi\rangle = \langle \phi \circ (mn_\alpha), g\rangle$$
$$= \langle \phi mn_\alpha, g\rangle \to \langle \phi mn, g\rangle$$
$$= \langle mn, g\phi\rangle = \langle mn, f\rangle. \qquad \square$$

Now we are ready for the main theorem of this section.

**Theorem 3.11.** $Z(UC_l(K)^*) = M(K)$

**Proof.** By Lemmas 3.5 and 3.7, it is enough to show that $Z(UC_l(K)^*) \subseteq M(K)$. Let $m \in Z(UC_l(K)^*)$, then by Lemma 3.2, $m = \mu + m_1$, for some $\mu \in M(K)$ and $m_1 \in C_0(K)^\perp$. It is enough to show that $m_1 = 0$. Let $\phi \in L^1(K)$. Since $C_0(K)^\perp$ is an ideal in $UC_l(K)^*$ (Lemma 3.2) $\phi m_1 \in C_0(K)^\perp$ and $\phi m_1 \in Z(L^1(K)^{**})$, by Lemma 3.10. Hence $\phi m_1 = 0$ (Lemma 2.7). Let $f \in UC_l(K)$, then $f = g\phi$, for some $g \in UC_l(K)$, and $\phi \in L^1(K)$ ([19, Lemma 2.2] and Lemma 2.4(ii)), and

$$\langle m_1, f\rangle = \langle m_1, g\phi\rangle = \langle \phi \circ m_1, g\rangle = \langle \phi m_1, g\rangle = 0.$$

Hence $m_1 = 0$, as desired. □

**Corollary 3.12.** If $K$ is commutative, then $M(K)$ is the algebraic center of $UC_l(K)^*$.

**Corollary 3.13.** Let $m \in UC_l(K)^*$ be such that $L_m$ is weak$^*$-weak$^*$ continuous on any bounded sphere of $UC_l(K)^*$, then $m \in M(K)$.

**Definition 3.14.** A bounded continuous function $f$ is called weakly almost periodic if $\{{}_xf : x \in K\}$ is relatively weakly compact in the space of all bounded continuous functions on $K$. We denote the Banach space of all weakly almost periodic functions on $K$ by $WAP(K)$.

The following corollary was proved by Skantharajah for hypergroups in [20, Theorem 4.2.7, p. 94], and by Granirer for groups in [7, p. 62-64]. Another version of this Corollary was proved in [15, Theorem 19]. A.T. Lau has also proved it in [12, Corollary 4].

**Corollary 3.15.** Let $K$ be a locally compact hypergroup. Then $K$ is compact if and only if $UC_l(K) = WAP(K)$.

**Proof.** If $K$ is compact, then by [8, 2.2D and 4.2F] we have $UC_l(K) = C(K) = WAP(K)$. For the converse, from $UC_l(K) = WAP(K) = Z(UC_l(K)^*) = M(K)$, it follows that $K$ is compact. □

For the following corollary in the group case, see [12, Corollary 5].

**Corollary 3.16.** Let $K$ be a locally compact hypergroup. Then $K$ is compact if and only if $UC_l(K)$ has a unique left invariant mean.

**Proof.** If $K$ is compact, then the normalized Haar measure is the unique left invariant mean on $UC_l(K) = C(K)$.

Conversely, let $m$ be the unique left invariant mean on $UC_l(K)$, then one can check that $mn$ is also left invariant mean on $UC_l(K)$, for each $n \in UC_l(K)^*$. Hence $mn = \lambda m$, for some complex number $\lambda$. Let $\{n_\alpha\}$ be a net in $UC_l(K)^*$ converging to $n$ in the weak$^*$-topology, and $mn_\alpha = \lambda_\alpha m$, $mn = \lambda m$, then $\lambda_\alpha = mn_\alpha(1) = n_\alpha(1)$ converges to $n(1) = mn(1) = \lambda$. Hence $L_m$ is weak$^*$-weak$^*$ continuous, and by Theorem 3.11 and Proposition 3.7, $m \in M(K)$ and by [8, 7.2B], $K$ is compact. □

## Acknowledgement

The author would like to sincerely thank the referees for their valuable comments and useful suggestions.

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