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Lower Bounds for Matrices on Weighted Sequence Spaces

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Abstract

This paper is concerned with the problem of finding a lower bound for certain matrix operators such as Hausdorff and Hilbert matrices on sequence spaces $l_p(w)$ and Lorentz sequence spaces $d(w,p)$, which is recently considered in [7,8], similar to [13] considered by J. Pecaric, I. Peric and R. Roki. Also, this study is an extension of some works which are studied before in [1,2,7,8].

Keywords: Inequality; Lower bound; Hausdorff matrix; Hilbert matrix; Weighted sequence space; Lorentz sequence space

Introduction

We study the lower bounds of certain matrix operators on $l_p(w)$ and Lorentz sequence spaces $d(w,p)$ considered in [1-4] and [12] on l_p spaces and in [7] and [8] on $l_p(w)$ and $d(w,p)$ for certain matrix operators such as Cesaro, Copson and Hilbert operators. The problem of finding an upper bound of such matrices on weighted sequence spaces considered by authors in a companion paper [11].

Let $0 < p < \infty$, l_p be the normed linear space of all sequences with finite norm $\|x\|_p$, where

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}.$$

If $w = (w_n)$ is a decreasing non-negative sequence, we define the weighted sequence space $l_p(w)$ as follows:

$$l_p(w) = \{ \ (x_n) \ : \ \sum_{n=1}^{\infty} w_n |x_n|^p < \infty \ \},$$

with norm $\|x\|_{p,w}$, which is defined as follows:

$$\|x\|_{p,w} = (\sum_{n=1}^{\infty} w_n |x_n|^p)^{1/p}.$$

Also, if $w = (w_n)$ is a decreasing non-negative sequence such that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$, then the Lorentz sequence space $d(w,p)$ is defined as follows:

$$d(w,p) = \{ \ (x_n) \ : \ \sum_{n=1}^{\infty} w_n (x^*_n)^p < \infty \ \},$$

where (x^*_n) is the decreasing rearrangement of $(|x_n|)$. In fact $d(w,p)$ is the space of null sequences x for which x^* is in $l_p(w)$, with norm $\|x\|_{d(w,p)} = \|x^*\|_{p,w}$.

Let B be a matrix with non-negative entries. We consider lower bounds of the form

$$\|Bx\|_{p,w} \geq L\,\|x\|_{p,v} \ (\|Bx\|_{d(w,p)} \geq L\,\|x\|_{d(v,p)}),$$

valid for every non-negative sequence x, where L is a

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constant which does not depend on x. We seek the largest possible value of L, and denote the best lower bound by $L_{p,v,w}$ for matrix operator from $l_p(v)$ into $l_p(w)$. Also it is denoted by $L_{p,w}(B)$ and $L_{d(w,p)}(B)$ on $l_p(w)$ and $d(w,p)$, respectively. We shall use all above notations when $p<1$.

In Section 2, we generalize two techniques obtained by Bennett in section 7 of [1] and deduce a lower bound for Hausdorff matrix. In section 3, we generalize Theorem 1 of [7] for matrix operator from $l_p(v)$ into $l_p(w)$ and deduce a lower bound for the Hilbert and Copson matrices.

Throughout this paper, we denote the transpose matrix of B by B^t, and we denote by p^* the conjugate exponent of p, so that $p^* = p/(p-1)$.

In a similar way, the first author considered the norm of some operators on weighted sequence spaces in [9] and [10].

Hausdorff Matrix

In this section we consider the Hausdorff matrix operator $H(\mu)=(h_{j,k})$, with entries of the form:

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1}\Delta^{j-k} a_k & if \quad 1 \le k \le j \\ 0 & if \quad k > j, \end{cases}$$

where Δ is the difference operator; that is

$$\Delta a_k = a_k - a_{k+1}$$

and (a_k) is a sequence of real numbers, normalized so that $a_1 = 1$.

If

$$a_k = \int_0^1 \theta^{k-1} d\mu(\theta) \qquad (k=1,2,\cdots),$$

where μ is a probability measure on $[0,1]$, then for all $j,k=1,2,\cdots$,

$$h_{j,k} = \begin{cases} \binom{j-1}{k-1}\int_0^1 \theta^{k-1}(1-\theta)^{j-k}\, d\mu(\theta) & if \quad 1 \le k \le j \\ 0 & if \quad k > j. \end{cases}$$

The Hausdorff matrix contains some famous classes of matrices. These classes are as follows:

i) Choice $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ gives Cesaro matrix of order α;

ii) Choice $d\mu(\theta) = po\text{int}$ *evaluation at* $\theta=\alpha$ point evaluation at gives Euler matrix of order α;

iii) Choice $d\mu(\theta) = \frac{|\log(\theta)|^{\alpha-1}}{\Gamma(\alpha)}d\theta$ gives Holder matrix of order α;

iv) Choice $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ gives Gamma matrix of order α.

The Cesaro, Holder and Gamma matrices have non-negative entries whenever $\alpha > 0$ and also the Euler matrix, when $0 \le \alpha \le 1$.

The following lemma is the key to the rest of this paper.

Lemma 2.1. Let $p \ge 0$ and $B=(b_{i,j})$ be a matrix with non-negative entries. The following condition is equivalent to the statement that Bx is decreasing for every decreasing non-negative sequence x in $d(w,p)$:

(1) $r_{i,n} = \sum_{j=1}^{n} b_{i,j}$ decreases with i for each n, and $(r_{i,n})_{n=1}^{\infty}$ is bounded for each i.

Proof. Let $x \in d(w,p)$ be a decreasing non-negative sequence and $y = Bx$. If (1) holds, by Abel summation, we have

$$y_i = \sum_{j=1}^{\infty} b_{i,j}x_j = \sum_{j=1}^{\infty} r_{i,j}(x_j - x_{j+1}).$$

It follows that Bx is decreasing. The converse is deduced from the fact that $y_i = r_{i,n}$ when $x = e_1 + \cdots + e_n$. □

The above lemma shows that for a matrix B with condition (1), we have

$$L_{p,w}(B) = L_{d(w,p)}(B).$$

In this section, we are seeking a lower bound for the Hausdorff matrix(general form) and also for the Cesaro, Holder and Gamma matrices.

Proposition 2.2. Let $B=(b_{n,k})$ be an upper-triangle matrix with non-negative entries and $0 < p \le 1$. If

$$\sup_n \sum_{k=n}^{\infty} b_{n,k} = R > 0,$$

$$\inf_k \sum_{n=1}^{k} b_{n,k} = C,$$

then $L_{p,w}(B) \geq R^{\frac{p-1}{p}} C^{\frac{1}{p}}$.

Proof. Suppose x is a non-negative sequence. Applying Holder's inequality, we have

$$\begin{aligned}\sum_{k=n}^{\infty} b_{n,k} w_k x_k^p &= \sum_{k=n}^{\infty} b_{n,k}^{1-p} (b_{n,k} w_k^{1/p} x_k)^p \\ &\leq (\sum_{k=n}^{\infty} b_{n,k})^{1-p} (\sum_{k=n}^{\infty} b_{n,k} w_k^{1/p} x_k)^p \\ &\leq R^{1-p} (\sum_{k=n}^{\infty} b_{n,k} w_k^{1/p} x_k)^p.\end{aligned}$$

Since B is an upper-triangle matrix with non-negative entries and w is decreasing, then we have

$$\begin{aligned}R^{1-p}\sum_{n=1}^{\infty} w_n (\sum_{k=1}^{\infty} b_{n,k} x_k)^p &= R^{1-p}\sum_{n=1}^{\infty} w_n (\sum_{k=n}^{\infty} b_{n,k} x_k)^p \\ &\geq R^{1-p}\sum_{n=1}^{\infty} (\sum_{k=n}^{\infty} b_{n,k} w_k^{1/p} x_k)^p \\ &\geq \sum_{n=1}^{\infty} (\sum_{k=n}^{\infty} b_{n,k} w_k x_k^p) \\ &= \sum_{k=1}^{\infty} w_k x_k^p (\sum_{n=1}^{k} b_{n,k}) \\ &\geq C \sum_{k=1}^{\infty} w_k x_k^p.\end{aligned}$$

Hence $\|Bx\|_{p,w}^p \geq R^{p-1} C \|x\|_{p,w}^p$ and so we have the desired conclusion.□

In the following statement, we seek lower bound for the quasi-Hausdorff matrix when sequences are non-negative. Recall that transpose of a Hausdorff matrix which is called a quasi-Hausdorff matrix.

Theorem 2.3. Let $H(\mu)$ be the Hausdorff matrix and $0 < p \leq 1$. Then

$$\|H^t x\|_{p,w} \geq (\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)) \|x\|_{p,w},$$

for every non-negative sequence x. This constant is the best possible choice.

Proof. Let $E(\alpha)$ be the Euler matrix of order α. Since the row sums of $E^t(\alpha)$ are all $1/\alpha$ and column sums are all 1, applying Proposition 2.2, we have

$$L_{p,w}(E^t(\alpha)) \geq \alpha^{\frac{1-p}{p}}.$$

We now apply the Minkowski's inequality to get:

$$\begin{aligned}\|H^t x\|_{p,w} &= (\sum_{n=1}^{\infty} w_n (\sum_{k=1}^{\infty} H_{n,k}^t x_k)^p)^{1/p} \\ &= (\sum_{n=1}^{\infty} w_n (\int_0^1 \sum_{k=1}^{\infty} E_{n,k}^t(\alpha) x_k \, d\mu(\alpha))^p)^{1/p} \\ &\geq \int_0^1 (\sum_{n=1}^{\infty} w_n (\sum_{k=1}^{\infty} E_{n,k}^t(\alpha) x_k)^p)^{1/p} d\mu(\alpha) \\ &= \int_0^1 \|E^t(\alpha) x\|_{p,w} \, d\mu(\alpha) \\ &\geq (\int_0^1 \alpha^{\frac{1-p}{p}} d\mu(\alpha)) \|x\|_{p,w}.\end{aligned}$$

This completes the proof of the above inequality. Therefore for any real number $\alpha > 0$, we have

$$\|H^t x\|_{p,w+\alpha} \geq (\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)) \|x\|_{p,w+\alpha}, \qquad (I)$$

for all non-negative sequence x in $l_p(w)$. We show that the above constant is the best possible. Let $\rho > \frac{1}{p}$ and n be a fixed integer such that $n \geq \rho$. We define x by

$$x_k = \begin{cases} 0 & if \quad k < n \\ \dfrac{\binom{k-\rho}{k-n}}{\binom{k}{n}} & if \quad k \geq n. \end{cases}$$

Since

$$x_k = \frac{(k-\rho)\cdots(n+1-\rho)}{k\cdots(n+1)} \approx k^{-p},$$

when $k \to \infty$, it follows that $\|x\|_p < \infty$ and $\|x\|_p \to \infty$ when $\rho \to \frac{1}{p}$. Since w is decreasing and also for all k, $w_k + \alpha \geq \alpha$, then we have

$$\alpha^{1/p} \|x\|_p \leq \|x\|_{p,w+\alpha} \leq (w_1 + \alpha)^{1/p} \|x\|_p.$$

So $\|x\|_{p,w+\alpha} < \infty$ and $\|x\|_{p,w+\alpha} \to \infty$ when $\rho \to \frac{1}{p}$.
Moreover, for all $m > n$ we have

$$(H^t x)_m = x_m \int_0^1 \theta^{\rho-1} d\mu(\theta).$$

Hence

$$\left\|H^t x\right\|_{p,w+\alpha}^p = \sum_{m=1}^{n} (w_m+\alpha)\left(\sum_{k=m}^{\infty} h_{k,m} x_k\right)^p + \sum_{m=n+1}^{\infty} (w_m+\alpha)(H^t x)_m^p$$
$$\le n(w_1+\alpha) \sup_{k,m} \left|h_{k,m}\right|^p \|x\|_1^p + \left(\int_0^1 \theta^{\rho-1} d\mu(\theta)\right)^p \|x\|_{p,w+\alpha}^p$$

and also

$$L_{p,w+\alpha}(H^t) \le \frac{n(w_1+\alpha)\sup_{k,m}\left|h_{k,m}\right|^p \|x\|_1^p}{\|x\|_{p,w+\alpha}^p} + \left(\int_0^1 \theta^{\rho-1} d\mu(\theta)\right)^p.$$

If $\rho \to \frac{1}{p}$, then

$$L_{p,w+\alpha}(H^t) \le \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta).$$

Therefore

$$L_{p,w+\alpha}(H^t) = \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)$$

and the constant in (I) is best possible. Hence for all m there is a non-negative sequence $y_m \in l_p(w)$, such that

$$\frac{\left\|H^t y_m\right\|_{p,w+\alpha}}{\|y_m\|_{p,w+\alpha}} < \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) + \frac{1}{m}.$$

Since $\left\|H^t y_m\right\|_{p,w} \le \left\|H^t y_m\right\|_{p,w+\alpha}$, we have

$$\frac{\left\|H^t y_m\right\|_{p,w+\alpha}}{\|y_m\|_{p,w+\alpha}} \ge \frac{\left\|H^t y_m\right\|_{p,w}}{\|y_m\|_{p,w+\alpha}}$$
$$= \frac{\|y_m\|_{p,w}}{\|y_m\|_{p,w+\alpha}} \cdot \frac{\left\|H^t y_m\right\|_{p,w}}{\|y_m\|_{p,w}}$$
$$\ge \frac{\|y_m\|_{p,w}}{\|y_m\|_{p,w+\alpha}} L_{p,w}(H^t)$$

and

$$\frac{\|y_m\|_{p,w}}{\|y_m\|_{p,w+\alpha}} L_{p,w}(H^t) \le \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) + \frac{1}{m}.$$

If $\alpha \to 0$, since $\|y\|_{p,w+\alpha} < \infty$, we have $\|y\|_{p,w+\alpha} \to \|y\|_{p,w}$ and so

$$L_{p,w}(H^t) \le \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta) + \frac{1}{m}.$$

Now, if $m \to \infty$, we have

$$L_{p,w}(H^t) \le \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta).$$

Therefore

$$L_{p,w}(H^t) = \int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta).$$

This establishes the proof of the theorem.□

In the following corollary we state one result of Theorem 2.3 on $d(w,p)$.

Corollary 2.4. Let $H(\mu)$ be the Hausdorff matrix satisfying condition (1) of Lemma 2.1. If $0 < p \le 1$, then

$$\left\|H^t x\right\|_{d(w,p)} \ge \left(\int_0^1 \theta^{\frac{1-p}{p}} d\mu(\theta)\right) \|x\|_{d(w,p)}$$

for all decreasing non-negative sequence x.

Proof. Applying Lemma 2.1 and Theorem 2.3, we deduce the statement.□

Example. We denote Gamma matrix of order 2 by $\Gamma(2)$. If $\Gamma^t(2) = \left(b_{i,j}\right)$ is the transpose of the Gamma matrix, then we have

$$b_{i,j} = \begin{cases} \dfrac{i}{\frac{1}{2} j(j+1)} & if \quad j \ge i \\ 0 & if \quad j < i. \end{cases}$$

Since $r_{i,n} = 2 - \frac{2i}{n+1}$, it is clear that

$r_{i+1,n} \le r_{i,n} \le 2.$

Hence $r_{i,n}$ decreases with i for each n and $(r_{i,n})_{n=1}^{\infty}$ is bounded for each i. Therefore $\Gamma^t(2)$ satisfies condition (1) of Lemma 2.1. Applying Corollary 2.4, we deduce that

$$L_{d(w,p)}(\Gamma^t(2)) \ge \frac{2p}{p+1}.$$

In the following statement, we find a lower bound for a quasi-Hausdorff matrix when sequences are non-negative.

Proposition 2.5. Let $0 < p,q < 1$ and B be a matrix with non-negative entries. Then

$$\|Bx\|_{q,w} \ge L\|x\|_{p,w}$$

for all non-negative x, if and only if

$$\|B^t y\|_{p^*,w} \ge L\|y\|_{q^*,w}$$

for all non-negative y, where p^*, q^* are the conjugate exponents of p and q, respectively.

Proof. Suppose u is a sequence with non-negative entries. First we show that

$$\|u\|_{t,w} = \inf\{<u,v>: \text{v is a non-negative sequence and } \|\text{v}\|_{t^*,w} \ge 1\} \qquad (I)$$

for $0 < t < 1$ or $t < 0$, where $<u,v> = \sum_{k=1}^{\infty} w_k u_k v_k$.

Let v be a non-negative sequence such that $\|\text{v}\|_{t^*,w} \ge 1$. Then applying Holder's inequality, we deduce that:

$$\begin{aligned} <u,v> &= \sum_{k=1}^{\infty} w_k u_k v_k \\ &= \sum_{k=1}^{\infty} w_k^{\frac{1}{t}+\frac{1}{t^*}} u_k v_k \\ &\ge (\sum_{k=1}^{\infty} w_k u_k^t)^{1/t} (\sum_{k=1}^{\infty} w_k v_k^{t^*})^{1/t^*} \\ &= \|u\|_{t,w}\|v\|_{t^*,w} \\ &\ge \|u\|_{t,w}. \end{aligned}$$

Hence $\inf <u,v> \ge \|u\|_{t,w}$.

We divide the proof of the converse inequality into two cases as follows:

Case 1. If $u > 0$, we take

$$\tilde{v}_k = u_k^{t-1} \quad , \quad v_k = \frac{\tilde{v}_k}{\|\tilde{v}\|_{t^*,w}}.$$

Hence $\|\tilde{v}\|_{t^*,w} = \|u\|_{t,w}^{t-1}$ and $<u,v> = \|u\|_{t,w}$ and so that

$$\inf <u,v> \le \|u\|_{t,w}.$$

Case 2. If some $u_k = 0$, we consider (i), (ii).

(i) For $t < 0$, $\|u\|_{t,w} = 0$ and set

$$v_n = \begin{cases} 0 & for \quad n \ne k \\ \frac{1}{w_k^{1/t^*}} & for \quad n = k. \end{cases}$$

(ii) For $0 < t < 1$, we set

$$\tilde{v}_k = \begin{cases} u_k^{t-1} & for \quad u_k > 0 \\ (\frac{\xi}{w_k 2^k})^{1/t^*} & for \quad u_k = 0 \end{cases}$$

and $v_k = \frac{\tilde{v}_k}{\|\tilde{v}\|_{t^*,w}}$, where ε is positive.

Hence $\|v\|_{t^*,w} = 1$, $\|\tilde{v}\|_{t^*,w} \ge \frac{1}{(\varepsilon + \|u\|_{t,w}^t)^{-1/t^*}}$ and also

$$<u,v> \le \|u\|_{t,w}^t (\varepsilon + \|u\|_{t,w}^t)^{-1/t^*}.$$

So that

$$\inf <u,v> \le \|u\|_{t,w}^t (\varepsilon + \|u\|_{t,w}^t)^{-1/t^*}.$$

In which if ε tends to zero, we have

$$\inf <u,v> \le \|u\|_{t,w}.$$

This completes the proof of (I).

Applying (I) twice, we deduce that:

$$\begin{aligned} \inf_{\|x\|_{p,w} \ge 1} \|Bx\|_{q,w} &= \inf_{\|x\|_{p,w} \ge 1} \inf_{\|y\|_{q^*,w} \ge 1} <Bx, y> \\ &= \inf_{\|x\|_{p,w} \ge 1} \inf_{\|y\|_{q^*,w} \ge 1} <x, B^t y> \\ &= \inf_{\|y\|_{q^*,w} \ge 1} \inf_{\|x\|_{p,w} \ge 1} <x, B^t y> \end{aligned}$$

$$= \inf_{\|y\|_{q^*,w} \geq 1} \|B^t y\|_{p^*,w}$$

and so we have the statement.□

In the following statement, we are seeking a lower bound of the Hausdorff matrix when sequences are non-negative.

Corollary 2.6. Let $p < 0$ and $H(\mu)$ be the Hausdorff matrix. Then

$$\|H^t x\|_{p,w} \geq (\int_0^1 \theta^{\frac{-1}{p}} d\,\mu(\theta)) \|x\|_{p,w}$$

for every non-negative sequence x. This constant is the best possible choice.

Proof. Since $0 < p^* < 1$, applying Theorem 2.3 and Proposition 2.5, we establish the statement.□

Corollary 2.7. Suppose $0 < p \leq 1$ and $H(\mu)$ is the Hausdorff matrix. Then

$$\|H^t x\|_p \geq (\int_0^1 \theta^{\frac{1-p}{p}} d\,\mu(\theta)) \|x\|_p$$

for every non-negative sequence x. This constant is the best possible choice.

Proof. By taking $w_n = 1$ for all n in the Theorem 2.3, we have the above inequality.□

Corollary 2.8. If $p > 0$ and $H(\mu)$ is the Hausdorff matrix, then

$$\sum_{n=1}^{\infty} w_n (\sum_{k=1}^{n} \frac{h_{n,k}}{|x_k|})^{-p} \leq (\int_0^1 \theta^{\frac{1}{p}} d\,\mu(\theta))^{-p} \sum_{k=1}^{\infty} w_k |x_k|^p$$

for every non-negative sequence, and this constant is best possible.

Proof. Let y be a sequence with non-negative entries. Since $-p < 0$, applying Corollary 2.6, we have

$$\|H^t y\|_{-p,w} \geq (\int_0^1 \theta^{\frac{1}{p}} d\,\mu(\theta)) \|y\|_{-p,w} .$$

Hence

$$\sum_{n=1}^{\infty} w_n (\sum_{k=1}^{n} h_{n,k} y_k)^{-p} \leq (\int_0^1 \theta^{\frac{1}{p}} d\,\mu(\theta))^{-p} \sum_{k=1}^{\infty} w_k |y_k|^{-p}.$$

By replacing y_k by $\frac{1}{|x_k|}$ for $k = 1, 2, \cdots,$ we get the required result.□

Lower Bound for Matrix Operators on $d(w,p)$ and $l_p(w)$

In this part of the study, we generalize Theorem 1 of [7] for matrix operators from $l_p(v)$ into $l_p(w)$ and deduce a lower bound for the Hilbert, Copson and Gamma matrices.

Lemma 3.1. [7, Lemma 2]. Let $p \geq 1$. Suppose that (a_j), (x_j) are non-negative sequences and that (x_j) is decreasing which tends to 0. Let $A_n = \sum_{j=1}^{n} a_j$ (with $A_0 = 0$) and $B_n = \sum_{j=1}^{n} a_j x_j$. Then

(i) $B_n^p - B_{n-1}^p \geq (A_n^p - A_{n-1}^p) x_n^p$ for all n.

(ii) If $\sum_{j=1}^{\infty} a_j x_j$ is convergent, then

$$(\sum_{j=1}^{n} a_j x_j)^p \geq \sum_{n=1}^{\infty} A_n^p (x_n^p - x_{n+1}^p).\ \square$$

Theorem 3.2. Suppose $A = (a_{i,j})$ is a matrix operator from $l_p(v)$ into $l_p(w)$ with non-negative entries. Let $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$, $S_n = \sum_{i=1}^{n} w_i r_{i,n}^p$ and $V_n = v_1 + \cdots + v_n$. Then

$$L_{p,v,w}(A)^p = \inf_n \frac{S_n}{V_n}.$$

Proof. Denote the stated infimum by C. Let x be in $l_p(v)$ such that $x_1 \geq x_2 \geq \cdots \geq 0$ and $y = A(x)$. By Lemma 3.1, we have

$$y_i^p \geq \sum_{n=1}^{\infty} r_{i,n}^p (x_n^p - x_{n+1}^p).$$

Hence

$$\begin{aligned} \sum_{i=1}^{\infty} w_i y_i^p &= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} r_{i,n}^p (x_n^p - x_{n+1}^p) \\ &= \sum_{n=1}^{\infty} (x_n^p - x_{n+1}^p) \sum_{i=1}^{\infty} w_i r_{i,n}^p \\ &= \sum_{n=1}^{\infty} S_n (x_n^p - x_{n+1}^p) \\ &\geq C \sum_{n=1}^{\infty} V_n (x_n^p - x_{n+1}^p) \\ &= C \sum_{n=1}^{\infty} v_n x_n^p . \end{aligned}$$

Therefore

$$\|Ax\|_{p,w}^{p} \geq C\,\|x\|_{p,v}^{p},$$

hence

$$L_{p,v,w}(A)^p \geq C.$$

To show that the constant C is the best possible, we take $x_1 = x_2 = \cdots = x_n = 1$ and $x_k = 0$ for all $k \geq n+1$. Then

$$\|x\|_{p,v}^{p} = V_n \quad , \quad \|Ax\|_{p,w}^{p} = S_n.$$

Therefore

$$L_{p,v,w}(A)^p = C. \square$$

Note 1. In the same way, one shows that if A is regarded as an operator from $l_p(v)$ into $l_p(w)$, where $p \geq q \geq 1$, then its lower bound is $\inf_n ({S_n^{1/q}}/{V_n^{1/p}})$.

Note 2. In the case $p = 1$, the sequence $({S_n}/{V_n})$ also determines the norm; in fact, $\|A\|_{1,v,w} = \sup_n ({S_n}/{V_n})$, see [11].

Write $u_n = \sum_{i=1}^{\infty} w_i a_{i,n}^p$. Since $v_n = V_n - V_{n-1}$, we have the following statement.

Proposition 3.3. If A satisfies the conditions of Theorem 3.2 and $(a_{i,j})$ decreases with j for each i, then

$$L_{p,v,w}(A)^p \geq \inf_n [n^p - (n-1)^p]\frac{u_n}{v_n}.$$

Proof. See Proposition 1 of [7].□

We recall that the Hilbert operator H is defined by the matrix

$$a_{i,j} = \frac{1}{i+j}.$$

In the following statement, we consider the lower bound of H.

Theorem 3.4. Suppose that $w_n = \frac{1}{n^{\alpha}}$ and $V_n = n^{1-\alpha}$ with $0 \leq \alpha \leq 1$ and let $p \geq 1$. Then

$$L_{p,v,w}(H)^p = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}(i+1)^p}.$$

Proof. We have $v_n = n^{1-\alpha} - (n-1)^{1-\alpha}$. Since $n^{1-\alpha} - n^{-\alpha} = n^{-\alpha}(n-1) \leq (n-1)^{1-\alpha}$, hence $v_n \leq n^{-\alpha}$. Also $n^p - n^{p-1} = n^{p-1}(n-1) \geq (n-1)^p$ and $n^p - (n-1)^p \geq n^{p-1}$. Therefore $\frac{n^p - (n-1)^p}{v_n} \geq n^{p+\alpha-1}$ and so

$$\inf_n \frac{n^p - (n-1)^p}{v_n} u_n \geq \inf_n n^{p+\alpha-1} u_n.$$

If $C_n = n^{p+\alpha-1} u_n$, a small change in the proof of ([6], Theorem 13) shows that $C_n \geq C_1$ for all n; hence $\inf_n C_n = C_1 = u_1$. Thus $L_{p,v,w}(H)^p \geq u_1$. Since $\|e_1\|_{p,v} = 1$ and $\|He_1\|_{p,w} = u_1$, we have $L_{p,v,w}(H)^p \leq u_1$. Therefore

$$L_{p,v,w}(H)^p = u_1 = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}(i+1)^p}. \square$$

Corollary 3.5. We have $L_p(H)^p = \xi(p-1)$.

Proof. If $\alpha = 0$, then $w_n = v_n = 1$ and applying the pervious theorem, we have the statement.□

If $w_n = v_n$, we obtain a lower bound for matrix operator on $d(w,p)$ and $l_p(w)$ which is considered in [7].

Corollary 3.6. Suppose $A = (a_{i,j})$ is a matrix operator from $l_p(w)$ into itself with non-negative entries. We write $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$, $S_n = \sum_{i=1}^{n} w_i r_{i,n}^p$ and $W_n = w_1 + \cdots + w_n$. Then

$$L_{p,w}(A)^p = \inf_n \frac{S_n}{W_n}. \square$$

As we stated in section two the Hausdorff matrix is contained the famous Cesaro and Gamma matrices. We denote the Cesaro matrix of order α by $C(\alpha)$ and the Gamma matrix of order α by $\Gamma(\alpha)$. If $\alpha = 2$, choice $d\mu(\theta) = 2(1-\theta)d\theta$ gives $C(2)$ with entries:

$$a_{n,k} = \begin{cases} \dfrac{n-k+1}{\frac{1}{2}n(n+1)} & if \quad k \leq n \\ 0 & if \quad k > n \end{cases}$$

and $d\,\mu(\theta) = 2\theta d\,\theta$ choice gives $\Gamma(2)$ with entries:

$$a_{n,k} = \begin{cases} \dfrac{k}{\frac{1}{2}n(n+1)} & if \quad k \le n \\ 0 & if \quad k > n. \end{cases}$$

If A^t is the transpose matrix of A, $C^t(\alpha)$ is called the Copson matrix of order α. For $\alpha = 1$, $\Gamma(1) = C(1)$. Hence for $w_n = \frac{1}{n^\alpha}$ where $0 < \alpha \le 1$, applying [8] we have

$$L_{1,w}(\Gamma^t(1)) = L_{1,w}(C^t(1)) = \frac{1}{\alpha}.$$

In the following statement, we find lower bound of $C^t(2)$ and $\Gamma^t(2)$ on $l_1(w)$. It is enough to consider the sequence $(\frac{s_n}{w_n})$ instead of $(\frac{S_n}{W_n})$, because of the well-known fact listed in the following lemma.

Lemma 3.7. If $m \le \frac{s_n}{w_n} \le M$ for all n, then $m \le \frac{S_n}{W_n} \le M$ for all n.

Proof. Elementary.□

Proposition 3.8. Let $0 < \alpha \le 1$. If $w_n = \frac{1}{n^\alpha}$, then

$$L_{1,w}(C^t(2)) = 1.$$

Proof. We show that $\frac{s_n}{w_n} \ge \frac{s_1}{w_1}$ for all n. Therefore applying Lemma 3.7, we have $\frac{S_n}{W_n} \ge \frac{S_1}{W_1} = s_1$. If we apply Corollary 3.6, then

$$L_{1,w}(C^t(2)) = 1.$$

We now show the first inequality. For all n, we have

$$\begin{aligned} \frac{s_n}{w_n} &= n^p \sum_{k=1}^{n} \frac{1}{k^p} \frac{n-k+1}{\frac{1}{2}n(n+1)} \\ &= \frac{2}{n(n+1)}(n^{p+1} + \frac{n^p}{2^p}(n-1) + \frac{n^p}{3^p}(n-2) + \cdots + 1) \\ &\ge \frac{2}{n(n+1)}(n + (n-1) + (n-2) + \cdots + 1) \\ &= 1 = s_1, \end{aligned}$$

the desired inequality.□

Proposition 3.9. Let $w_n = \frac{1}{n}$. Then

$$L_{1,w}(\Gamma^t(2)) = 1.$$

Proof. We show that $\frac{s_n}{w_n} \ge \frac{s_1}{w_1}$ for all n. Therefore applying Lemma 3.7, we have $\frac{S_n}{W_n} \ge \frac{S_1}{W_1} = s_1$. If we apply Corollary 3.6, then

$$L_{1,w}(\Gamma^t(2)) = 1.$$

We now show the first inequality. For all n, we have

$$\begin{aligned} \frac{s_n}{w_n} &= n \sum_{k=1}^{n} \frac{1}{k} \frac{k}{\frac{1}{2}n(n+1)} \\ &= \frac{2n}{n+1} \\ &\ge 1 = s_1, \end{aligned}$$

the required inequality.□

References

1. Bennett G. Factorizing the classical inequalities. *Mem. Amer. Math. Soc.*, 576 (1996).
2. Bennett G. Inequalities complimentary to Hardy. *Quart. J. Math. Oxford(2)*, **49**: 395-432 (1998).
3. Bennett G. Lower bounds for matrices. *Linear Algebra and Appl.*, **82**: 81-98 (1986).
4. Bennett G. Lower bounds for matrices II. *Canadian J. Math.*, **44**: 54-74 (1992).
5. Hardy G.H., Littlewood J.E., and Polya G. *Inequalities*. 2nd Edition, Cambridge University Press, Cambridge (2001).
6. Jameson G.J.O. Norms and lower bounds of operators on the Lorentz sequence spaces. *Illinois J. Math*., **43**: 79-99 (1999).
7. Jameson G.J.O. and Lashkaripour R. Lower bounds of operators on weighted spaces and Lorentz sequence spaces. *Glasgow Math. J.* **42**: 211-223 (2000).
8. Lashkaripour R. Lower bounds and norms of operators on Lorentz sequence spaces. Doctoral Dissertation, Lancaster (1997).
9. Lashkaripour R. Weighted Mean Matrix on Weighted Sequence Spaces. *WSEAS Transaction on Mathematics. Issue 4*, Volum 3, 789-793 (2004).
10. Lashkaripour R. Transpose of the Weighted Mean Operators on Weighted Sequence Spaces. *WSEAS Transaction on Mathematics.* Issue 4, Volum 4, 380-385 (2005).
11. Lashkaripour R. and Foroutannia D. Inequalities involving upper bounds for certain matrix operators. Proceeding of the Indian Academy of Sciences (Math. Sci.), **116**(3): 325-336 (2006).
12. Lyons R. A lower bound for the Cesaro operator. *Proc. Amer. Math. Soc.*, **86**: 694 (1982).
13. Pecaric J., Peric I., and Roki R. On bounds for weighted norms for matrices and integral operators. *Linear Algebra and Appl.*, **326**: 121-135 (2001).