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ON THE GENERALIZATION OF N-PLE MARKOV PROCESSES

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Abstract

The notion of N-ple Markov process is defined in a quite general framework and it is shown that N-ple Markov processes are linear combinations of some martingales.

Introduction

Let $\{X_t, t\geq 0\}$ be a real valued Gaussian process on some probability space (Ω, F, P) with zero mean and continuous in quadratic mean. Let

$$\sum_t^- = \bigcap_n \sigma\left\{X_s : s < t + \frac{1}{n}\right\},$$

$$\sum_t^+ = \bigcap_n \sigma\left\{X_s : s > t - \frac{1}{n}\right\},$$

$$\Gamma_t = \bigcap_n \sigma\left\{X_s : |t-s| < \frac{1}{n}\right\}$$

where $\sigma\{\ldots\}$ is the smallest σ-field with respect to which the elements of $\{\ldots\}$ are measurable. We say the process has Germ field Markov property if given Γ_t, the two σ-fields $\sum_t^-$ and $\sum_t^+$ are conditionally independent. If the process is $(N-1)$ times differentiable and given $G(t) = \sigma\{X(t), X'(t), \ldots, X^{(N-1)}(t)\}$ the two σ-fields $\sum_t^-$ and $\sum_t^+$ are independent, then one says that the process is an N-ple Markov process in the sense of DOOB. Here it is understood that $X(t), \ldots, X^{(N-1)}(t)$ are linearly independent as elements of $L^2(\Omega, \mathcal{G}_t, P)$, where $\mathcal{G}_t = \bigcap_n \sigma\{X_s : |t-s| < \frac{1}{n}\}$. In this paper, we generalize this notion by generalizing the structure of $G(t)$.

Definitions and Results

Let $X = \{Xt, t\geq 0\}$ be a process defined on some probability space (Ω, F, P).

Definition. The process X is called a generalized N-ple Markov process with respect to the process $\{Y_i(t), t\geq 0\}_{i=1,\ldots,N}$ if:

(i) for each $t\geq 0$, $Y_1(t), \ldots, Y_N(t)$ are linearly independent as elements of $L^2(\Omega, \mathcal{G}t, P)$, where $\mathcal{G}_t = \bigcap_n \sigma\{X_s : |t-s| < \frac{1}{n}\}$. Moreover, we assume that the process $\{Z(t), t\geq 0\}$ is continuous in quadratic mean, where for each t, $Z(t) = (Y_1(t), \ldots, Y_n(t))$,

(ii) $\sum_t^+ \perp \sum_t^- \mid \Gamma_t$ where:

$$\sum_t^+ = \bigcap_{\varepsilon>0} \sigma\{X_u : u > t-\varepsilon\},$$

$$\sum_t^- = \bigcap_{\varepsilon>0} \sigma\{X_u : u < t-\varepsilon\}$$

$$\Gamma_t = \sigma\{Y_1(t), \ldots, Y_N(t)\},$$

We have the following result concerning the process $Z(t) = (Y_1(t), \ldots, Y_N(t))^*$.

Theorem 1. If $\{Xt, t\geq 0\}$ is a Gaussian generalized N-ple Markov process with respect to $\{Y_i(t)\}_{i=1,\ldots,N}$ then the process $Z(t) = (Y_1(t), \ldots, Y_N(t))^*$ is a Markov process.

Proof. By assumption we have

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*means the transosope of matrix.

$$\sigma\{X_u : u \geq s\} \perp \sigma\{X_u : u \leq s\} \mid \sigma(Z(s)),$$

where $A \perp B \mid G$ means that given G, A and B are conditionally independent. For each $\varepsilon > 0$ we have:

$$\sigma\{Z_u : u \geq s+\varepsilon\} \subset \sigma\{X_u : u \geq s\}$$

and

$$\sigma\{Z_u : u \leq s-\varepsilon\} \subset \sigma\{X_u : u \leq s\},$$

therefore

$$\sigma\{Z_u : u \geq s+\varepsilon\} \perp \sigma\{Z_u : u \leq s-\varepsilon\} \mid \sigma\{Z(s)\},$$

Therefore

$$\bigvee_{\varepsilon>0} \sigma\{Z_u : u \geq s+\varepsilon\} \perp \bigvee_{\varepsilon>0} \sigma\{Z_u : u \leq s-\varepsilon\} \mid \sigma\{Z(s)\},$$

thus

$$\sigma\{Z_u : u > s\} \perp \sigma\{Z_u ; u < s\} \mid \sigma(Z(s)).$$

Finally by continuity assumption of $Z(t)$, we get

$$\sigma\{Z_u : u \geq s\} \perp \sigma\{Z_u : u \leq s\} \mid \sigma(Z(s))$$

and this completes the proof.

This simple fact leads us to a Goursat type ([1], p. 74) representation of generalized N-ple Markov processes.

Theorem 2. Let $\{X_t, t \geq 0\}$ be a Gaussian generalized N-ple Markov process with respect to the Gaussian processes $\{Y_i(t), t \geq 0\}_{i=1,\ldots,N}$. If the covariance matrix $\Gamma(t,s) = E(Z(t)Z^*(s))$ of $Z(t) = (Y_1(t),\ldots,Y_N(t))^*$ is nonsingular, then:

$$X_t = \sum_{i=1}^{N} \psi_i(t)\, U_i(t)$$

where $\psi_i(t), i=1,\ldots,N$, are N real functions and $\underline{U}(t) = (U_1(t),\ldots,U_N(t))$ is an N-variate martingale.

Proof. From Theorem 1, $Z(t)$ is an N-variate Gaussian Markov process.

Therefore by (3.1[2]), it has the following representation:

$$Z(t) = \Phi(t)\, \underline{U}(t)$$

where $\Phi(t)$ is an $N \times N$ non-singular matrix and $\underline{U}(t)$ is an N-variate martingale. On the other hand, by the Markov property of $\{X_t\}$ we have:

$$\begin{aligned} X_t &= E(X_t / X_u : u \leq t) \\ &= E(X_t \mid Z(t)) \\ &= A(t)\, Z(t) \end{aligned}$$

where $A(t)$ is a $1 \times N$ matrix, so we have:

$$\begin{aligned} X_t &= A(t)\, \Phi(t)\, \underline{U}(t) \\ &= \psi(t)\, \underline{U}(t) \\ &= \sum_{i=1}^{N} \psi_i(t)\, U_i(t) \end{aligned}$$

where $\psi(t) = (\psi_1(t),\ldots,\psi_N(t)) = A(t)\,\Phi(t)$, and $\underline{U}(t) = (U_1(t),\ldots,U_n(t))^*$.

References

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