ON THE POWER FUNCTION OF THE LRT AGAINST ONE-SIDED AND TWO-SIDED ALTERNATIVES IN BIVARIATE NORMAL DISTRIBUTION

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Abstract

This paper addresses the problem of testing simple hypotheses about the mean of a bivariate normal distribution with identity covariance matrix against restricted alternatives. The LRTs and their power functions for such types of hypotheses are derived. Furthermore, through some elementary calculus, it is shown that the power function of the LRT satisfies certain monotonicity and symmetry properties. We treat two cases, the case of one-sided alternatives restricted to some closed convex cone, and the case of two-sided alternatives restricted to a two-sided cone.

1. Introduction

Consider a bivariate normal vector (X, Y) with mean $\theta=(\theta_1, \theta_2)$ and identity covariance matrix. Let V_1 be a one-sided convex closed cone and V_2 a two-sided closed cone. These cones are subsets of R^2 with vertex (0,0). It is required to test the following hypotheses:

$$P_1 \quad H_0: (\theta_1, \theta_2)=(0,0) \quad \text{versus} \quad H_1: (\theta_1, \theta_2) \in V_1 \setminus \{(0,0)\}. \tag{1}$$

$$P_2 \quad H_0: (\theta_1, \theta_2)=(0,0) \quad \text{versus} \quad H_1: (\theta_1, \theta_2) \in V_2 \setminus \{(0,0)\}. \tag{2}$$

Many authors considered testing problems against restricted alternatives. Bartholomew [3] considered a test for homogeneity of means against ordered alternatives. He devoted his work to deriving the LRT and finding its null distribution. Due to the problem of solving certain recurrence relations, the distribution is not completely determined. After this problem had been solved by Miles [9], Bartholomew [3] extended his previous work and gave some further properties of the null distribution. Kudo [8] and independently Nüesch [11] considered a testing problem for which the alternative space is the non-negative quadrant. They derived the LRT and investigated some of its properties.

Perlman [12] considered the testing problem that Bartholomew [3] discussed, but under the assumption that the covariance matrix is completely unknown. Groeneboom and Truax [6] established a monotonicity property of the power functions of some multivariate tests. They considered a multivariate normal X: (μ, Σ), for which it is assumed that the matrix $\Lambda = \mu\mu'\Sigma^{-1}$ is diagonal of the form λI. Let $L = (L_1, L_2, \ldots, L_p)$ be the characteristic roots of Λ. Define a monotone test to be one that accepts the null hypothesis for small values of a function g (L) which is non-decreasing in each of its arguments. They showed that the power function of a monotone test is non-decreasing in λ.

Das Gupta *et al.* [7] gave two sufficient conditions for the power function of an invariant test of general linear hypothesis to be monotone increasing in each of the non-

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entrality parameters. These conditions are given in terms of convexity and symmetry of certain sections of the acceptance region. Anderson and Gupta [2] extended one of these conditions to invariant tests of the hypothesis of independence between two sets of random variables.

Mudholkar [10] showed that the power functions of a class of invariant tests based on statistics generated by symmetric functions of convex increasing functions of the maximal invariant statistics are monotone increasing in the relevant non-centrality parameters. Srivastava [13] established a monotonicity property of the power functions of the LRTs for testing homogeneity of the variance and for testing sphericity. Carter and Srivastava [5] showed that the modified LRT for testing homogeneity of variances and the sphericity test possess monotone non-decreasing power functions. Srivastava *et al.* [13] considered testing the hypothesis of equality of two covariance matrices Σ_1 and Σ_2 of two multivariate normal populations. They showed that the power function of the modified LRT increases in λ_i, where the λ_i's are the latent roots of $\Sigma_1 \Sigma_2^{-2}$.

In this paper, we show that the power functions of the LRTs for the testing problems P_1 and P_2 have the following monotonicity property: If the alternative (θ_1, θ_2) is represented in polar coordinates as (Δ, γ), where $\Delta = \theta_1^2 + \theta_2^2$, it is shown that for a fixed Δ, the power function is increasing in γ for $\gamma < \gamma_0$ and decreasing for $\gamma > \gamma_0$, where γ_0 is half the angle of the cone. Although Bartholomew [4] motivated this property by some numerical examples for certain cones, the result has not yet been established theoretically. He could only conjecture that this property holds for any convex closed cone. In Section 2, we present the LRTs for problems P_1 and P_2. Section 3 is devoted to proving the monotonicity property for the LRT for problem P_1. Section 4 contains the proof of the monotonicity of the LRT for problem P_2.

2. Derivation of the LRT for Problems P_1 and P_2

Let V_1 and V_2 be the closed convex cones in R^2 with vertex at (0,0) which are given by (representation in polar coordinates)

$$V_1 = \{(r, \beta): r \geq 0, 0 \leq \beta \leq \beta^*\} \quad (3)$$

and

$$V_2 = \{(r, \beta): r \in R, 0 \leq \beta \leq \beta^*\} \quad (4)$$

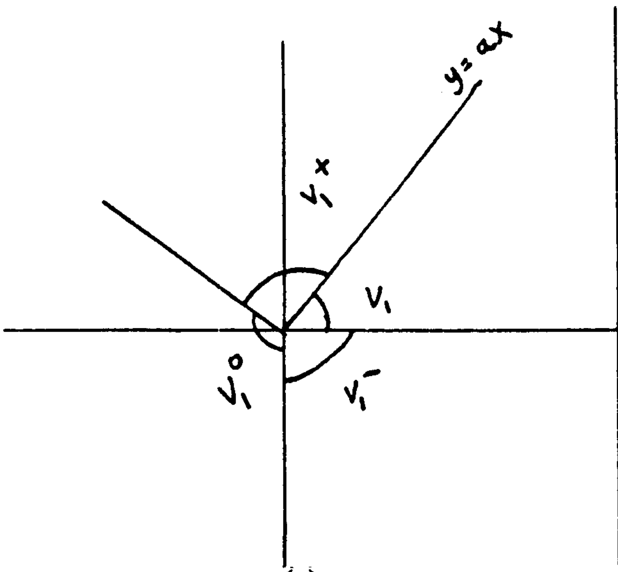
where β^* satisfies $0 \leq \beta^* \leq \pi$.

Bartholomew's approach [3] to deriving the LRT was to partition the space R^2 into four regions V_1, V_1^0, V_1^- and V_1^+ which are illustrated graphically in Figure 2-1-a.

Through some algebra it can be easily shown that the LRT for the problem P_1 is given by:

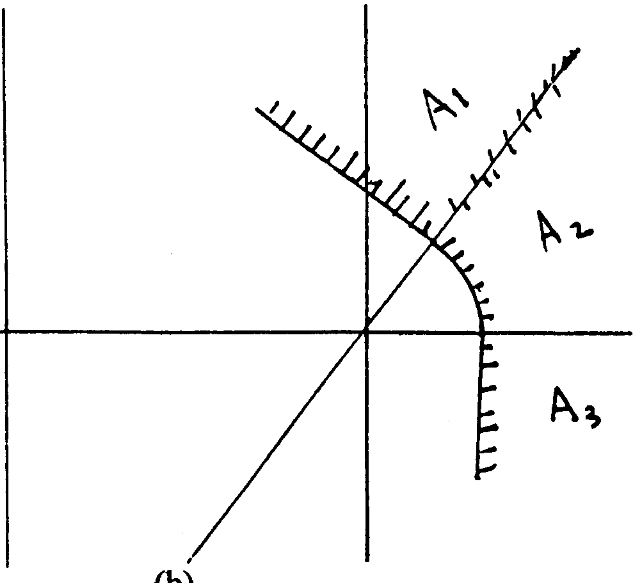
$$\Phi_1 = \begin{cases} 1 & \text{if } \bar{X}^{2} < k \\ 0 & \text{if } \bar{X}^{2} \geq k \end{cases} \quad (5)$$

where



(a)

Figure 2-1-a. The four cones that form a partition of R^2



(b)

Figure 2-1-b. The critical region of the LRT Φ_1

$$\bar{X}^2=\begin{cases} x^2+y^2 & \text{if } (x,y)\in V_1 \\ 0 & \text{if } (x,y)\in V_1^0 \\ x^2 & \text{if } (x,y)\in V_1^- \\ \dfrac{(x+by)^2}{1+b^2} & \text{if } (x,y)\in V_1^+ \end{cases} \qquad (6)$$

and b= tan β^*. The critical region of the LRT Φ_1 is depicted in Figure 2-1-b.

Now we will derive the LRT for the problem P_2. We partition the space R^2 into six regions denoted by $V_2^+, V_2^{++}, V_2^-, V_2^{--}, V_2^{+-}$ and V_2^{-+} which are illustrated in Figure 2-2-a.

The LRT Φ_2 has the form:

$$\Phi_2=\begin{cases} 1 & \text{if } \bar{X}^2<k \\ 0 & \text{if } \bar{X}^2\geq k \end{cases} \qquad (7)$$

where

$$\bar{X}^2=\begin{cases} x^2+y^2 & \text{if } (x,y)\in V_2^-\cup V_2^+ \\ x^2 & \text{if } (x,y)\in V_2^{-+}\cup V_2^{+-} \\ \dfrac{(x+by)^2}{1+b^2} & \text{if } (x,y)\in V_2^{--}\cup V_2^{++} \end{cases} \qquad (8)$$

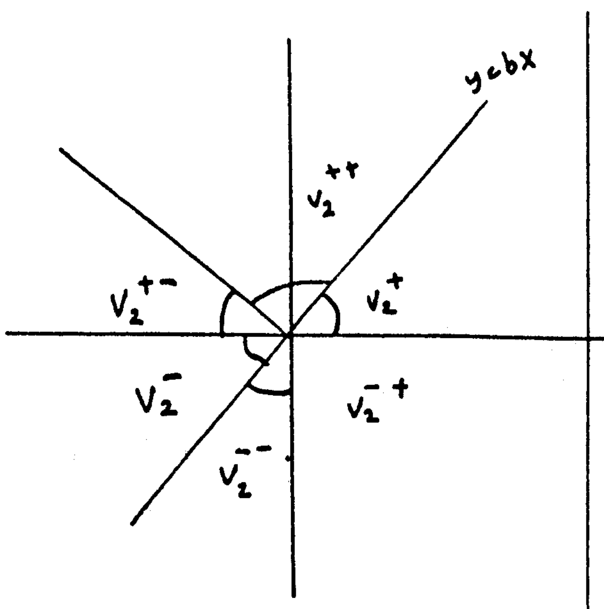
and b = tan β^*. The critical region of the LRT is depicted in Figure 2-2-b.

3. The Monotonicity of the Power Function of the LRT Φ_1

In this section, we shall explicitly obtain the power function of the LRT Φ_1. In addition, we will prove that the power function is symmetric about $\beta^*/2$. Furthermore, we show that the power function satisfies a monotonicity property with respect to the angle of the alternative (θ_1, θ_2). First, we obtain the power function of the test Φ_1. To proceed we adopt the following notation:

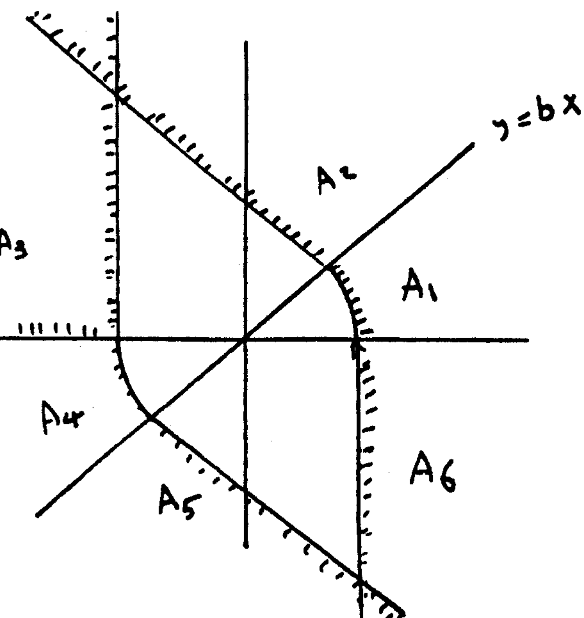
$$Q(w)=\int_{-\infty}^{\infty} z(t)dt. \qquad (9)$$

where z(t) is the standard normal distribution.



(a)

Figure 2-2-a. The cones V_2^+, V_2^{++}, V_2^-, V_2^{--}, V_2^{+-} and V_2^{-+}



(b)

Figure 2-2-b. The critical region for the LRT Φ_2

Let (Δ, γ) and (r, ζ) be the polar transformation of (θ_1, θ_2) and (x, y) respectively. We divide the critical region of Φ_1 into three parts A_1, A_2, A_3 as shown in Figure 2-1-b. We use this in the proof of the following theorem.

Theorem 3.1. The power function of the LRT Φ_1 is given by

$$\eta_1(\Delta, \gamma) = G(\Delta, \beta^* - \gamma) + G(\Delta, \gamma) + H^*(\Delta, \gamma) \tag{10}$$

where

$$G(\Delta, \gamma) = Q(k^{1/2} - \Delta\cos\gamma)\, Q(\Delta\sin\gamma), \tag{11}$$

$$H^*(\Delta, \gamma) = \frac{\exp(\frac{\Delta^2}{2})}{2\pi} \int_{-\gamma}^{\beta^* - \gamma} H(k, \Delta\cos\zeta)\, d\zeta \tag{12}$$

and

$$H(k,t) = \exp(-t^2/2) \int_{k^{1/2}}^{\infty} r \exp(-\frac{1}{2}(r-t)^2\, dr. \tag{13}$$

Proof.

The power function is given by:

$\eta_1(\Delta, \gamma) = P(A_1) + P(A_2) + P(A_3)$.

Through some elementary calculus, it can be easily shown that

$P(A_1) = G(\Delta, \beta^* - \gamma)$,

$P(A_3) = G(\Delta, \gamma)$

and

$P(A_2) = H^*(\Delta, \gamma)$.

The following theorem presents a symmetry property of the power function $\eta_1(\Delta, \gamma)$.

Theorem 3.2. For a fixed Δ, the power function $\eta_1(\Delta, \gamma)$ is symmetric in γ about $\beta^*/2$, i.e.,

$$\eta(\Delta, \beta^* - \gamma) = \eta(\Delta, \gamma) \text{ for } 0 \le \gamma \le \beta^*/2 \tag{14}$$

Proof.

Notice that $G(\beta^* - \gamma) + G(\gamma)$ is symmetric about $\beta^*/2$. Thus, it is left to show that H^* is symmetric. However, in the definition of H^*, if we change the variable of integration from ζ into $-\zeta$, we obtain

$H^*(\Delta, \beta^* - \gamma) = H^*(\Delta, \gamma)$.

This completes the proof of the theorem.

Through some numerical computations, concerning the special case $\beta^* = \pi/3$, Bartholomew [4] noticed that for fixed Δ, the power function $\eta_1(\Delta, \gamma)$ has a bell shape in γ. In the followıng theorem, we snow that this is true for all values of $\beta^* \le \pi$.

Theorem 3.3. For a fixed Δ, the power function $\eta_1(\Delta, \gamma)$ is an increasing function of γ for $\beta^*/2 - \pi \le \gamma \le \beta^*/2$ and decreasing for $\beta^*/2 \le \gamma \le \beta^*/2 + \pi$.

Proof.

By the symmetry of η about $\beta^*/2$, it is enough to show that η is increasing for $\beta^*/2 - \pi \le \gamma \le \beta^*/2$.

Let B denote the acceptance region of the LRT Φ_1. Therefore

$$\eta_1(\Delta, \gamma) = 1 - \iint_B \frac{1}{2\pi} \exp(-\frac{1}{2}(x^2 + y^2) - \frac{\Delta^2}{2} + \Delta x\cos\gamma + \Delta y \sin\gamma)\, dxdy.$$

Take the derivative of η_1 with respect to γ to get

$$\frac{\partial \eta_1(\Delta, y)}{\partial \gamma} = -\iint_B \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x^2 + y^2) - \frac{\Delta^2}{2} + \Delta x\cos\gamma + \Delta y\sin\gamma\right] [-\Delta x\sin\gamma + \Delta y\cos\gamma]\, dxdy.$$

Now make the transformation of (x,y) into (x', y'), by a rotation of the angle γ. It can be shown that (x', y') has the bivariate normal distribution with mean vector $(\Delta, 0)$ and identity covariance matrix. Therefore

$$\frac{\partial \eta_1(\Delta, \gamma)}{\partial \gamma} = \iint_{B'} M(x', y')\, dx'dy',$$

where

$$M(x', y') = \frac{1}{2\pi} \Delta y' \exp[y'^2 + (x' - \Delta)^2]$$

and B' is the image of B in x'y'-plane. The region B' is depicted in Figure 3-1. It can be seen from the figure that B' can be divided into three disjoint subregions B_1, B_2 and B_3 as illustrated in the same figure.

Because B_1 is a mirror image of B_2 and because of the structure of M(x',y') we have

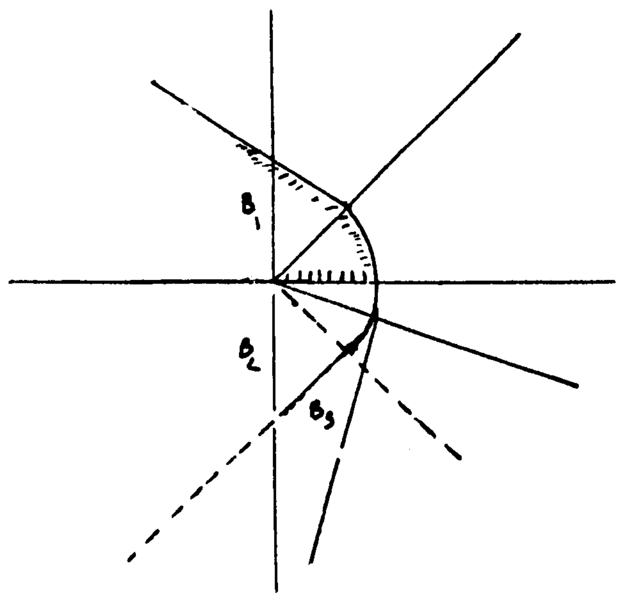


Figure 3-1. The set B' and its partition B_1, B_2 and B_3

$$\iint_{B_1} M(x', y')\, dx'dy' = -\iint_{B_2} M(x', y')\, dx'\, dy'$$

also, because $y' < 0$ on B_3,

$$\iint_{B_3} M(x', y')\, dx'dy' < 0.$$

Thus

$$\frac{\partial \eta_1(\Delta, y)}{\partial \gamma} > 0 \qquad \text{for} \qquad \beta^*/2 - \pi \le \gamma \le \beta^*/2$$

This completes the proof.

4.The Monotonicity of the Power Function of the LRT Φ_2

Following the style of Section 3, we will obtain the power function of LRT Φ_2 and then we will show that it has monotonicity and symmetry properties.

Theorem 4.1. The power function of the LRT Φ can be expressed as $\eta_2(\Delta, \gamma) = \beta_1(\Delta, \gamma) + \beta_1(\Delta, \gamma) - \beta_3(\Delta, \gamma)$ where

$$\beta_1(\Delta, \gamma) = \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi}\left[\int_{-\gamma}^{\beta^* - \gamma} (H(k, \Delta\cos\xi) + H(k, -\Delta\sin\xi)\, d\xi\right],$$

$\beta_2(\Delta, \gamma) = G(\Delta, \gamma) + G(\Delta, \beta^* - \gamma) + G(\Delta, -\gamma) + G(\Delta, \gamma - \beta^*)$,

$$\beta_3(\Delta, \gamma) = \int_{\sqrt{k}}^{\infty}\int_{-\infty}^{\frac{1}{a}(x+\sqrt{k(1+a^2)})} (z(x - \Delta\cos\gamma)\, z(y - \Delta\sin\gamma) + z(x+\Delta\cos\gamma) z(y+\Delta\sin\gamma)\, dydx,$$

$H(k, t)$ is given in (11), and $G(\Delta, \gamma)$ is given in (13).

Proof.

The proof of the theorem is based on dividing the critical region into subregions A_1, A_2, A_3, A_4, A_5 and A_6 illustrated in Figure 2-2-b.

Hence,

$$\beta_2(\Delta, \gamma) = \sum_{i=1}^{6} \Pr(A_i) - \Pr(A_2 \cap A_3) - \Pr(A_5 \cap A_6).$$

Using a similar argument as in the proof of Theorem 3-1, it can be shown that

$$P(A_1) = \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi}\left[\int_{-\gamma}^{\beta^* - \gamma} (H(k, \Delta\cos\xi_1) d\xi_1\right]$$

and

$$P(A_4) = \frac{e^{-\frac{1}{2}\Delta^2}}{2\pi}\left[\int_{-\gamma}^{\beta^* - \gamma} (H(k, -\Delta\cos\xi_1) d\xi_1\right]$$

Also

$P(A_2) = G(\gamma)$,
$P(A_5) = G(-\gamma)$,
$P(A_3) = G(\gamma - \beta^*)$,

and

$P(A_6) = G(B^* - \gamma)$.

Furthermore,

$$P(A_2 \cap A_3) = \int_{\sqrt{k}}^{\infty}\int_{-\infty}^{\frac{1}{a}(x+\sqrt{k(1+a^2)})} z(x - \theta_1)\, z(y - \theta_2)\, dydx$$

and

$$P(A_5 \cap A_6) = \int_{\sqrt{k}}^{\infty} {}^{-\frac{1}{a}(x+\sqrt{k(1+a^2)})}\, z(x - \theta_1)\, z(y - \theta_2)\, dydx.$$

This completes the proof.

The following theorem establishes the symmetry of the power function.

Theorem 4.2. For a fixed $\Delta \ge 0$, the power function $\eta_2(\Delta, \gamma)$ is symmetric about $\gamma = \beta^*/2$, i.e., $\eta_2(\Delta, \gamma) = \eta_2(\Delta, \beta^* - \gamma)$, $\forall \gamma < \beta^*/2$.

Proof.

Using a similar argument as in Theorem 3-2, we can show that $\beta_1(\Delta, \gamma)$ and $\beta_2(\Delta, \gamma)$ are symmetric about $\gamma = \beta^*/2$. We want to show that $\beta_3(\Delta, \gamma)$ is symmetric about $\gamma = \beta^*/2$. To accomplish this, transform (x, y) into (x', y') by a rotation through an angle $\beta^*/2$ clockwise. It is easy to see that (x', y') is distributed as $N((\theta'_1, \theta'_2), I)$ where, $\theta'_1 = \Delta\cos\gamma'$ and $\theta'_2 = \Delta\sin\gamma'$, and $\gamma' = \gamma - \beta^*/2$.

Notice that the regions $(A_2 \cap A_3)$ and $(A_5 \cap A_6)$ in the xy-plane are transformed into M_1 and M_2 in the x'y'-plane. Now let

$\beta_3(\Delta, \gamma') = \beta_{31}(\Delta, \gamma') + \beta_{32}(\Delta, \gamma')$

where

$$\beta_{31}(\Delta, \gamma') = \iint_{M_1} z(x' - \Delta\cos\gamma')\, z(y' - \Delta\sin\gamma')\, dy'dx'$$

and

$$\beta_{32}(\Delta, \gamma') = \iint_{M_2} z(x' - \Delta\cos\gamma')\, z(y' - \Delta\sin\gamma')\, dy'dx'.$$

But, if we transform y' into $-y'$ we get

$$\beta_{31}(\Delta, -\gamma') = \beta_{32}(\Delta, \gamma') \tag{2.3.2}$$

therefore, $\beta_3(\Delta, \gamma')$ is symmetric about $\gamma' = 0$. By the

definition of γ' we can conclude that $\beta_3(\Delta, y)$ is symmetric about $\beta^*/2$.
This completes the proof.

The following theorem establishes the monotonicity of the power function.

Theorem 4.3. The power function of the LRT Φ_2 increases in γ for all $\beta^*/2-\pi \le \gamma \le \beta^*/2$ and decreases for all $\beta^*/2 \le \gamma \le \beta^*/2+\pi$.

Proof.

Let A* be the acceptance region of the LRT Φ_2, then $\eta_2(\Delta, \gamma) = 1 - \Pr(A^*)$.
From this it can be seen that

$$\frac{\partial \eta_2(\Delta,\gamma)}{\partial \gamma} = -\iint_{A^*} \left(\frac{-x\,\Delta \sin\gamma + y\,\Delta\cos\gamma}{2\pi}\right)\exp\left(-\frac{1}{2}(x^2+y^2) - \frac{\Delta^2}{2} + x\,\Delta\cos\gamma + y\,\Delta\sin\gamma\right)dxdy.$$

Transform (x,y) by a rotation of angle $\gamma \le \beta/2$ into (x', y'). Then (X', Y') is distributed as N ((Δ, 0), I). Assume that A** is the image of the A* in the new x'y'-plane. Hence

$$\frac{\partial \eta_2(\Delta,\gamma)}{\partial \gamma} = -\iint_{A^{**}} M(x',y')\,dx'dy'$$

where,

$$M(x',y') = \frac{\Delta y'}{2\pi}\exp\left(-\frac{1}{2}((x'-\Delta)^2 + y'^2)\right).$$

Now, A** can be partitioned into four disjoint regions B_1, B_2, B_3 and B_4 which are illustrated in Figure 4-1.

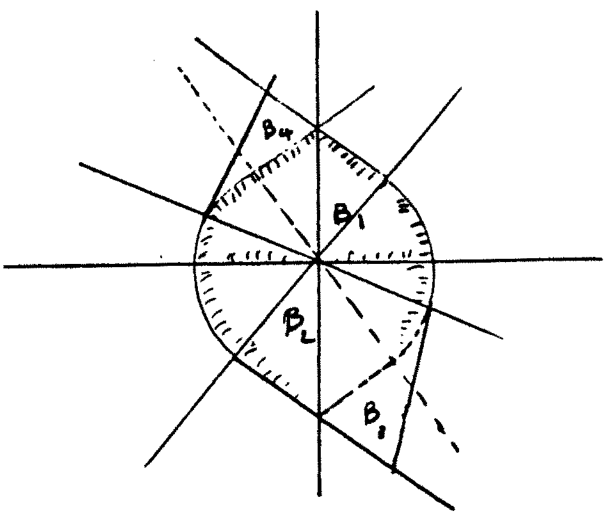


Figure 4-1. Illustration of the four regions B_1, B_2, B_3 and B_4

By symmetry and structure of M (x', y'),

$$\iint_{B_1} M(x',y')\,dx'dy' = -\iint_{B_2} M(x',y')dx'dy'.$$

This implies that

$$\frac{\partial \eta_2(\Delta,\gamma)}{\partial \gamma} = -\iint_{B_3 \cup B_4} M(x',y')dx'dy'.$$

Notice that by transforming (x',y') into (-x',-y'), we get that

$$\iint_{B_3} M(x',y')\,dx'dy' = \iint_{B_4} M(-x',-y')\,dx'dy'.$$

Therefore,

$$\iint_{B_3 \cup B_4} M(x',y')dx'dy' = -\iint_{B_4} M(x',y') - M(-x',-y')\,dx'dy'.$$

However, if $(x',y') \in B_4$ then $x' \le 0, y' > 0$ and $\Delta \ge 0$, so this implies that

$$\exp\left[-\frac{1}{2}(x'-\Delta)^2\right] \le \exp\left[-\frac{1}{2}(x'+\Delta)^2\right].$$

Hence,

$M(x',y') - M(-x',-y') \le 0$ for all $(x',y') \in B_4$.

This means that

$$\iint_{B_3 \cup B_4} M(x',y')dx'dy' \le 0.$$

Therefore,

$$\frac{\partial \eta_2(\Delta,\gamma)}{\partial \gamma} \ge 0,$$

so $\eta_2(\Delta,\gamma)$ is increasing for $\gamma < \beta^*/2$. This completes the proof.

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