J. Sci. I. R. Iran

Vol.7 No.4
Autumn 1996

AMENABLE WEIGHTED HYPERGROUPS

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Abstract

In this paper among many other things we prove that the topological left amenability and left amenability of a weighted hypergroup (K, ω) are equivalent. For a normal subgroup H of K, we define a weight function ω° on K/H and obtain connection between left amenability of (K, ω) and $(K/H, \omega^{\circ})$. Let H be a compact subhypergroup of K. We define the weight function ω° on $K//H$ and obtain connection between left amenability of (K, ω) and $(K//H, \omega^{\circ})$.

Introduction

Throughout this paper, K will denote a hypergroup with a fixed left Haar measure λ. Unless otherwise specified our notation will follow that of [3]. The following notations are different from those in [3].

δ_x The point mass at $x \in K$

χ_A The characteristic function of the non-empty set $A \subseteq K$

$\|f\|_\infty = ess\ sup\ |f|$

The involution on K is denoted by $x \to x^-$. If f is a Borel function on K and $x,y \in K$ the left translation ${}_xf$ or L_xf and the right translation f_y or R_yf are defined by $L_xf(y) = {}_xf(y) = R_yf(x) = \int_K f\, d(\delta x * \delta y) = f(x*y)$, if the integral exists. The function f^- is given by $f^-(x) = f(x^-)$. The integral $\int \ldots d\lambda(x)$ is often denoted by $\int \ldots dx$.

Let K be a hypergroup and let $\rho: K \to (0, \infty)$ be a Borel function. Let X be a Banach space of measures of equivalence classes of functions on K. We define the corresponding weighted space as $X(\rho) = \{f \mid \rho f \in X\}$. We norm $X(\rho)$ so that the map $f \to \rho f: X(\rho) \to X$ becomes an isometry.

In what follows we shall use one of the following spaces for X.

$M(K) = \{\mu \mid \mu$ is a regular Borel measure, $\|\mu\| = |\mu|(K) < \infty\}$,

$L^1(K) = \{f \mid f$ Borel measurable and $\int |f|\, dx < \infty\}$,

$L^\infty(K) = \{f \mid f$ Borel measurable and $\|f\|_\infty < \infty\}$,

$RUC(K) = \{f \in L^\infty(K) \mid x \to {}_xf$ is continuous from K to $(L^\infty(K), \|\ \|_\infty)\}$,

$LUC(K) = \{f \in L^\infty(K) \mid x \to f_x$ is continuous from K to $(L^\infty(K), \|\ \|_\infty)\}$,

$UC(K) = RUC(K) \cap LUC(K)$.

Definition 1. A function $\omega : K \to [1, \infty)$ is called a Borel weight function on K, if

(i) for every $t \in spt(\delta_x * \delta_y)$, $\omega(t) \leq \omega(x)\omega(y)$,

(ii) $w(e) = 1$ (e is the unique element of K such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$, for all $x \in K$),

(iii) w is Borel measurable and locally bounded.

We shall use $M_\omega(K)$, $L^\infty_\omega(K)$, $L^1_\omega(K)$, $RUC_\omega(K)$, $LUC_\omega(K)$ and $UC_\omega(K)$ instead of $M(K)(\omega)$, $L^\infty(K)(\frac{1}{\omega})$, $L^1(K)(\omega)$, $RUC(K)(\frac{1}{\omega})$, $LUC(K)(\frac{1}{\omega})$ and $UC(K)(\frac{1}{\omega})$. We shall use $\|\ \|_\omega$ and $\|\ \|^1_\omega$ respectively for norms of $L^\infty_\omega(K)$ and $L^1_\omega(K)$.

Lemma 2. $L^\infty_\omega(K)$ is a translation invariant Banach space containing the constant functions and ω.

Proof. Let $f \in L^\infty_\omega(K)$ and $s \in K$.

$$\left|\frac{{}_sf}{\omega}(t)\right| = \left|\frac{f(s*t)}{\omega(t)}\right| = \left|\int_K \frac{f(x)}{\omega(t)} d(\delta_s * \delta_t)(x)\right| \leq \int_K \frac{|f(x)|}{\omega(x)} \frac{\omega(x)}{\omega(t)} d(\delta_s * \delta_t)(x)$$

Keywords: Amenable; Hypergroup; Weight

1990 AMS Mathematics Subject Classification: Primary 43A07

$\leq \omega(s)\int_K \frac{|f|}{\omega}(x)\, d(\delta_s * \delta_t)(x) = \omega(s)\, {}_s(\frac{|f|}{\omega})(t)$.

Since $\frac{f}{\omega} \in L^\infty(K)$, ${}_s(\frac{|f|}{\omega}) \in L^\infty(K)$ and $\left\| {}_s(\frac{|f|}{\omega})\right\|_\infty \leq \left\| (\frac{f}{\omega})\right\|_\infty$ [3, 3.1 B]. Thus ${}_sf \in L^\infty_\omega(K)$ and $\|{}_sf\|_\omega \leq \|f\|_\omega\, \omega(s)$. Hence $L^\infty_\omega(K)$ is translation invariant.

The proof of other statements are clear. □

Let K be a hypergroup with a fixed left Haar measure λ and ω be a Borel weight function on K. Let X be a left translation invariant subspace of $L^\infty_\omega(K)$ containing the constant functions. A linear functional m on X is called a mean if $f \geq 0 (\lambda a.e)$ implies $m(f) \geq 0$ and $m(1) = 1$. A mean m on X is called a left invariant mean [LIM] if $m(L_x f) = m(f)$ for all $x \in K, f \in X$. We call (K, ω), the hypergroup K with the weight function ω and (K, ω) is called left amenable, if there is a LIM on $L^\infty_\omega(K)$.

Let $f \in L^\infty_\omega(K)$ and $p \in L^1_\omega(K)$, the functions $f.p, p.f \in L^\infty_\omega(K)$ are defined by,

$$<f.p,q> = <f,p*q>,\ <p.f,q> = <f,q*p>\quad q \in L^1_\omega(K). \quad (1)$$

This is in fact the module operation defined on page 50 of [1].

Let $P_\omega(K) = \{p \in L^1_\omega(K) / p \geq 0, \int p(x)dx = 1\}$. A mean m on $L^\infty_\omega(K)$ is said to be a topological left invariant mean [TLIM] if $m(f.p) = m(f)$ $(f \in L^\infty_\omega(K), p \in P_\omega(K))$. (K,ω) is called topologically left amenable, if there is a TLIM on $L^\infty_\omega(K)$.

Henceforth, the weight function is continuous.

Lemma 3. Let $f \in L^\infty_\omega(K)$, $p \in L^1_\omega(K)$ and $s, t \in K$. Then,

(a) $p.f(x) = <f, {}_x\text{-}p>$,

(b) $p.f(s*t) = <f, {}_s\text{-}({}_t\text{-}p)>$,

(c) $\|(p.f)t - p.f\|_\omega \leq \|f\|_\omega \|{}_t\text{-}p - p\|_\omega^1$,

(d) $f.p(x) = <f, \Delta(x^-)p_{x^-}>$,

(e) $f.p(s*t) = <f, \Delta(s^-)\Delta(t^-)(p_{s^-})_{t^-}>$,

(f) $\|{}_s(f.p) - f.p\|_\omega \leq \|f\|_\omega \|\Delta(s^-)p_{s^-} - p\|_\omega^1$

(g) Let f be continuous. Then

$$\left|\frac{f}{\omega}(s*t) - \frac{f(s*t)}{\omega(s)}\right| \leq \|f\|_\omega \sup\ \{|1 - \omega(t)|, |\frac{1}{\omega(t^-)} - 1|\}$$

Proof. We only prove statements (d), (e), (f) and (g). The proof of other statements are similar.

Proof of (d). By definition of $f.p$, $f.p(x) = \int f(y*x)p(y)dy$. Since the modular function Δ [3] is constant on spt $(\delta_x * \delta_y)$ with value $\Delta(x)\Delta(y)$, for all $x, y \in K$, then $p\Delta(y*x) = p(y*x)\Delta(y*x)$. Hence

$$\int f(y)p_{x^-}(y)\Delta(x^-)dy = \int f(y)p(y*x^-)\Delta(y*x^-)\Delta(y^-)dy$$
$$= \int f(y)p\Delta(y*x^-)\Delta(y^-)dy = \int f(y*x)p\Delta(y)\Delta(y^-)dy$$
$$= \int f(y*x)\, p(y)\Delta(y)\Delta(y^-)dy = \int f(y*x)p(y)\, dy. \quad (3)$$

Formulas (2) and (3) imply $<f, \Delta(x^-)p_{x^-}> = f.p(x)$.

Proof of (e). $f.p(s*t) = \int f.p(z)d(\delta_s*\delta_t)(z) = \int d(\delta_s*\delta_t)(z) \int f(y)\Delta(z^-)p_{z^-}(y)dy$

$= \int f(y)dy \int \Delta(z^-)p_{z^-}(y)d(\delta_s*\delta_t)(z) = \int f(y)\Delta(s^-)\Delta(t^-)_y p(t^-*s^-)dy$

$= \int f(y)\Delta(s^-)\Delta(t^-)(p_{s^-})_{t^-}(y)dy = <f, \Delta(s^-)\Delta(t^-)(p_{s^-})_{t^-}>$.

Proof of (f). $\left|\frac{({}_s(f.p) - (f.p))(t)}{\omega(t)}\right| = \left|\frac{f.p(s*t) - f.p(t)}{\omega(t)}\right| =$

$\left|\frac{<f, \Delta(t^-)\Delta(s^-)(p_{s^-})_{t^-}> - <f, \Delta(t^-)p_{t^-}>}{\omega(t)}\right|$

$= \left|\frac{<f, (\Delta(s^-)p_{s^-} - p)_{t^-}\Delta(t^-)>}{\omega(t)}\right|$.

Thus $\|{}_s(f.p) - f.p\|_\omega \leq \|f\|_\omega \|\Delta(s^-)p_{s^-} - p\|_\omega^1$

Proof of (g).

$\left|\frac{f}{\omega}(s*t) - \frac{f(s*t)}{\omega(s)}\right| = \left|\int f(x)\left(\frac{1}{\omega(x)} - \frac{1}{\omega(s)}\right) d(\delta_s * \delta_t)(x)\right| \leq$

$\|f\|_\omega \max\ \{|1 - \frac{\omega(x)}{\omega(s)}|\ x \in spt\, \delta_s * \delta_t\}$

If $\frac{\omega(x)}{\omega(s)} \geq 1$, since $x \in spt(\delta_s * \delta_t)$, $\omega(x) \leq \omega(s)\,\omega(t)$, we get $0 \leq \frac{\omega(x)}{\omega(s)} - 1 \leq \omega(t) - 1$. If $\frac{\omega(x)}{\omega(s)} \leq 1$ since $x \in spt(\delta_s * \delta_t)$, $s \in (\delta_x * \delta_{t^-})$, thus $\omega(s) \leq \omega(x)\,\omega(t^-)$. So,

$0 \leq 1 - \frac{\omega(x)}{\omega(s)} \leq 1 - \frac{1}{\omega(t^-)}$. Therefore

$|1 - \frac{\omega(x)}{\omega(s)}| \leq \max\{|1 - \frac{1}{\omega(t^-)}|, |\omega(t) - 1|\}$. Hence,

$\left|\frac{f}{\omega}(s*t) - \frac{f(s*t)}{\omega(s)}\right| \leq \|f\|_\omega \sup\ \{|1 - \omega(t)|, |\frac{1}{\omega(t^-)} - 1|\}$ □

Proposition 4. $L^\infty_\omega(K).L^1_\omega(K) = RUC_\omega(K) = \{f \in L^\infty_\omega(K) / x \to {}_xf$ is norm continuous at e$\}$.

Proof. Let $f \in L^\infty_\omega(K)$, $p \in L^1_\omega(K)$ and $s, t \in K$

$$|{}_t(\frac{f.p}{\omega})(s) - \frac{f.p}{\omega}(s)| = |\frac{f.p}{\omega}(t^*s) - \frac{f.p}{\omega}(s)| \le |\frac{f.p}{\omega}(t^*s) - \frac{f.p(t^*s)}{\omega(s)}|$$

$$+ |\frac{f.p(t^*s)}{\omega(s)} - \frac{f.p(s)}{\omega(s)}| \le \|fp\|_\omega \sup\{|1-\omega(t)|, |1-\frac{1}{\omega(t^-)}|\}$$

$+ \|f\|_\omega \|\Delta(t^-) p_{t^-} - p\|^1_\omega$. [cf. Lemma 3 f,g]

Since ω is continuous at e and $t \to \Delta(t)p_{t^-}$ from K to $L^1_\omega(K)$ is continuous, then $f.p \in RUC_\omega(K)$. The rest of the proof is similar to the proof of proposition 1.3 of [2]. □

Lemma 5. Let K be a hypergroup and let ω be a weight function on K. Each of the spaces $UC_\omega(K)$, $RUC_\omega(K)$ and $LUC_\omega(K)$ is a norm closed, conjugate closed and translation invariant subspace of $L^\infty_\omega(K)$ containing the constant functions and ω.

Proof. If $g \in RUC_\omega(K)$, write $g = f.p$ where f $\in L^\infty_\omega(K)$, $p \in L^1_\omega(K)$. Then, for $x \in K$ ${}_xg = {}_x(f.p) = f.\Delta(x^-)p_{x^-} \in RUC_\omega(K)$. So $RUC_\omega(K)$ is left translation invariant, and it is easily seen to be right translation invariant.

By lemma 3(g), 1 $\in RUC_\omega(K)$. The other statements are similar. □

Theorem 6. The following statements are equivalent.

(i) (K, ω) is topologically left amenable

(ii) (K, ω) is left amenable

(iii) There is a left invariant mean on $UC_\omega(K)$.

Proof. (iii) $\to$ (i). Let μ_0 be a left invariant mean on $X = UC_\omega(K)$ and $p \in P_\omega(K)$ with compact support. Since the mapping $x \to {}_xf$ $(f \in X)$ from K to $(X, \|.\|_\omega)$ is continuous and the point evaluation functionals in X^* separate the points of X, we have

$$f.p = \int {}_xf p(x)dx.$$

Thus

$$\mu_0(f.p) = \mu_0(f)$$

It is proved similar to non-weighted case that for all $f \in LUC_\omega(K)$ and $p_1, p_2 \in P_\omega(K)$ with compact support

$$\mu_0(f.p_1) = \mu_0(f.p_2)$$

Let V be a compact neighbourhood of e and set $p = \frac{\chi_V}{\lambda(V)}$.

Then $\mu(f) = \mu_0(p.f.p)$ defines a TLIM on $L_\omega(K)$. □

A subgroup H of K is called normal if $xH = Hx$, for all $x \in K$. Let H be a normal subgroup of K and let K/H be the set of all cosets xH, $x \in K$, equipped with the quotient topology with respect to the natural map $p(x) = xH$. Then K/H becomes a hypergroup under the convolution

$$\int_{K/H} fd(\delta_{xH} * \delta_{yH}) = \int_K f\,o\,p\,d(\delta_x * \delta_y)\ (x, y \in K\ f \in C_c(K/H)).$$

Let ω be a weight function on K. The function $\omega^\circ(xH) = \inf\{\omega(t) | t \in xH\}$ is an upper semicontinuous weight function on K/H [5].

Proposition 7. If (K, ω) is left amenable, so is $(K/H, \omega^\circ)$.

Proof. Let m be a LIM on $L^\infty_\omega(K)$ and write $\langle M, f\rangle = \langle m, f\,o\,p\rangle$ $(f \in L^\infty_{\omega^\circ}(K/H))$. Since f is Borel measurable and p is continuous, $f\,o\,p$ is Borel measurable and since $\omega^\circ(xH) \le \omega(x)$, $f\,o\,p \in L^\infty_\omega(K)$. We have ${}_x(f\,o\,p)(y) = {}_{xH}f\,o\,p(y)$, hence M is a LIM on $L^\infty_{\omega^\circ}(K/H)$. □

We have shown in [4] that there is a group K with a normal subgroup H and a weight ω on K such that (K, ω) is amenable, but (H, ω) is not amenable. However, we have the following.

Proposition 8. If (H, ω) and $(K/H, \omega^\circ)$ are left amenable, then so is (K, ω).

Proof. Let m_1 be a LIM on $L^\infty_\omega(H)$ and m_2 be a LIM on $L^\infty_{\omega^\circ}(K/H)$. For $f \in UC_\omega(K)$ write, $f_1(x) = \langle m_1, {}_xf|_H\rangle$ $(x \in K)$. Then f_1 is continuous and constant on the cosets of H in K. For $t \in xH$,

$$|f_1(x)| = |f_1(t)| \le \|m_1\| \|f\|_\omega \omega(t), \text{ then } \left\|\frac{f_1(x)}{\omega^\circ(xH)}\right\|_\infty < \infty.$$

Hence we can write $f_1 = F\,o\,p$, $F \in L^\infty_{\omega^\circ}(K/H)$. Put $\langle m, f\rangle = \langle m_2, F\rangle$, then m is a LIM on $UC_\omega(K)$ [6, prop 3.6]. Thus by Theorem 6, (K, ω) is left amenable. □

Let J and L be hypergroups with left Haar measures. The $J \times L$ with convolution, $\delta_{(x,y)} * \delta_{(x_1, y_1)} = (\delta_x * \delta_{x_1}) \times (\delta_y * \delta_{y_1})$, is a hypergroup with a left Haar measure [3]. If ω_1 and ω_2 are weights on J and L respectively, then $\omega = \omega_1 \times \omega_2$ given by $\omega(x, y) = \omega_1(x)\omega_2(x)$ for $(x, y) \in J \times L$ is a weight on $J \times L$.

Proposition 9. $(J \times L, \omega)$ is left amenable if and only if both (J, ω_1) and (L, ω_2) are left amenable.

Proof. See proof of proposition 3.8 of [6]. □

Let H be a compact subhypergroup of K, and let λ be a Haar measure on K and σ be the normalized Haar measure on H. For $x, y \in K$, let $HxH = H*x*H$. The set $K//H = \{HxH | x \in K\}$ is the set of all double cosets of H in K. This set equipped with the quotient topology with respect to the projection $\pi: K \to K//H$, $\pi(x)= HxH$ and convolution, $\delta_{HxH} * \delta_{HyH} = \int_H \delta_{HtH}(\delta_x * \sigma * \delta_y)dt$ is a hypergroup. For details see [3, 14.2F].

Lemma 10. Let ω be a weight function on K which is equal to 1 on H, then $\omega^\circ(HxH)= inf\{\omega(z) \mid z \in HxH\}$ is an upper semicontinuous weight function on K//H and $\omega^\circ(HxH)= \omega(x)$.

Proof. Since $x \in HxH$, $\omega^\circ(HxH) \leq \omega(x)$. If $z \in HxH$, then $HzH= HxH$. Hence $x \in HzH$ and $\omega(x) \leq \omega(z)$. Therefore $\omega^\circ(HxH)= \omega(x)$. Let $HzH \in spt(\delta_{HxH} * \delta_{HyH}) = \{HzH \mid z \in xHy\}$. Then $\omega^\circ(HzH)= \omega(z) \leq \omega(x)\omega(y)= \omega^\circ(HxH)\omega^\circ(HyH)$. Thus ω is a weight function on $K//H$. □

Proposition 11. If (K, ω) is left amenable, so is $(K//H, \omega^\circ)$. If $\delta_x * \sigma = \sigma * \delta_x$, for each $x \in K$ where σ is the normalized Haar measure of H, then the converse is also true.

Proof. Let m be a TLIM on $L^\infty_\omega(K)$, and $f \in L^\infty_{\omega^\circ}(K//H)$, write $<M, f> = <m, f \circ \pi>$ where π is the projection of K onto $K//H$. Let $p \in P_{\omega^\circ}(K//H)$, then an easy computation shows that,

$(f.p) \circ \pi = (f \circ \pi).(p \circ \pi)$.

Since $\omega^\circ(HxH)= \omega(x)$, $\int p \circ \pi(x)dx = 1$ and $\int p \circ \pi(x)\omega(x)dx < \infty$, $p \circ \pi \in P_\omega(K)$. Thus M is a TLIM on $L^\infty_{\omega^\circ}(K//H)$. The rest of the proof is similar to the proof of proposition 3, 10 of [6]. □

Let H be a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$ where e is the identity of both hypergroups. Let $K= H \vee J$ be the joint hypergroup of H and J [6]. Let ω_1 be a weight function on J, then the function,

$$\omega(x) = \begin{cases} 1 & x \in H \\ \omega_1(x) & x \in J \end{cases}$$

is a weight function on K.

Corollary 12. (K, ω) is left amenable if and only if (J, ω_1) is left amenable.

Proof. [6, 3.12]. □

Acknowledgements

I would like to express my deep gratitude to Dr. A.R. Medghalchi for his guidance and encouragement.

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