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ON THE STRUCTURE OF FINITE PSEUDO-COMPLEMENTS OF QUADRILATERALS AND THEIR EMBEDDABILITY

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Abstract

A pseudo-complement of a quadrilateral D of order n, n, > 3, is a non-trivial (n+1)-regular linear space with $n^2 - 3n + 3$ points and $n^2 + n - 3$ lines. We prove that if n > 18 and D has at least one line of size n - 1, or if n > 25, then the set of lines of D consists of three lines of size n -1, 6(n - 2) lines of size n - 2, and $n^2 - 5n + 6$ lines of size n - 3. Furthermore, if n > 21 and D has at least one line of size n - 1, then D is embeddable in a unique projective plane of order n. These results improve the results of the author.

Introduction

A simple graph G consists of a non-empty finite set $V(G)$, called the set of vertices, and a function m from the set of unordered pairs of elements of $V(G)$ into the set $\{0,1\}$ such that for every vertex P, $m(P,P) = 0$. Two vertices P and Q are joined if $m(P,Q) = 1$. Then PQ is called an edge of G. Given a vertex P of G, the number of edges through P is called the degree of P and is denoted by $d(P)$. Also, for two vertices P and Q of G, the total number of vertices joined to both P and Q is denoted by $l(P,Q)$.

A claw at a vertex P of G is an ordered pair (P, S) such that S is a subset of $V(G)$, P is joined to all vertices in S, and no two vertices in S are joined. A claw (P,S) is extendable if there is a vertex Q not in S, which is joined to P and not joined to any vertex in S. Otherwise, (P,S) is a maximal claw.

A set of pairwise joined vertices of G is called a clique. A clique K is a maximal clique if no vertex outside K is joined to all vertices in K.

A structure D is an ordered triplet (P, B, I) in which P and B are non-empty disjoint finite sets, called the sets of points and lines, respectively, and I is a subset of $P \times B$. We say a point X is contained in a line y if (X, y) belongs to I. The number of points common to two lines y and z is denoted by $/y,z/$. Two distinct lines y and z are disjoint if $[y, z] = 0$, otherwise they intersect.

Given a structure $D = (P, B, I)$, the structure $D' = (B, P, I^t)$ is called the dual of D.

A structure D is called a non-trivial (n+1)-regular linear space, n>1, if in D:

(i) a point is contained in exactly n+1 lines;

(ii) two distinct points are contained in a unique common line;

(iii) no line contains all points of D.

Then n is called the order of D.

A projective plane of order n, $n > 1$, is a non-trival $(n+1)$ - regular linear space in which all lines have the same size $n+1$. A set of four lines of a projective plane of order n is called a quadrilateral if no three of them contain a common point. A pseudo-complement of a quadrilateral of order n,n>3, is a non-trivial (n+1) - regular linear space with $n^2 - 3n + 3$ points and $n^2 + n - 3$ lines. Examples of pseudo-complements of quadrilaterals of order n are obtained by removing quadrilaterals from projective planes of order n. A structure D is said to be embeddable into a larger structure D′ if D can be extended into D' by addition of new points and new lines.

In this paper, we show that if n >18 and there exists at least one line of size n -1, or, if n >25, then the set of lines of a pseudo - complement of a quadrilateral of

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order n consists of 3 lines of size n -1, 6(n -2) lines of size n -2 and $n^2 - 5n + 6$ lines of size n -3. Also, if n >21 and it has at least one line of size n -1, then it is embeddable in a unique projective plane of order n. These results improve the results of [1, 2].

2. On the Structure of Finite Pseudo - Complements of Quadrilaterals

Now on, D will denote a pseudo-complement of a quadrilateral of order n, n >3.

We call a line of D a β-Line if y is of size $n - \beta$.

It can be easily verified (see,[2]) that:

Lemma 2.1. In D,

(i) a β-Line is disjoint with exactly $n(\beta+1) - 4$ lines;

(ii) a β-Line y and a δ-Line $z, y \neq z$, are mutually disjoint with exactly $(n-1)(1+|y,x|)+(\beta+|y,z|)(\delta+|y,z|) - 4$ lines;

(iii) a point P not in a β-Line y is contained in exactly $\beta + 1$ lines disjoint with y.

Let a_i be the number of (n -i)-Lines of D. Then $a_{n+1} = 0$. Also

$$\sum_i a_i = n^2 + n - 3, \quad (1)$$

and, by simple counting methods,

$$\sum_i ia_i = (n+1)\ (n^2 - 3n + 3)$$
$$\sum_i i(i-1)\ a_i = (n^2 - 3n + 3)(n^2 - 3n + 2) \quad (2)$$

Hence

$$\sum_i (i - n + 2)(i - n + 3) = 6$$

Thus

$$3a_n + a_{n-1} + a_{n-4} + 3a_{n-5} = 3 \quad (3)$$

and for i, $i \leq n - 6$, $a_i = 0$

Lemma 2.2. D cannot have a 0-Line.

Proof. Let $a_n \geq 1$. Then, by (3), $a_n = 1$, and therefore, $a_{n-1} = a_{n-4} = a_{n-5} = 0$. Let y be the 0-Line of D. Then, by Lemma 2.1.(i,iii), y is disjoint with exactly n - 4 lines which are mutually disjoint and each of which is either a 2-Line or a 3-Line. Let α of these be 2-Lines. Then, by counting the total number of flags (P,z), $P \in y$, $|y,z| = 0$, we get.

$\alpha(n - 2) + (n - 4 - \alpha)(n - 3) = n^2 - 4n + 3$,

whence, a = 3n - 9. But $\alpha \leq n - 4$. Thus, we must have

$$3n - 9 \leq n - 4,$$

from which, $n \leq 2$, a contradiction.

Lemma 2.3 If n > 18, then D cannot have a 5-Line.

Proof. Let $a_{n-5} \geq 1$. Then by [2], $a_{n-5} = 1$, and therefore, $a_{n-1} = a_{n-4} = 0$. Let y be the 5 - Line. Then, by Lemma 2.1 (i,iii), y is disjoint with exactly 6n - - 4 lines each of which is a 2-Line or a 3-Line. Besides, each point not in y is contained in exactly 6 lines disjoint with y. Hence, if α and β denote the number of the z-lines and the 3-Lines disjoint with y, respectively, we have

$$\alpha + \beta = 6n - 4$$

Also, by counting as in Lemma 2.2,

$$\alpha(n-2) + \beta(n - 3) = 6(n^2 - 4n + 8)$$

Hence,

$$\alpha = -2n + 36$$

But, as $\alpha \geq 0$, we get $n \leq 18$, which is a contradiction.

Lemma 2.4. In D, no 1-Line can be disjoint with any 4-Line.

Proof. Suppose y is a 1-Line of D and α, β, γ and δ be the number of 1-Lines, 2-Lines, 3-Lines and 4-Lines disjoint with y respectively. Then, by (3), $0 \leq \alpha + \delta \leq 2$ also, by Lemma 2.1 (i), we have

$$\alpha + \beta + \gamma + \delta = 2n - 4$$

Now, if we count as in the Lemma 2.2, we get

$$\alpha(n - 1) + \beta(n - 2) + \delta(n - 3) + \gamma(n - 4) = 2(n - 2)^2,$$

whence,

$$\gamma = \alpha - 2\delta$$

But, $\gamma \geq 0$, and therefore, $\alpha \geq 2\delta$, which forces $\delta = 0$.

Using the same techniques as in the proof of Theorem 3 [2], and the Lemma 2.4, one can easily conclude that:

Lemma 2.5. If $n > 9$ and y is a 1-Line of D, then any line of D disjoint with y is a 2-Line.

Lemma 2.6. If $n > 25$, or, if $n > 18$ and D contains at least one 1-Line, then D cannot have a 4-Line.

Proof. Let y be a 4-line of D. Then, by [2], $1 \leq a_{n-4} \leq 3$, and $0 \leq a_{n-1} \leq 2$. Also, by Lemmas 2.1 (i,iii) and 2.4, y is disjoint with exactly 5n - 4 other lines, each of which is an i-Line, i = 2, 3, 4. Besides, each point not on y is contained in exactly 5 lines disjoint with y. Thus, if α, β and γ are the number of 2-Lines, 3-Lines and 4-Lines disjoint with y, respectively, then we have

$$\alpha + \beta + \gamma = 5n - 4$$

Also, by counting as in Lemma 2.2, we get

$$\alpha(n - 2) + \beta(n - 3) + \gamma(n - 4) = 5\ (n^2 - 4n + 7)$$

whence,

$$n = 23 + \gamma - \alpha \qquad (4)$$

Case I. D contains no 1-Line. Then, by [2] and [3], $n \leq 25$, which is a contradiction.

Case II. D has at least one 1-Line. Then, by Lemmas 2.1 (ii), 2.4 and 2.5, $\alpha = 6$, and thus, by [3], $n \leq 18$, which is again a contradiction.

Now, using Lemmas 2.1, 2.3, and 2.6, we can state the following theorem:

Theorem 2.7. Let D be pseudo-complement of a quadrilateral of order n. If $n > 25$, or, if $n > 18$ and D contains at least one 1-Line, then, the set of lines of D consists of 3 1-Lines, $6(n - 2)$ 2-Lines, and $n^2 - 5n + 6$ 3-Lines.

3. Embedding

Throughout this section, D will denote a pseudo-complement of a quadrilateral of order n, containing at least one 1-Line. Then, by theorem 2.7, if $n > 18$, the set of lines of D consists of three 1-Lines, $6(n - 2)$ 2-Lines, and $n^2 - 5n + 6$ 3-Lines. We define a simple-graph G in which $V(G)$ is the set of lines of D and two vertices are joined if and only if the corresponding lines of D are disjoint. Then, we call G the line graph of D. We call a vertex P of G a β-vertex if its corresponding line of D is a β-line. Then, by Lemma 2.1:

Lemma 3.1.

(i) If P is a β-vertex, then $d(P) = n(\beta + 1) - 4$.

(ii) If P is a β-vertex and Q is a δ-vertex with $P \neq Q$, then

$$l(P,Q) = \left\{ \begin{array}{ll} n - 5 + \beta\delta & \text{if } m(P,Q) = 1, \\ (\beta + 1)(\gamma + 1) - 4 & \text{otherwise.} \end{array} \right\}$$

(iii) For a β-vertex P, there exists a claw (P, S) of order $\beta + 1$.

It has been proved in [1] that:

Lemma 3.2. If $n > 14$, then a 1-vertex P is contained in exactly two maximal cliques H and K of size $n - 1$ such that each contains $n - 2$ 2-vertices, $H \cap K = \{P\}$, and no 2-vertex in H is joined to any 2-vertex in K.

Lemma 3.3. If $n > 14$, then every i-vertex, $i = 2$, 3, of G not joined to a 1-vertex P of G is joined to exactly $i - 1$ vertices in every maximal clique containing P.

Lemma 3.4. Let P be a 2-vertex of G.

(i) If $n > 19$, then there exists a claw $(P, \{R_1, R_2, R_3\})$ in which R_1 is a 1-vertex and the others are 2-vertices. Furthermore, such a claw is not extendable.

(ii) If $n > 21$, then there does not exist a claw $(P, \{R_1, R_2, R_3, R_4\})$ in which R_i, $i = 1, 2$, is an i-vertex, and each of the others is a 3-vertex.

Lemma 3.5. If $n > 21$, then every 2-vertex P is contained in exactly three maximal cliques K_i, $i = 1, 2, 3$, such that K_1 consists of a unique 1-vertex and $n - 2$ 2-vertices, and each of the others consists of three 2-vertices and $n-3$ 3-vertices. Furthermore, $K_i \cap K_j = \{P\}, 1 \leq i \neq j \leq 3$.

Proof. Let $(P, \{R_1, R_2, R_3\})$ be a claw in which R_1 is a 1-vertex and the others are 2-vertices. By Lemma 3.4 (i), such a claw exists and is maximal. Clearly, P and R_1 are contained in a unique maximal clique K_1 of size $n - 1$ containing $n - 2$ 2-vertices. Let $M = \{R_1, R_2\}$ and N be the set of all vertices joined to P but not to any vertex in M. Then, by Lemmas 3.1 and 3.4, N is a clique and $R_3 \in N$. Let $T = V(G) - M$, and consider the expression

$$A = \sum_{X \in T} m(P, X)\,(1 - m(R_1, X) - m(R_{2,x}))$$

It is easily seen that the contribution of each vertex X to A is 1 if $X \in N$, and is non-positive otherwise. Hence, by Lemma 3.1 (i,ii),

$$|N| \geq A = n - 2.$$

Thus, if K_3 is a maximal clique including $\{P\} \cup N$, then $|K_3| \geq n - 1$. In a similar fashion, one can prove that R_2 is also contained in a maximal clique K_2 of size at least $n - 1$.

Suppose $X \in (K_i \cap K_J) - \{P\}$, $1 \leq i \neq j \leq 3$. Then, as by Lemma 3.1 (ii),

$$|K_i \cup K_j| \leq l(P, X) + 2 \leq n + 3, |K_i \cap K_j|, \leq l(R_i, R_j) \leq 5,$$

we have

$$2(n - 1) \leq |L_i| + |K_J| \leq n + 8,$$

whence $n \leq 10$. Thus, $K_i \cap K_J = \{P\}$, $1 \leq i \neq j \leq 3$.

Case I. There is only one vertex X not in $\cup_{i=1}^{3} K_i$. Then, one of the K_i's, i = 2, 3, say K_2, must be of size n, and thus, $|K_3| = n - 1$.

Suppose X is a 2-vertex. Then, by Lemmas 3.1 (ii), 3.3, and the maximality of the K_i's, i = 2,3, X can be joined to at most 14 vertices in common with P. Hence, by Lemma 3.1 (ii), we should have

$$n = 1 \leq 14,$$

whence, $n \leq 15$, which is a contradiction.

Suppose X is a 3-vertex. Then, as the structure corresponding to G, i.e., D, has $n^2 - 3n + 3$ points, and by Lemmas 3.1(ii), 3.2 and 3.3, the total number of points on the lines corresponding to the vertices in

K_2 is also $n^2 - 3n + 3$, X can be disjoint with exactly one 1-vertex in $K_1 - \{P\}$ and 2 vertices in $K_2 - \{P\}$. Also, by Lemma 3.1(ii) and the maximality of K_3, X cannot be disjoint with more than 11 vertices in $K_3 - \{P\}$. Thus, by Lemma 3.1(ii),

$$n + 1 \leq 14,$$

whence, $n \leq 13$, which is a contradiction.

Case II. There are two vertices X and Y, outside $\cup_{i=1}^{3} K_i$. Suppose one of X and Y, say X, is a 2-vertex. Then, by arguing as in the second paragraph of case I, we get

$$n - 1 \leq 15$$

whence, $n \leq 16$, a contradiction.

Let both X and Y be 3-vertices. Then, as the structure corresponding to G, i.e. D, has $n^2 - 3n + 3$ points, and by Lemma 3.2 and 3.3, the total number of points on the lines corresponding to the vertices in each of K_2 and K_3 is

$$(n - 3)(n - 3) + 3(n - 2) = n^2 - 4n + 6,$$

there are at most 4 vertices in $K_2 \cup K_3 - \{P\}$, commonly joined to X and Y. But, by Lemmas 3.1(*ii*) and 3.3, each of X and Y must be joined to at least n - 1 vertices in $K_2 \cup K_3 - \{P\}$, and $n>21$. Hence, there must be a vertex Z in $K_2 - \{P\}$ joined to X but not to Y, and a vertex T in $K_3 - \{P\}$ joined to Y but not to X such that z and T are not joined. Then, $(P,. \{R_1, Z, T\})$ is a maximal claw. Therefore, if we argue as in the first paragraph, we conclude that X should be contained in a maximal clique K of size at least $n - 3$ containing P, and $K \cap K_i = \{P\}$, unless $n \leq 19$. So, as $n>21$, $K \cap K_i = \{P\}$, $i = 1,2,3$ and thus, every vertex joined to P must be in one of the K_i's. Also, as $n > 21$, and the structure corresponding to G, i.e. D, has $n^2 - 3n + 3$ points, we should have $|K_i| \leq n$, $i = 2,3$. Thus $|K_i| = n$, $i = 2,3$.

Now we prove that K_2 and K_3 are unique. Suppose K is a maximal clique of size n which contains P and is different from both K_2 and K_3. Clearly, K must intersect at least one of $K_2 - \{P\}$ and $K_3 - \{P\}$, say, $K_2 - \{P\}$. Then, by Lemma 3.1 (*ii*), $|K \cap K_2| \leq 12$, $|K \cap K| \leq 3$, and therefore, $n = |K| \leq 15$, a contradiction. Now, by Lemmas 3.3 and 3.4 the Lemma follows.

By arguments exactly the same as in Lemma 3.10 of the author [1], we get:

Lemma 3.6. If $n>21$, then a 3-vertex P of G is contained in exactly four maximal cliques K_i, $1 \leq i \leq 4$, of size n. Each consists of three 2-vertices and n - 3 3-vertices. Furthermore,

$$K_i \cap K_j = \{P\},\ 1 \leq i \neq j \leq 4$$

In terms of structures, Lemmas 3.2, 3.5 and 3.6, can be stated as follows:

Lemma 3.7. A 1-line y of a pseudo-complement of a quadrilateral of order $n, n>14$ is contained in exactly two classes H and K of size $n-1$ such that:

(i) each class contains n-2 2-lines;

(ii) $H \cap K = \{y\}$;

(iii) two lines are in the same class if and only if they are disjoint.

Lemma 3.8. A 2-line y of a pseudo-complement of a quadrilateral of order, $n > 21$, containing at least one 1-line, is contained in exactly 3 classes K_i, $1 \leq i \leq 3$, such that:

(i) K_1 consists of one 1-line and n-2 2-lines;

(ii) K_i, $i = 2,3$, consists of three 2-lines and n - 3 3-lines;

(iii) any two distinct lines in the same class are disjoint;

(iv) no line in K_i is disjoint with all lines in K_j, $1 \leq i \neq j \leq 3$;

(v) $K_i \cap K_J = \{y\}$, $1 \leq i \neq j \leq 3$.

Lemma 3.9. A 3-line y of a pseudo-complement of a quadrilateral of order $n, n>21$, containing at least one 1-line is contained in exactly four classes $K_i 1 \leq i \leq 4$, of size n such that:

(i) each K_i, consists of three 2-lines and n-3 3-lines;

(ii) $K_i \cap K_J = \{y\}$, $1 \leq i \neq j \leq 4$;

(iii) any two distinct lines in the same class are disjoint;

(iv) no line in K_i is disjoint with all lines in K_j, $1 \leq i \neq j \leq 4$.

By Lemmas 2.1(iii), 3.7, 3.8 and 3.9:

Corollary 3.10. Every point of a pseudo-complement of a quadrilateral of order n, $n>21$, containing at least one 1-line is contained in a unique line of every class described in Lemmas 3.7, 3.8 and 3.9.

By arguments exactly the same as in Lemma 3.10 of the author [1], we have:

Lemma 3.11. A pseudo - complement of a quadrilateral of order n, $n > 21$, containing at least one 1-line, has exactly $4n$ - 2 classes of the type described in Lemmas 3.7, 3.8 and 3.9.

Let D be a pseudo-complement of a quadrilateral of order n, $n > 21$, containing at least one 1-line. If we add as new points all classes described in Lemmas 3.7, 3.8 and 3.9 to every line contained in them, then by Corollary 3.10 and Lemma 3.11, we get a larger structure D' whose dual $(D')^t$ is a non-trivial $(n+1)$-regular linear space with $n^2 + n - 3$ points and $n^2 + n + 1$ lines. But, as Vanstone [3] has proved, $(D')^t$ is uniquely embeddable into a projective plane of order n and the dual of a finite projective plane is also a finite projective plane.

Thus:

Theorem. A pseudo-complement of a quadrilateral of order n, n >21, containing at least one 1-line is uniquely embeddable into a projective plane of order n.

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