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On Commutators of Isometries and Hyponormal Operators

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Abstract

A sufficient condition is obtained for two isometries to be unitarily equivalent. Also, a new class of M-hyponormal operator is constructed.

The aim of this article is to extend results due to Hoover [5] about the equivalence of quasisimilar isometries and to Clary [1] about the equality of the spectra of quasisimilar hyponormal operators. For $S \in B(H)$ and $T \in B(K)$, let $C(S,T): B(K,H) \longrightarrow B(K,H)$ denote the commutator of S and T defined by $C(S,T)A = SA - AT$, where K and H are separable Hilbert spaces. A linear transformation A is called a quasiaffinity if it is injective and has a dense range. Hoover [5] showed that if S and T are isometries, and if A and B are quasiaffinities such that,

(1) $C(S,T)A=0$ and $C(T,S)B=0$,

then S and T are unitarily equivalent. Our Theorem 1 shows that Hoover's result remains true if condition (1) is replaced by the weaker condition

(2) $C^n(S,T)A=0$ and $C^n(T,S)B=0$,

where n is a natural number. (Here $C^n(S,T)$ denotes n times application of $C(S,T)$.) (See also [3, pp. 217 - 226].)

Also, Clary [1] showed that if S and T are hyponormal and A and B are quasiaffinities satifying (1), then $\sigma(S) = \sigma(T)$. In Theorem 2, we will show that the equality of spectra remains valid if condition (1) is replaced by the following condition

(3) $\lim_n \| C^n(S,T)A \|^{1/n} = \lim_n \| C^n(T,S)B \|^{1/n} = 0.$

(Note that similar subnormal operators need not be unitarily equivalent [4, Problem 156]; see also example 14.9 on page 220 of [3].)

In the proof of Theorem 2, hyponormality of S and T are used only to prove that certain manifolds are closed and for this, M-hyponormality is quite sufficient. Our Theorem 3, introduces new examples of M-hyponormal operators, and hence, extends Theorem 2 for a wider class of operators.

To prove the main results we need the following lemmas. Throughout the remainder of this article E_N denotes the resolution of the identity for a normal operator N, and D_T denotes the domain of a linear transformation T.

Lemma 1 (Colojoara-Foias [2, page 48]).
Let $S \in B(H)$, $T \in B(K)$, and $A \in B(K,H)$ be such that $\lim_n \|C^n(S,T)A\|^{1/n}=0$. Let $f: G \subset C \to K$ be an analytic function such that $(z-T)f(z) \equiv x$ for some $x \in K$ and let

(4) $g(z)=\sum_{n=0}^{\infty}(-1)^n C^n(S,T)Af^{(n)}(z)/n!$

for $z \in G$. Then $(z-S)g(z) \equiv Ax$, and g is analytic on G.

The following lemma is well-known.

Lemma 2. Let $S \in B(H)$ and m be a natural number. Then $x \in \ker S^m$ if and only if there exist vectors $a_1, \ldots, a_m \in H$ such that

$$(z-S)(a_1/z+a_2/z^2+\ldots+a_m/z^m)=x$$

for all $z \neq 0$.

Lemma 3. Let $N \in B(H)$ be a normal operator and let $x \in H$. Assume there exists an analytic function $f: G \longrightarrow H$ such that $(z-N)f(z) \equiv x$. Then $E_N(G)x = 0$.

The proof is well-known. It follows from the fact that if σ is any closed subset of G and $T = N|E_N(\sigma)H$, then $E_N(\sigma)f(z)$ and $(z-T)^{-1}E_N(\sigma)x$ together define an entire function g such that $(z-T)g(z) \equiv E_N(\sigma)x$. Hence $E_N(\sigma)x=0$ and thus $E_N(G)x=0$.

The following lemma extends a result of Conway [3, page 222].

Lemma 4. Let S and T be subnormal and A and B be quasiaffinities satisfying condition (3). Then S and T have unitarily equivalent normal parts. Moreover, A (resp. B) maps D_N (resp. D_M) into D_M (resp. D_N), and $AN=MA|D_N$ (resp. $BM=NB|D_M$), where M and N are the normal parts of S and T, respectively.

Proof. Let $A_1: D_N \to D_M$ and $Y: D_N \to D_M^{\perp}$ be such that $Ax=A_1x+Yx$ for all $x \in D_N$. Let $S=M \oplus S_1$ and $T=N \oplus T_1$, where S_1 and T_1 are the pure parts of S and T, respectively. Then

$$\lim \|C^n(M,N)A_1\|^{1/n} = \lim \|C^n(S_1,N)Y\|^{1/n}=0.$$

By [10, Corollary 1], $S_1 \,|\, (YD_N)^-$ is normal and hence $Y=0$. Thus A_1 is injective and again, by [10, Lemma 2], N is unitarily equivalent to a reducing part of M. Similarly, M is unitarily equivalent to a reducing part of N and hence M and N are unitarily equivalent [6]. The rest of the proof follows from [10, Corollary 1].

Now, we are ready to prove our main results.

Theorem 1. Let S and T be isometries and let A and B be quasiaffinities satisfying condition (2). Then S and T are unitarily equivalent.

Proof. By Lemma 4, S and T have unitarily equivalent unitary parts M and N; moreover,

$$A=\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix},$$

where $A_3:D_N^{\perp}\rightarrow D_M^{\perp}$ and $B_3:D_M^{\perp}\rightarrow D_N^{\perp}$ have dense ranges. Let, up to unitary equivalence, $W^{(k)}$ and $W^{(j)}$ be respectively the pure parts of S and T, where W is the simple unilateral shift and k and j are finite or countable cardinalities. (W^0 means the zero operator on the zero space). We have to show that $k=j$.

Note that A^*_3 and B^*_3 are injective and

$$C^*(W^{*(j)},W^{*(k)})A_3^*=0,\ C^*(W^{*(k)},W^{*(j)})B_3^*=0.$$

Let m be any natural number and let $x \in \ker (w^{*(k)})^m$. By Lemma2, there exists a function $f(z)=a_1z^{-1}+\ldots+a_mz^{-m}$ such that

$$(z-W^{*(k)})f(z)=x \qquad (z\neq 0)$$

Now, Lemma 1 implies that $(z-W^{*(j)})\,g(z)=A^*_3x$, where $g(z)=b_1z^{-1}+b_2z^{-2}+\ldots+b_{m+n}z^{-m-n}$. Thus $A^*_3x \in \ker (W^{*(j)})^{m+n}$ and hence $mk \leqslant (m+n)j$. Letting $m \longrightarrow \infty$, we see that $k \leqslant j$. Similarily, $j \leqslant k$ and hence $k=j$. Therefore, S and T are unitarily equivalent and the proof is complete.

Theorem 2. Let $S \in B(H)$ and $T \in B(K)$ be hyponormal. Let A and B be quasiaffinities satisfying condition (3). Then $\sigma(S)=\sigma(T)$.

Proof. Since condition (3) is unaffected if S and T are replaced by z-S and z-T respectively, it is sufficient to show that S is invertible if and only if T is invertible. (Here z is any complex number.)

Assume S is invertible and let $D=D(0;r)$ be an open neighbourhood of 0 contained in the complement of $\sigma(S)$. Let $x \in H$ and $f(z)=(z-S)^{-1}x$ for $z \in D$. By Lemma 1, there exists an analytic function $g: D \longrightarrow K$ such that $(z-T)\,g(z)\equiv Bx$. Thus B maps H into the set M_T consisting of all vectors $y \in K$ for which there exists an analytic function $g_y: D \longrightarrow K$ such that $(z-T)g_y(z)\equiv y$. Thus M_T is dense in K. By [7, Proposition 1] the set M_T is closed and hence $M_T = K$ and $\sigma(T) \cap D = \emptyset$ [2, page 23]. Similarly, S is invertible if T is so. The proof is complete.

As it is obvious from the proof, the hyponormality of S and T is used only to show that M_S and M_T are closed. In view of [9, Remark 3], these are closed if S and T are merely M-hyponormal. An operator T is called M-hyponormal if M is a positive constant and

(6) $\|(z-T)^*x\|\leqslant M\|(z-T)x\|$

for all $z\in\mathbb{C}$ and all $x \in D_T$. The following theorem gives new examples of M-hyponormal operators and hence extends Theorem 2 to a wider class of known operators. (See [14; 15] for previous known examples.)

Theorem 3. Let $T \in B(H)$ be a weighted shift with weights $\{a_1, a_2,\ldots\}$ such that $0 < a_n \leqslant 1$ $(n=1, 2, 3,\ldots)$. Define $b_1=0$ and $b_n=\max\left\{0,\ a^2_{n-1}-a^2_n\right\}$.

Assume

(7) $\sum\limits_{n=1}^{\infty} r^{-2n}b_n<\infty$

for some $r \in (0,1)$. Let $a = \liminf a_i$. Then the following assertions are true.

(a) If $a > r$, then T is M-hyponormal.

(b) If $a=0$, then $\lim a_i = 0$ and T is not M-hyponormal.

Proof. (a) We first note that condition (6) is equivalent with the following condition

(8) $TT^*-T^*T\leqslant(M^2-1)(z-T)^*(z-T) \quad (Z\in\mathbb{C})$.

Let $x \in H$ be an arbitrary unit vector. Let $\{e_1, e_2,\ldots\}$ be the orthonormal basis such that $Te_n = a_n e_{n+1}$ $(n=1,2,\ldots)$. If $x=\Sigma t_ie_i$, then

(8′) $((TT^*-T^*T)x,x)\leqslant\sum_{i>1}b_i|t_i|^2=g(x)$,

and

$$\|(z-T)x\|^2=|zt_1|^2+\sum_{i\geqslant 2}|a_{i-1}t_{i-1}-zt_i|^2=f(x,z),$$

say. We will show that

$$0<p=\inf\{f(x,z)/g(x):x\in H,\|x\|=1,\ |z|\geqslant r,\ g(x)\neq 0\}.$$

Assume, if possible, that p=0 and let

$$(1-r^2)^{-1}d^2\sum_{i\geqslant 1}b_i r^{-2i}<1,$$

for some $d>0$. Choose x and z such that $f(x, z)/g(x) <d^2$, $g(x) > 0$, $\|x\|=1$, and $|z|\geqslant r$. Let $g=g(x)$. Let

$s=r^{-1}$, $d_1=|zt_1|g^{-1/2}$ and $d_i=|a_{i-1}t_{i-1}-zt_i|g^{-1/2}$

for $i=2, 3,\ldots$. Then $\sum d_i^2 \leqslant d^2$ and, by induction on i,

$$g^{-1/2}|t_i|\leqslant sd_i+s^2d_{i-1}a_{i-1}+\cdots+s^id_1a_{i-1}\cdots a_1$$

for $i \geqslant 2$. Using Schwartz inequality and the fact that $a_i \leqslant 1$, it follows that

$$g^{-1/2}|t_i|\leqslant s^{i+1}(s^2-1)^{-1/2}d,$$

for $i \geqslant 2$. Thus

$$g(x)=\sum b_i|t_i|^2<g(x)d^2\sum b_ir^{-2i}(1-r^2)^{-1}$$

a contradiction. Hence $p > 0$ and

(9) $g(x)\leqslant p^{-1}f(x,z)$,

for all x and z such that $\|x\|=1$ and $|z|\geqslant r$. (The inequality (9) trivially holds for $g(x) = 0$.) Now, it follows from (8′) and (9) that if $|z|\geqslant r$, then $(TT^*-T^*T)\leqslant(M^2-1)(z-T)^*(z-T)$, where $M^2-1=p^{-1}$.

Now, assume $a > r$ and $|z| \leqslant r$. In view of [4, Problem 76], T is similar to a weighted shift S with weights $\{r_1, r_2,\ldots\}$ such that

$|r_n|>(r+a)/2$ for $n=1,2,\ldots$. Thus, if $T=ASA^{-1}$,

$$\begin{aligned}\|(z-T)x\| &\geqslant\|A^{-1}\|^{-1}(\|SA^{-1}x\|-|z|\,\|A^{-1}x\|)\\ &\geqslant\|A^{-1}\|^{-1}\Big[|z|-(r+a)/2\Big]\|A^{-1}x\|\\ &\geqslant\|A\|^{-1}\|A^{-1}\|^{-1}(a-r)/2=E>0,\end{aligned}$$

for all $x \in H$ with $\|x\|=1$ and all z with $|z|\leqslant r$. Hence $k(z-T)^*(z-T)\geqslant kE^2\geqslant TT^*-T^*T$, where $k=E^{-2}\|TT^*-T^*T\|$. In view of (8), the proof of (a) is complete.

(b) Assume $a=0$ and, if possible, $s=\limsup a_i > 0$. Let $\epsilon = s^2/3$ and choose natural numbers k and N such that $k > N$, $\Sigma_{n\geqslant N}\, b_n < \epsilon$, and $a_k^2>s^2-\epsilon$. Then $a_i^2 \geqslant a_k^2-\sum_{n\geqslant k}b_n>s^2-2\epsilon=\epsilon$ for all $i\geqslant k$, a contradiction. Thus $\lim a_i = 0$ and T is a completely nonnormal, compact, quasinilpotent operator. Such operators cannot be M-hyponormal [12; 15]. The proof of the theorem is complete.

Corollary (Thatte-Joshi [14], Wadhwa [15]). The weighted shift T with weights $\{a_1, a_2,...\}$ is an M-hyponormal operator if $|a_n| = |a_{n+1}| = |a_{n+2}| = ...$ for some natural number n.

Examples. (i) Let $a_1 = 1$ and $a_k = (1 - \sum_{n=2}^{k} n^{-n})^{1/2}$, k=2, 3,.... Then the weighted shift T with weights $a_1, a_2, ...$ is M-hyponormal.

(ii) The weighted shift T with weights $a_n = (r/2)^n$ is not M-hyponormal ($0 < r \leqslant 1$). However, in view of [12; 13], T is dominant, i.e., for each $z \in C$ there exists $M_z > 0$ such that

$$\|(z-T)^* x\| \leqslant M_z \|(z-T)x\| \text{ for all } x \in D_T.$$

(iii) The weighted shift T with weights $a_n = n^{-n}$ is neither M-hyponormal nor dominant.

(iv) The weighted shift T with weights $\{2^{1/2}, 2^{-1/2}, 2^{1/2}, 2^{-1/2}, ...\}$ is similar to the simple unilateral shift but is not M-hyponormal. The similarity follows from [4, Problem 76]. The fact that T is not M-hyponormal is true for any T such that $|T^*T - TT^*|$ is invertible. If T is M-hyponormal, then it follows from [8, Theorem 2] that

$$(z-T)^* (z-T) \geqslant k^2 \ |T^*T - TT^*|^2$$

for all $z \in ℂ$, where k is a constant independent of z. Now, if $|T^*T - TT^*|$ is invertible, then $\|(z-T)x\| \geqslant \epsilon k$ for all $z \in ℂ$, where $\epsilon = \inf \sigma(|T^*T - TT^*|)$. Thus T has no approximate point spectrum, a contradiction.

Remarks. (i) We do not know whether or not Lemma 4 is true for hyponormal operators S and T. However, if S and T are hyponormal (or even dominant), if $C^n(S, T)A=0$, and if $C^n(T, S)B=0$ for some natural number n, then S and T have unitarily equivalent normal parts. For a proof, observe that, with the notation of the proof of Lemma 4, $C^n(S_1, N)Y=0$. By [10, Theorem 1], Y=0. The rest of the proof is the same as that of Lemma 4.

(ii) Assume S and T satisfy condition (3) for some quasiaffinities A and B. It follows from [10, Lemma 2] that S and T are unitarily equivalent if they are normal. We do not know whether S and T are unitarily equivalent if they are isometries. However, for general subnormal operators S and T even their similarity need not imply their unitary equivalence [4, Problem 156].

(iii) A revision of Theorem 2 reveals that if S is M-hyponormal, if T is arbitrary and if A is a quasiaffinity such that $\lim_n \|C^n(S,T)A\|^{1/n} = 0$, then $\sigma(S) \subset \sigma(T)$.

(iv) In Theorem 1, the proof of the unitary equivalence of the pure parts of S and T is based on a counting argument. Therefore, it cannot be applied to operators S and T satisfying condition (3). However, we can generalize Hoover's result in a different way. Assume

$$S = M \oplus W^{(k)} \text{ and } T = N \oplus W^{(j)}$$

where M and N are normal, and $W^{(i)}$ denotes the direct sum of i copies of a cyclic subnormal operator W for some countable cardinality i. Suppose S and T are quasisimilar, i.e., they satisfy condition (1) for some quasiaffinities A and B. Then S and T are unitarily equivalent. The equivalence of M and N follows from Lemma 4. Let A_3 and B_3 be as in the proof of Theorem 1. Then A_3^* and B_3^* are injective and

$$C(W^{*(j)}, W^{*(k)})A_3^* = 0, \ C(W^{*(k)}, W^{*(j)})B_3^* = 0.$$

By [11, Theorem 1], k=j and hence $W^{(k)} = W^{(j)}$.

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