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SOME RESULTS ON REDEFINED FUZZY SUBGROUPS

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Abstract

In this article we will use the redefined definition of a fuzzy subgroup as in [1], and will mainly prove that if G, G′ are two groups and f: G ⟶ G′ is an epimorphism, then there is a bijection between the set of all fuzzy subgroups of G/Ker f and that of G′.

Introduction

The concept of a fuzzy subset was introduced by L. A. Zadeh in [6]. In [4], A. Rosenfeld used this notion and developed the theory of fuzzy subgroups and obtained many basic results. Since then several other works were continued in this direction, for example, P. S. Das [2], N. P. Mukherjee and P. Bhattacharya [3], who have proved interesting results on fuzzy normal subgroups and fuzzy cosets . J. M. Anthony and H. Sherwood in [1], redefined the fuzzy subgroups in the light of a notion called t-norms. This notion was introduced by B. Schweizer and A. Sklar [5], in order to generalise the ordinary triangle inequality in a metric space to the more general probabilistic metric space.

In this article we will use the redefined definition of fuzzy subgroups as in [1], and will prove some results, among which a similar one may be seen as in theorem 4.8 of [3].

Preliminaries

We review briefly some definitions and results. For more details see references.

Definition 2. 1. Let S be a set. A fuzzy subset μ of S is a function μ: S ⟶ [0, 1].

Definition 2. 2. Let G be a group. A fuzzy subset μ of G is said to be a fuzzy subgroup of G, if

i) $\mu(xy) \geqslant \mathrm{Min}(\mu(x), \mu(y)); \forall x, y \in G$.

ii) $\mu(x^{-1}) = \mu(x); \forall x \in G$.

Definition 2. 3. A t-norm is a function $T: [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying, for each ϵ, λ, ξ in [0, 1]:

i) $T(0,0)=0, T(\lambda,1)=\lambda=T(1,\lambda)$,

ii) $T(\epsilon,\lambda) \leqslant T(\xi,\mu)$, if $\epsilon \leqslant \xi$ and $\lambda \leqslant \mu$,

iii) $T(\epsilon,\lambda)=T(\lambda,\epsilon)$,

iv) $T(\epsilon,T(\lambda,\xi))=T(T(\epsilon,\lambda),\xi)$.

Obviously the function Min defined on [0, 1] × [0, 1] is a t-norm. Other t-norms which are frequently encountered in the study of probabilistic metric spaces are T_m and Prod defined by $T_m(\lambda,\mu)=\mathrm{Max}(\lambda+\mu-1,0)$, Prod $(\lambda,\mu)=\lambda\cdot\mu$, for each λ,μ in [0,1].

Definition 2.4. Let G be a group. A fuzzy subset μ of G is said to be a fuzzy subgroup of G with respect to a t-norm T, if

i) $\mu(xy) \geqslant T(\mu(x),\mu(y)); \forall x,y \in G$,

ii) $\mu(x^{-1})=\mu(x); \forall x \in G$.

In definition 2.4, if G is only a groupoid and condition ii) is removed, then μ is called a fuzzy subgroupoid of G with respect to t-norm T.

In [1], there are given examples which are not fuzzy subgroup with respect to t-norm Min, but they are so with respect to T_m.

Definition 2.5. A t-norm T is continuous if it is continuous function (with respect to the usual topologies) from [0, 1] × [0, 1]. into [0, 1].

Notice that Min, T_m, and Prod are all continuous t-norms.

Definition 2.6. If μ is a fuzzy subset of S, and f: S ⟶ S′ is a surjection function, we let μ^f: S′ ⟶ [0, 1], be defined by,

$$\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x); \ \forall y \in S'.$$

The fuzzy subset μ^f of S′ is called the image of μ under f.

Lemma 2.7. If T is any continuous t-norm and Λ, Λ' are any index sets, then

$$\sup_{\alpha\in\Lambda,\ \beta\in\Lambda'} T(x_\alpha,y_\beta) = T(\sup_{\alpha\in\Lambda} x_\alpha, \sup_{\beta\in\Lambda'} y_\beta).$$

Proof. Let $x^* = \sup_{\alpha\in\Lambda} x_\alpha$ and $y^* = \sup_{\beta\in\Lambda} y_\beta$. Since

$$x_\alpha \leqslant x^*; \forall \alpha \in \Lambda,\ y_\beta \leqslant y^*; \forall \beta \in \Lambda'$$

So, by definition 2.3 (ii), we have

$$T(x_\alpha, y_\beta) \leqslant T(x^* y^*);\ \forall \alpha \in \Lambda,\ \forall \beta \in \Lambda',$$

thereby,

$$(1) \qquad \sup_{\alpha \in \Lambda,\ \beta \in \Lambda'} T(x_\alpha, y_\beta) \leqslant T(x^*, y^*).$$

Now we prove the converse of (1). Since T is continuous, it follows that for all $\epsilon > 0$, there is $\delta > 0$, such that

$$|T(x,y) - T(x^*, y^*)| < \varepsilon,$$

whenever,

$$|x - x^*| < \delta \text{ and } |y - y^*| < \delta$$

Choose $\alpha \in \Lambda$, $\beta \in \Lambda'$ such that $x_\alpha > x^* - \delta$ and $y_\beta > y^* - \delta$. Then

$$T(x^*, y^*) - T(x_\alpha, y_\beta) < \varepsilon.$$

Thus,

$$T(x_\alpha, y_\beta) > T(x^*, y^*) - \varepsilon.$$

Therefore,

$$\sup_{\alpha \in \Lambda,\ \beta \in \Lambda'} T(x_\alpha, y_\beta) > T(x^*, y^*) - \varepsilon; \forall \varepsilon > 0.$$

Letting $\epsilon \to 0$, we get

$$(2) \qquad \sup_{\alpha \in \Lambda,\ \beta \in \Lambda'} T(x_\alpha, y_\beta) \geqslant T(x^*, y^*).$$

Hence the proof is completed by using (1) and (2).

Results

In order to prove the main result of this part in Theorem 3.9, we will prove several lemmas.

Lemma 3.1 Let G and G′ be two groups and f: $G \to G'$ be an epimorphism. If μ is a fuzzy subgroups of G with respect to a continuous t-norm T, then μ^f is a fuzzy subgroup of G′ with respect to T.

Proof. For $y_1, y_2 \in S'$, let $A_i = f^{-1}(y_i)$; $i = 1,2$, $A_{12} = f^{-1}(y_1 y_2)$, and $A_1 A_2 = \{x_1 x_2 | x_i \in A_i;\ i = 1,2\}$. If $x \in A_1 A_2$, then $x = x_1 x_2$ for some $x_i \in A_i$; $i = 1,2$.

So we have $f(x) = f(x_1 x_2) = f(x_1) f(x_2) = y_1 y_2$.

Thereby, $x \in A_{12}$. Hence, $A_1 A_2 \subset A_{12}$, and

$$\mu^f(y_1 y_2) = \sup_{x \in f^{-1}(y_1 y_2)} \mu(x) \geqslant \sup_{x_1 \in A_1,\ x_2 \in A_2} \mu(x) = \sup_{\substack{x_1 \in A_1,\ x_2 \in A_2 \\ x = x_1 x_2}} \mu(x_1 x_2)$$

$\geqslant \sup_{x_1 \in A_1,\ x_2 \in A_2} T(\mu(x_1), \mu(x_2))$; since μ is a fuzzy subgroup of G with respect

$= T(\sup_{x_1 \in A_1} \mu(x_1), \sup_{x_2 \in A_2} \mu(x_2))$; by Lemma 2.7

$= T(\mu^f(y_1), \mu^f(y_2))$; by Definition 2.6

That is,

$$\text{i)}\ \mu^f(y_1 y_2) \geqslant T(\mu^f(y_1), \mu^f(y_2));\ \forall y_1, y_2 \in S'.$$

Since F is an empimorphism, We have

$$x^{-1} \in f^{-1}(y) \Leftrightarrow f(x^{-1}) = y \Leftrightarrow (f(x))^{-1} = y \Leftrightarrow f(x) = y^{-1} \Leftrightarrow x \in f^{-1}(y^{-1}).$$

Therefore,

$$\mu^f(y^{-1}) = \sup_{x \in f^{-1}(y^{-1})} \mu(x) = \sup_{x \in f^{-1}(y^{-1})} \mu(x^{-1}) = \sup_{x^{-1} \in f^{-1}(y)} \mu(x^{-1}) = \sup_{t \in f^{-1}(y)} \mu(t) = \mu^f(y)$$

That is, ii) $\mu^f(y^{-1}) = \mu^f(y); \forall y \in S'$

Hence the proof is complete by definition 2.4.

Corollory 3.2. If f: $G \to G'$ in Lemma 3.1 is an isomorphism then μ^f is a fuzzy subgroup of G′ such that for any $y \in S'$, $\mu^f(y) = \mu(x)$, where $x = f^{-1}(y)$.

Remark 3.3. Note that Lemma 3.1 is a generalisation for Proposition 4 of [1]. Although a part of Lemma 3.1 is proved in [1], but here in the light of Lemma 2.7, it is given for the convenience of the reader.

Notation . For a group G, and a t-norm T, we let F (G) = $\{\mu | \mu$ is a fuzzy subgroup of G with respect to T$\}$.

Lemma 3.4. Let f: $G \to G'$ be an isomorphism between two groups G and G′, and T be a continuous t-norm. Then there exists a bijection between F (G) and F (G′), and we write $F(G) \rightleftarrows F(G')$.

Proof. Define $\bar{f}$: $F(G) \to F(G'); \mu \to \mu^f$, then by Corollory 3.2, $\bar{f}$ is well-defined.

$\bar{f}$ is surjective:

If $\mu' \in F(G')$, then for $(\mu' f^{-1}) \in F(G)$, $\bar{f}(\mu' f^{-1}) = \mu'$, because for any $y \in G'$, by Corollory 3.2,

$$\bar{f}(\mu' f^{-1})(y) = (\mu' f^{-1})^f(y) = \mu' f^{-1}(x) = \mu'(y);$$

where $y = f(x)$, for the unique $x \in G$.

$\bar{f}$ is injective:

If $\bar{f}(\mu_1) = \bar{f}(\mu_2)$, Then $\mu_1^f = \mu_2^f$, so $\mu_1^f(y) = \mu_2^f(y)$ for any $y \in G$, therefore, by Corollory 3.2, we get

$$\mu_1(x) = \mu_2(x).$$

If N is a normal subgroup of a group G, we write $N \lhd G$.

Lemma 3.5. If $N \lhd G$, and μ is a fuzzy subgroup of G with respect to a continuous t-norm T, then

$$\bar{\mu}: G/N \to [0,1]$$
$$\bar{g} \to \sup_{h \in \bar{g}} \mu(h),$$

is a subgroup of G/N with respect to T, where $\bar{g}$ is the right coset of g.

Proof. i) For any $\bar{g}_1, \bar{g}_2 \in G/N$,

$$\bar{\mu}(\bar{g}_1 \bar{g}_2) = \bar{\mu}(\overline{g_1 g_2}) = \sup_{h \in \overline{g_1 g_2}} \mu(h) = \sup_{h \in \bar{g}_1 \bar{g}_2} \mu(h) = \sup_{\substack{h_1 \in \bar{g}_1,\ h_2 \in \bar{g}_2 \\ h = h_1 h_2}} \mu(h_1 h_2)$$

$\geqslant \sup_{h_1 \in \bar{g}_1,\ h_2 \in g_2} T(\mu(h_1), \mu(h_2))$, by Definition 2.4,

$= T(\sup_{h_1 \in \bar{g}_1} \mu(h_1), \sup_{h_2 \in \bar{g}_2} \mu(h_2))$, by Lemma 2.7

$= T(\bar{\mu}(\bar{g}_1), \bar{\mu}(g_2))$

ii) For any $\bar{g} \in G/N$, note that $h \in (\bar{g})^{-1} \Leftrightarrow h^{-1} \in \bar{g}$, so

$$\bar{\mu}((\bar{g})^{-1}) = \sup_{h \in (\bar{g})^{-1}} \mu(h) = \sup_{h \in (\bar{g})^{-1}} \mu(h^{-1}) = \sup_{h^{-1} \in \bar{g}} \mu(h^{-1}) = \sup_{t \in \bar{g}} \mu(t) = \bar{\mu}(\bar{g}).$$

Remark 3.6. It is easy to see that if $N \lhd G$, and μ_0 is a fuzzy subgroup of G/N with respect to a continuous t-norm, T, then $\mu: G \longrightarrow [0,1]; \; g \longrightarrow \mu_0(\bar{g})$ is a fuzzy subgroup of G with respect to T, induced by μ_0.

Lemma 3.7. For $N \lhd G$, if we let

$\bar{F}(G/N) = \{\bar{\mu}$ as in Lemma 3.5, where $\mu \in F(G)\}$.

then $\bar{F}(G/N) = F(G/N)$

Proof. By Lemma 3.5, we have $\bar{F}(G/N) \subset F(G/N)$. Suppose $\mu_0 \varepsilon F(G/N)$, then by Remark 3.6, μ defined by $\mu'(g) = \mu_0(\bar{g})$; $g \varepsilon G$, is in $F(G)$.

But $\mu(h) = \mu(g)$

for all $h \varepsilon \bar{g}$, since $\mu(h) = \mu_0(\bar{h}) = \mu_0(\bar{g}) = \mu(g)$. Hence:

$$\bar{\mu}(g) = \sup_{h \varepsilon \bar{g}} \mu(h) = \mu(g) = \mu_0(g); \; \forall g \varepsilon G.$$

Thereby, $\mu_0 = \bar{\mu} \epsilon \bar{F}(G/N)$. Therefore $F(G/N) \subset \bar{F}(G/N)$, and the proof is complete.

Remark 3.8. The map $\varphi: F(G) \longrightarrow \bar{F}(G/N); \; \mu \longrightarrow \bar{\mu}$ is surjective.

In the next theorem we will prove a similar result like the first isomorphism theorem of groups as follows:

Theorem 3.9. If G, G′ are groups, $f: G \longrightarrow G'$ is an epimorphism and T is a continuous t-norm, then $\bar{F}(G/\mathrm{Ker}\, f) \simeq F(G')$.

Proof. Since G/Ker f and G′ are isomorphic, so by Lemmas 3.4 & 3.7 the proof is complete.

Definition 3.10. Let μ be a fuzzy subgroup of G with respect to a t-norm $T\mu$ is called a fuzzy normal subgroup of G with respect to T, if

$\mu(xy) = \mu(yx); \; \forall x, y \varepsilon G$.

We denote it by $\mu_T^{\lhd} G$.

Corollory 3.11. If $N \lhd G$ and $\mu_T^{\lhd} G$, then $\bar{\mu}_T^{\lhd} G/N$, where $\bar{\mu}$ is as in Lemma 3.5.

Proof. for any $\bar{g}_1, \bar{g}_2 \epsilon G/N$

$$\bar{\mu}(\bar{g}_1\bar{g}_2) = \bar{\mu}(\overline{g_1 g_2}) = \sup_{h \epsilon \overline{g_1 g_2}} \mu(h) = \sup_{h \epsilon \bar{g}_1 \bar{g}_2} \mu(h) = \sup_{h_1 \epsilon \bar{g}_1 \; h_2 \epsilon \bar{g}_2 \; h = h_1 h_2} \mu(h_1 h_2)$$

$$= \sup_{h_2 \epsilon \bar{g}_2 \; h_1 \epsilon \bar{g}_1} \mu(h_2 h_1), \text{ since } \mu_T^{\lhd} G$$

$$= \sup_{\substack{k = h_2 h_1 \\ h_2 \varepsilon \bar{g}_2, \, h_1 \varepsilon \bar{g}_1}} \mu(k) = \sup_{k \varepsilon \bar{g}_2 \bar{g}_1} \mu(k) = \sup_{k \varepsilon \overline{g_2 g_1}} \mu(k) = \bar{\mu}(g_2 g_1) = \bar{\mu}(\bar{g}_2 \bar{g}_1).$$

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