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THE INTERNAL IDEAL LATTICE IN THE TOPOS OF M-SETS

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Abstract

We believe that the study of the notions of universal algebra modelled in an arbitarry topos rather than in the category of sets provides a deeper understanding of the real features of the algebraic notions. [2], [3], [4], [5], [6], [7], [13], [14] are some examples of this approach. The lattice Id(L) of ideals of a lattice L (in the category of sets) is an important ingredient of the category of lattices. In this paper, we construct the (internal) ideal lattice $\mathcal{I}$(A) of a lattice A in the topos of M-sets for a monoid M. The process of the construction of $\mathcal{I}$(A) is so that it can also be done in any arbitrary topos whose ingredients are known. Finally, we consider the lattice structure of $\mathcal{I}$(A) for some special kind of lattices A in the topos of M-sets and show, among other things, that if A is an internally complete M-Boolean algebra then $\mathcal{I}$(A) is an M-Stone lattice.

1. The Topos of M-Sets

1.1. Recall that for a monoid M with identity e, a (left) M-set is a set X together with a function $\lambda: M x X \to X$, called the action of M on X, such that for $x \in X$ and m, $n \in M$ (denoting $\lambda(m, x)$ by mx)

i) ex=x

ii) (mn) x=m (nx)

A map $f: X \to Y$ between M-sets X,Y, such that for $x \in X$, $m \in M$, f(mx) = mf(x) is called an equivariant map. For any monoid M, the class of all M-sets and equivariant maps between them form a category denoted by MSet.

It is proved that the category MSet is isomorphic to the topos (see [9]) Set^M, where M is considered as a category with one object. So, MSet is a topos whose ingredients (limits, subobject classifier, exponentiation) are followed from the general topos Set^C, where C is a small category (see [8]).

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1.2. Limits and colimits in MSet are calculated in the same way as in the category Set, by defining the action of M on them in a natural way. In particular, the singleton {0} with the obvious M-action is the terminal object 1 of MSet. Monomorphisms (epimorphisms) in MSet are exactly one-one (onto) equivariant maps.

1.3. The subobject classifier Ω in MSet is the set L_M of all the left ideals of M (i.e. subsets S of M satisfying $mx \in S$, for $m \in M$, $x \in S$) with the action of M on $\Omega = L_M$ defined by $mS = \{x \in M : xm \in S\}$. Also, the truth arrow $t: 1 \to \Omega$ is given by t(0)=M.

Note that, M is a group if and only if $\Omega = \{0, M\}$.

1.4. The exponentiation B^A, for A, $B \in$ MSet, is the set Hom_{MSet} (MxA, B) with the action of M on it given by

$$(mf)(s, a) = f(sm, a)$$

for $m \in M$, $f \in B^A$.

The evaluation arrow $ev: B^A x A \to B$ is defined by

$ev(f, a) = f(e, a)$, for $f \in B^A$, $a \in A$. In particular, for any M-set B, $Hom_{MSet}(M, B) \cong B$, since $B^1 \cong B$.

Note that, if M is a group then the exponentiation B^A is isomorphic to the M-set $Hom_{Set}(A, B)$ with the action given by $(mg)(a)=mg(m^{-1}a)$, for $g:A \to B$, $m \in m$, $a \in A$.

1.5. The power object PA of A in the topos MSet is

$$\Omega^A = Hom_{MSet}(MxA, \Omega) \cong Sub(MxA)$$

where Sub (MxA) is the set of all subobjects (in MSet) of MxA with the action of M defined by

$$sX = \{(m, a) : (ms, a) \in X\}$$

for $X \in Sub(MxA)$, $s \in M$.

Note that, any subset X of MxA can be written as

$$x = \bigcup_{m \in M} \{m\} x X_m$$

where $X_m = \{a \in A: (m, a) \in X\}$, so we identify X by a family $(X_m)_{m \in M}$. Now, since any $X \in Sub(MxA)$ is a subset of MxA which is closed under the action of M, (i.e. $(sm, sa) = s(m, a) \in X$, for $s \in M$, $(m, a) \in MxA$), we have that the family $X=(X_m)_{m \in M}$ is in Sub (MxA) if and only if $sX_m \subseteq X_{sm}$, for every s, $m \in M$, where $sX_m = \{sa: a \in X_m\}$. Also, the action of M on $\Omega^A \cong$ Sub (MxA) then translates to $sX = (X_{ms})_{m \in M}$, for $s \in M$, $X=(X_m)_{m \in M} \in Sub(MxA)$.

It is easy to see that when M is a group, $\Omega^A \cong \rho(A)$ the power set of A with the action of M given by $mY=\{ma: a \in Y\}$, for $m \in M$, $Y \subseteq A$.

1.6. The membership relation $\in_A$, for $A \in MSet$; that is the pullback of $t:1 \to \Omega$ and $ev: \Omega^A x A \to \Omega$; is

$$\in_A = \{(X, a) : X = (X_m)_{m \in M} \in \Omega^A, a \in X_e\}$$

with the same action of M on $\Omega^A x A$. In fact, $\in_A = ev^{-1}\{M\}$.

2. Lattices in MSet

2.1. An (internal) lattice in the topos MSet is an M-set A which is also a lattice whose lattice maps, $\vee, \wedge$: $A \times A \to A$ are equivariant. We call such a lattice an M-lattice. (For convenience we take M-lattices with 0, and $m0=0$, for all m in M)

The class of all M-lattices and equivariant lattice homomorphisms between them form a category denoted by MLatt.

2.2. An (internal) bounded lattice in MSet or an M-bounded lattice is an M-lattice which has a greatest element 1 and a least element 0 (as a lattice) such that $m0=0$, $m1=1$, for every $m \in M$.

Example. For any monoid M and $A \in MSet$, the power object Ω^A is an M-bounded lattice with componentwise lattice operations. That is, for $X=(X_m)_{m \in M}$, $Y=(Y_m)_{m \in M} \in \Omega^A$, we have

$$X \vee Y = (X_m \cup Y_m)_{m \in M}, \; X \wedge Y = (X_m \cap Y_m)_{m \in M}$$

$$0 = (0_m)_{m \in M}; 0_m = 0, \forall m \in M$$

$$1 = (A_m)_{m \in M}, A_m = A, \forall m \in M$$

2.3. An M-bounded lattice A which is pseudo-complemented as a lattice (i.e. for every $a \in A$ there is an element $a^* \in A$ which is the largest element satisfying $a \wedge a^* = 0$) and whose unary operation $(\ )^*$: $A \to A$ is equivariant, is called an M-pseudo complemented lattice.

Example. Ω is an M-pseudo complemented lattice in MSet, which union (as join), intersection (as meet), $\phi = 0$, $M = 1$ and the pseudo-complementation given by $S^* = \{m \in M: mS = \phi\}$, for $S \in \Omega$.

2.4. An M-pseudo complemented lattice A which is a distributive lattice satisfying $a^* \vee a^{**} = 1$, for every $a \in A$, is called an M-Stone Lattice.

Note that Ω is not generally an M-Stone lattice. It is proved that Ω is an M-Stone lattice if and only if M satisfies the Ore Condition (i.e. for every a, $b \in M$ there exist s, $t \in M$ such that $sa=tb$).

2.5. An M-bounded lattice A which is a Boolean lattice whose unary operation $(\ )'$: $A \to A$ is equivariant, is called an M-Boolean lattice.

The class of all M-Boolean lattices and equivariant Boolean homomorphisms between them is a category denoted by MBoo.

2.6. Definition. Let $A \in MLatt$ and $X=(X_m)_{m \in M} \in \Omega^A$.

A supremum of X is an element a of A such that

i) ma is an upper bound of X_m, $\forall m \in M$. That is $x \leq ma$, $\forall x \in X_m$.

ii) for every $s \in M$, if mb is an upper bound of X_{ms} for all $m \in M$, then $sa \leq b$. That is $x \leq mb$, $\forall x \in X_{ms}$, implies $sa \leq b$.

If a supremum of $X \in \Omega^A$ exists it is clearly unique, and we denote it by $\vee X$. In particular, for a, $b \in A$, $a \vee b$ is the supremum of the family $(\{ma, mb\})_{m \in M}$.

Notice that, if M is a group then $a \in A$ is a supremum of $X=(X_m)_{m \in M}$ if and only if ma is the supremum of X_m, $\forall m \in M$.

2.7. An M-lattice A is said to be internally complete or M-complete if $\vee X$ exists, for every $X \in \Omega^A$.

It can be shown that A is internally complete if and only if there exists an order-preserving equivariant map $\vee:\Omega^A \to A$ such that, for $X = (X_m)_{m \in M} \in \Omega^A$, $b \in A$

$$\vee X \leq b \Leftrightarrow X_m \subseteq \downarrow mb, \forall m \in M.$$

It is proved that an internally complete lattice in MSet is a complete lattice (in Set). But, the converse is not necessarily true. For example, $2=\{0, 1\}$ is a complete lattice but 2 is an internally complete lattice in MSet if and only if M satisfies the left Ore condition.

Example. The M-lattices Ω and Ω^A are internally complete. In fact, the supremum of $X = (X_m)_{m\in M}\in \Omega^{\Omega}$ is given by $\vee X = \{m\in M: X_m=\Omega\}$; and the supremum of $X = (X_m)_{m\in M} \in \Omega^{\Omega^A}$, usually denoted by $\cup X$, has $\{a\in A: \exists Y\in X_m, a\in Y_e\}$ as its mth component.

2.8. An internally complete M-lattice A is called an internal locale or an M-locale if, for every $a\in A$, $X=(X_m)_{m\in M}\in \Omega^A$

$$a\wedge \vee X = \vee (\{ma\wedge x : x\in X_m\})_{m\in M}.$$

3. Construction of $\mathcal{I}$ (A)

We want to get the internal version of the notion "ideal" for lattices in the topos MSet. That is, for an (internal) lattice A in MSet, we construct an M-set, $\mathcal{I}$(A) which has properties similar to Id(A) and such that whenever $m=\{e\}$ and hence MSet $\cong$Set it is equal to Id(A). To do this, first we translate the definition of an ideal of a lattice into categorical terms and using the ingredients of the topos MSet we get the definition of $\mathcal{I}$(A), for a lattice A in MSet.

The importance of this construction is that this work can be done in an arbitrary topos $\mathcal{C}$ (instead of MSet) in a similar way.

3.1. Remark. Let A be a lattice. Recall that an ideal I of A is a non-empty subset of A such that

i) $a\in I$, $b\in I \Rightarrow a\vee b\in I$

ii) $a\in A$, $b\in I \Rightarrow a\wedge b\in I$.

We denote the lattice of all ideals of a lattice A by Id(A). It is clear that Id(A) is characterized by the following conditions:

(i)' $\{(X, a, b) : X\in \mathrm{Id}(A), a, b\in X\} \subseteq \{(X, a, b) : X\in \mathrm{Id}(A), a, b\in X, a\vee b\in X\}$

(ii)' $\{(X, a, b) : X\in \mathrm{Id}(A), a\in A, b\in X\} \subseteq \{(X, a, b) : X\in \mathrm{Id}(A), a\in A, b\in X, a\wedge b\in X\}$

Now, recall that the membership relation $\in_A$, in the topos Set (of sets) is the **set**

$$\in_A = \{(X, a) : X \subseteq A, a\in X\}$$

Also, consider the projection arrows $p, q: \wp(A)xAxA \to \wp(A)\, xA$ given by $p(X, a, b) = (X, a)$,

$q = (X, a, b) = (X, b)$. Then it easily follows that (i)', (ii)' are respectively equivalent to

(i)" $p^{-1} (\in_A\cap(\mathrm{Id}(A)\ xA)) \cap q^{-1} (\in_A\cap(\mathrm{Id}(A)\ xA)) \subseteq (idx\vee)^{-1} (\in_A\cap(\mathrm{Id}(A)xA))$

(ii)" $q^{-1} (\in_A\cap (\mathrm{Id}(A)\ xA)) \subseteq (idx\wedge)^{-1} (\in_A\cap (\mathrm{Id}(A)xA))$

where $idx\wedge$, $idx\vee$: $\wp(A)\ xAxA \to \wp(A)xA$ are given by

$(idx\wedge)(X, a, b) = (X, a\wedge b)$, $(idx\vee)(X, a, b) = (X, a\vee b)$. So, categorically, Id(A) is the **subset** (subobject in Set) of $\wp$(A) which satisfies (i)", (ii)", and

$\mathrm{Id}(A)x\{0\} \subseteq \in_A\cap(\mathrm{Id}(A)xA$ (i. e. $0\in X$, for all X in Id(A).)

3.2. Construction. Let A be an M-lattice and $\mathcal{I}$(A) be the internal version of Id(A) in MSet. We internalize the conditions (i)", (ii)" of the above remark, in MSet, to get the conditions which characterize $\mathcal{I}$(A).

Recall that the internal version of $\wp$(A) and $\in_A$ in MSet are Ω^A and $\in_A = \{(X, a) : X\in\Omega^A, a\in X_e\}$, respectively (see 1.5, 1.6). So, $\mathcal{I}$(A) is a **subobject** (a subset which is closed under the action of M) of Ω^A which satisfies

i) $p^{-1}(\in_A\cap (\mathcal{I}(A)\ xA)) \cap q^{-1} (\in_A\cap (\mathcal{I}(A)\ xA)) \subseteq (idx\vee)^{-1} (\in_A\cap(\mathcal{I}(A)\ xA))$;

ii) $q^{-1} (\in_A\cap(\mathcal{I}(A)\ xA)) \subseteq (idx\wedge)^{-1} (\in_A\cap (\mathcal{I}(A)\ xA))$;

iii) $\mathcal{I}(A)\ x\ \{0\} \subseteq \in_A\cap (\mathcal{I}(A)\ xA)$.

where p, q, $idx\vee$, $idx\wedge$: $\Omega^A xAxA \to \Omega^A xA$ are defined as in Set (see 1.2), and hence we have

$p^{-1} (\in_A\cap (\mathcal{I}(A)\ xA)) = \{(X, a, b) : X\in\mathcal{I}(A), a\in X_e, b\in A\}$

$q^{-1} (\in_A\cap (\mathcal{I}(A)\ xA)) = \{(X, a, b) : X\in\mathcal{I}(A), a\in A, b\in X_e\}$

$(idx\vee)^{-1} (\in_A\cap(\mathcal{I}(A)\ xA)) = \{(X, a, b) : X\in\mathcal{I}(A), a, b\in A, a\vee b\in X_e\}$

$(idx\wedge)^{-1} (\in_A\cap(\mathcal{I}(A)\ xA)) = \{(X, a, b) : X\in\mathcal{I}(A), a, b\in A, a\wedge b\in X_e\}$.

Thus, $\mathcal{I}$(A) is a subobject of Ω^A which satisfies

i) $X = (X_m)_{m\in M}\in\mathcal{I}(A)$ and $a, b\in X_e \Rightarrow a\wedge b\in X_e$

ii) $X = (X_m)_{m\in M}\in\mathcal{I}(A)$ and $a\in X_e$, $b\in A \Rightarrow a\wedge b\in X_e$

iii) $X = (X_m)_{m\in M}\in\mathcal{I}(A) \Rightarrow 0\in X_e$.

But, the above conditions yield that for $X = (X_m)_{m\in M}\in\mathcal{I}(A)$, $X_e\in \mathrm{Id}(A)$. On the other hand, the fact that $\mathcal{I}$(A) is a subobject of Ω^A guarantees that, for every $X=(X_s)_{s\in M}\in\mathcal{I}(A)$ and $m\in M$, $mX=(X_{sm})_{s\in M}$ is in $\mathcal{I}(A)$. Therefore, for every $X\in\mathcal{I}(A)$ and $m\in M$, we have $X_m=(mX)_e\in Id(A)$ (note that $0\in(mX)_e=X_m$). So, we have the following definition.

3.3. Definition. Let $A\in$ MLatt. The set

$\mathcal{I}(A)=\{X=(X_m)_{m\in M}\in\Omega^A : X_m\in Id(A), \forall m\in M\}$ is an M-lattice with the same action of M as on Ω^A and the componentwise lattice operations, and is called the internal ideal lattice of A. Each member of $\mathcal{I}(A)$ is called an internal ideal of A.

Notice that if M is a group then the restriction of the isomorphism $\Omega^A\cong\wp(A)$ (given by $X\to X_e$) on $\mathcal{I}(A)$ gives the isomorphism $\mathcal{I}(A)\cong Id(A)$.

Example. consider the monoid $M'_3=\{e, a, b\}$ with the binary operation given by $xy=y$, for $x, y\in M'_3$, with $y\neq e$. Then the lattice $2=\{0, 1\}$ with the identity action of M'_3 on it is an M'_3-lattice.
The action of M'_3 on $M'_3 x 2$ is then given by

	(e, 0)	(e, 1)	(a, 0)	(a, 1)	(b, 0)	(b, 1)
e	(e, 0)	(e, 1)	(a, 0)	(a, 1)	(b, 0)	(b, 1)
a	(a, 0)	(a, 1)	(a, 0)	(a, 1)	(b, 0)	(b, 1)
b	(b, 0)	(b, 1)	(a, 0)	(a, 1)	(b, 0)	(b, 1)

It can be seen that

$X_1=(\{0\}, \{0\}, \{0\})$, $X_2=(\{0\}, \{0\}, 2)$, $X_3=(\{0\}, 2, \{0\})$

$X_4=(\{0\}, 2, 2)$, $X_5=(2, 2, 2)$

are all the elements of $\mathcal{I}(2)$. So, $\mathcal{I}(2)$ is the M'_3-lattice

with the action of M'_3 given by

	X_1	X_2	X_3	X_4	X_5
e	X_1	X_2	X_3	X_4	X_5
a	X_1	X_1	X_5	X_5	X_5
b	X_1	X_5	X_1	X_5	X_5

4. Some Properties of $\mathcal{I}(A)$

4.1. Definition. Let $A\in$ MLatt. An ideal I of A is called an M-ideal of A if I is closed under the action of M, that is $mI\subseteq I$, for every $m\in M$, where $mI=\{ma: a\in I\}$.

4.2. Remark. For any M-lattice A, the global elements of $\mathcal{I}(A)$ (i. e. the equivariant maps from 1 to $\mathcal{I}(A)$) are in one-one correspondence with the M-ideals of A. In this correspondence, a global element $f:1\to\mathcal{I}(A)$ corresponds to the M-ideal X_e, where $f(0)=(X_m)_{m\in M}$. Conversely, an M-ideal I corresponds to the global element $g:1\to\mathcal{I}(A)$ given by $g(0)=(X_m)_{m\in M}$, where $X_m=I$, for every $m\in M$.

4.3. Recall that, for any lattice A, there is an embedding (in the category of lattices)

$\downarrow: A\to Id(A)$ given by $a\to\downarrow a=\{X\in A: X\leq a\}$.

Notice that defining the action of M on Id(A) by $m.I=\langle mI\rangle$, the ideal generated by $mI=\{ma: a\in I\}$, we can make Id(A) into an M-lattice in such a way that $\downarrow$ is an embedding in the category MLatt.

The internalization of the above embedding for the lattices in the topos MSet is as follows.

4.4. Lemma. Every $A\in$ MLatt can be embedded into $\mathcal{I}(A)$ by the map $[.]: A\to\mathcal{I}(A)$ given by $a\to[a]=(\downarrow ma)_{m\in M}$

Proof. Using the facts that $\downarrow$ is a one-one morphism in MLatt and the operations of $\mathcal{I}(A)$ are defined componentwise we easily get that [.] is a one-one morphism in MLatt.

4.5. Remark. For every $A\in$ MLatt, the embedding [.] factors through Id(A); that is there is a morphism $g: Id(A)\to\mathcal{I}(A)$ in MLatt such that $g^{\circ}\downarrow=[.]$. In fact, the assignment $I\to I^{\#}=(m.I)_{m\in M}$, where $m.I=\langle mI\rangle$, defines a (one-one) morphism $g: Id(A)\to\mathcal{I}(A)$ in MLatt. And for $a\in A$, we have

$$(\downarrow a)^{\#}=(m.\downarrow a)_{m\in M}=(\downarrow ma)_{m\in M}=[a]$$

that is, the triangle

$$A\xrightarrow{[.]}\mathcal{I}(A)$$

Id(A)

is commutative.

4.6. Recall that a lattice A is complete if and only if, for every $I\in Id(A)$, the supremum of I exists. Now, an M-lattice A is internally complete if and only if the supremum of $J=(J_m)_{m\in M}$ exists for every $J\in\mathcal{I}(A)$. This follows from the fact that for every $X=(X_m)_{m\in M}\in\Omega^A$, the supremum of x exists if the supremum of the internal ideal generated by x exists (and they are equal).

At the end, we consider the lattice structure of $\mathcal{I}(A)$, for some special kind of M-lattices A.

4.7. Theorem. Let A be an M-bounded lattice (in fact 0 is what we need). Then

a) $\mathcal{I}(A)$ is an internally complete M-bounded lattice.

b) If A is distributive then $\mathcal{I}(A)$ is an M-locale.

c) If A is distributive then $\mathcal{I}(A)$ is a distributive M-pseudo complemented lattice.

d) If A is an internally complete M-Boolean lattice then $\mathcal{T}(A)$ is an M-Stone lattice.

Proof. a) If 0 is the least element of A then $0=(0_m)_{m\in M}$, where $0_m=\{0\}$, is the least element of $\mathcal{T}(A)$. Moreover, $1=(A_m)_{m\in M}$, where $A_m=A$, is the greatest element of $\mathcal{T}(A)$. Hence, $\mathcal{T}(A)$ is an M-bounded lattice. Further, $\mathcal{T}(A)$ is internally complete since, for every $X=(X_m)_{m\in M}\in\Omega^{\mathcal{T}(A)}$, $\vee x=<Ux>$, the internal ideal generated by Ux.
In fact, the mth component of $\vee x$ is

$$\{b\in A : \exists Y_1,\ldots, Y_n\in X_m , [b]\leq Y_1\vee\ldots\vee Y_n\}.$$

b) By (a), $\mathcal{T}(A)$ is internally complete. Let $J=(J_m)_{m\in M}\in\mathcal{T}(A)$ and $X=(X_m)_{m\in M}\in\Omega^{\mathcal{T}(A)}$, we have to show that

$$J\wedge\vee x=\vee(\{mJ\wedge I : I\in X_m\})_{m\in M}$$

Let $m\in M$ and b be in the mth component of $J\wedge\vee x$, that is $b\in J_m\cap(\vee x)_m$. Then $b\in J_m$ and there are $I_1,\ldots, I_n\in x_m$ such that $[b]\leq I_1\vee\ldots\vee I_n$. So, since A and hence Id(A) is distributive,

$$[b]=[b]\wedge[I_1\vee\ldots\vee I_n) = ([b]\wedge I_1)\vee\ldots\vee([b]\wedge I_n).$$

But $b\in J_m$ implies $sb\in J_{sm}$ and hence $\downarrow sb\subseteq J_{sm}$, $\forall s\in M$, that is $[b]\leq mJ$. Thus

$$[b]\leq(mJ\wedge I_1)\vee\ldots\vee(mJ\wedge I_n)$$

where the right-hand side parentheses are members of $\{mJ\vee I: I\in X_m\}$, and so b is in the mth component of $\vee(\{mJ\wedge I:I\in x_m\})_{m\in M}$. Therefore, $J\wedge\vee x\leq\vee(\{mJ\wedge I: I\in x_m\})_{m\in M}$. The converse is trivially true because, for every $m\in M$, $x_m\subseteq\downarrow m\vee X$ and hence $mJ\wedge I\leq mJ\wedge m\vee X =m(J\wedge\vee X)$, for every $I\in x_m$, that is $\{mJ\wedge I:I\in X_m\}\subseteq\downarrow m(J\wedge\vee X)$.
Consequently, $\mathcal{T}(A)$ is an M-locale.

c) This follows from part (b) using the fact that any M-locale is M-pseudo-complemented. In fact, for $J=(J_m)_{m\in M}\in\mathcal{T}(A)$, we have

$$J^*=\vee(\{I\in\mathcal{T}(A) : I\wedge mJ=0\})_{m\in M}$$

and so the mth component of J* is

$$\{b\in A:sb\in(J_{sm})^* , \forall s\in M\}$$

where $(J_{sm})^* = \{a\in A : a\wedge j=0 , \forall j\in J_{sm}\}$ is the pseudo-complement of J_{sm} in Id(A). Note that distributivity of $\mathcal{T}(A)$ follows from the fact that Id(A) is distributive and the operations of $\mathcal{T}(A)$ are defined componentwise.

d) By (c), $\mathcal{T}(A)$ is distributive and M-pseudo-complemented. Now using the fact that A is an internally complete M-Boolean lattice, it can easily be seen that the definition of J* (in part (c)), for $J=(J_m)_{m\in M}\in\mathcal{T}(A)$, will be reduced to

$$J^*=[(\vee J)']$$

and hence $J^{**}=[(\vee J)']^*=[((\vee J)')']=[\vee J]$. Therefore, for every $J\in\mathcal{T}(A)$,

$$J^*\vee J^{**}=[(\vee J)']\vee[\vee J]=[1_A]=1_{\mathcal{T}(A)}$$

That is $\mathcal{T}(A)$ is an M-Stone lattice.

4.8. Corollary. Each M-lattice A (with 0) can be embedded into an internally complete M-lattice.

4.9. Remark. By the above corollary, every M-bounded lattice A is embedded in the internally complete M-lattice $\mathcal{T}(A)$.
We have shown that, for any $A\in$ MBoo, the subset $N(A)=\{J\in\mathcal{T}(A):J=J^{**}\}$ of $\mathcal{T}(A)$ is an M-Boolean algebra which is the minimal (normal) completion of A.

Now, recall that completeness and injectivity are equivalent notions for ordinary Boolean algebras (in Set), and injective hulls are exactly minimal completions. However, it is proved that there are internally complete M-Boolean algebras which are not injective in MBoo (see [7]). So, one can ask the following questions:
1) For which monoid M, N(A) is injective, and hence the injective hull of A?
2) Characterize the monoids M for which injectivity in MBoo is the same as internal completeness.

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