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B-AND B_r - COMPLETENESS IN LOCALLY CONVEX ALGEBRAS AND THE $E \times \phi$ THEOREM

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Abstract

Let E be a B-complete (B_r-complete) locally convex algebra and ϕ the topological direct sum of countably many copies of the scalar field with a jointly continuous algebra multiplication. It has been shown that $E \times \phi$ is also B-complete (B_r-complete) for componentwise multiplication on $E \times \phi$. B-and Br-completeness of E_1, the unitization of E, and also of $E \times \phi$ for other multiplications on $E \times \phi$ has been studied.

1. Introduction

B- and B_r-complete spaces appear to serve as a range space in closed graph theorem (see for example [3] and [4]). Also the closed graph and open mapping theorems have been generalized in various directions such as in [1] and [4]. A result of Savgulidze [8] and Smoljanov [9] asserts that if E is a B-complete locally convex space and ϕ is the topological direct sum of countably many copies of the scalar field then $E \times \phi$ is B-complete. We gave a new proof of this result along with some extensions and applications via the closed graph theorem in [7]. The aim of the present article is to adapt our methods to give versions of the Savgulidze-Smoljanov result when E and $E \times \phi$ are locally convex algebras and the B-completeness property is reformulated to take account of the algebra structure.

The following definitions were given by Rosa in [6] where he demonstrated their importance in connection with a long-standing problem concerning the algebra of continuous functions on a completely regular space. (Here, the term locally convex algebra means a Hausdorff locally convex space with a jointly continuous algebra multiplication, we shall allow the scalar field IK to be IR or $\mathbb{C}$.)

Keywords: B-completeness in algebras; Locally convex algebras

Definitions. A locally convex algebra E is B-complete (B_r-complete) if every continuous (continuous and one-to-one) nearly open algebra homomorphism from E onto a locally convex algebra is open.

We note that a B-complete algebra need not be complete and that there are B_r-complete algebras which are not B-complete ([6], end of Section 2 and Example 4.2). As we shall make several applications of Theorem 2.4 of [6] we record it here for easy reference.

Theorem 0([6], Theorem 2.4). Let B be a dense subalgebra of a locally convex algebra A.
(a) B is a B_r-complete algebra if and only if A is a B_r-complete algebra and B has non-zero intersection with every non-zero closed ideal of A.
(b) B is a B-complete algebra if and only if A is a B-complete algebra and $B \cap I$ is dense in I for every closed ideal I of A.

Starting with a locally convex algebra E and any multiplication on ϕ, which is necessarily jointly continuous ([2], §40.5 (3)), there are various ways of constructing a jointly continuous multiplication on $E \times \phi$ with respect to which E and ϕ(identified as usual with their canonical embeddings) are subalgebras. If we multiply componentwise on $E \times \phi$, we get a complete analogue of the Savgulidze-Smoljanov result for both B-and B_r-completeness (Theorem 1). However, with a more exotic construction, we find that the results depend on properties of E or the particular multiplication on ϕ(Theorems 4 and 6 and Section 6(v)). In some cases, our results and methods are also meaningful for algebras of the form $E \times IK^n$, we use the symbol Φ to denote either ϕ or IK^n when it is unnecessary to be specific. An important special case is the unitization E_1 of E which we study in Section 4. Our final section is devoted to illustrative examples and counter-examples. Generally we follow the notation and conventions of [5], [6] and [7].

2. Preliminaries

In this section, we give our basic lemma which is essentially a reformulation of some of the results in Section 2 of [7]. We limit ourselves to some comments on slight differences in the proofs and otherwise refer the reader to the appropriate parts of [7].

Lemma 1. Let E be a locally convex algebra and let $E \times \Phi$ have a multiplication which induces the given multiplication on E. Suppose that t is a continuous nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F. Then:
(a) $t|_E$ is a continuous nearly open algebra homomorphism of E onto t(E);
(b) if t(E) is closed in F and if either E is a B-complete algebra or E is a B_r-complete algebra and $t|_E$ is one-to-one, then t is open.

Proof. (a) Since E is a subalgebra of $E \times \Phi$, we have that $t|_E$ is an algebra homomorphism. The rest is immediate from ([7], Lemma 1) since t is linear. (Although ([7], Lemma 1) deals with the case $\Phi=\phi$, its proof also covers the case $\Phi= IK^n$).
(b) If E is a B-complete algebra or if E is a B_r-complete algebra and $t|_E$ is one-to-one, we deduce from (a) that $t|_E$ is open as a mapping onto t(E). If t(E) is closed in F, the argument of ([7], Corollary to Lemma 1) now shows that F is the locally convex direct sum of t(E) and any supplement H. With this fact, we may follow the method of proof of ([7], Theorem 1) to establish that t is open.

3. Componentwise Multiplication

Let E be a locally convex algebra and let Φ have any multiplication. The expression

$$(x,(\lambda_n))(y,(\mu_n))=(xy,(\lambda_n)(\mu_n)) \quad (x,y \in E,(\lambda_n),(\mu_n) \in \Phi)$$

defines a jointly continuous multiplication on $E \times \Phi$. We assume throughout this section that Φ has a fixed but arbitrary multiplication and that if E is a locally convex algebra, $E \times \Phi$ has the corresponding componentwise multiplication defined above. We shall establish

Theorem 1. If E is a B-complete (B_r-complete) algebra, then so is $E \times \Phi$.

We divide the proof into a number of lemmas.

Lemma 2. Let E be a B_r-complete algebra. If x is in the completion $\hat{E}$ of E and $xy=yx=0$ for all $y \in \hat{E}$, then $x \in E$.

Proof. This is trivial if E is complete. Suppose therefore that E is incomplete and that there exists $x \in \hat{E}\backslash E$ such that $xy=yx=0$ for all $y \in \hat{E}$. Then $I=\{\lambda x: \lambda \in IK\}$ is a closed ideal in $\hat{E}$ such that $I \cap E = \{0\}$. But by Theorem 0 each non-zero closed ideal in $\hat{E}$ has non-zero intersection with E-contradiction.

Lemma 3. Let E be a B_r-complete algebra and let t be a continuous nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F such that $t|_E$ is one-to-one. Then t(E) is closed in F.

Proof. It follows from the above hypotheses and Lemma 1(a) that $t|_E$ is a topological isomorphism of E onto t(E). If E is complete then t(E) is complete and hence closed in F. Suppose E is not complete.

Let $\hat{t}$ be the extension of t by continuity to a homomorphism of $\hat{E} \times \Phi$ into $\hat{F}$. We show that ker t=ker $\hat{t}$. Let $(x, (\lambda_n)) \in$ ker $\hat{t}$. Since ker $\hat{t}$ is an ideal in $\hat{E} \times \Phi$, for all $y \in \hat{E}$ we have that $(x, (\lambda_n)) (y, 0) = (xy, 0)$ and $(y, 0) (x, (\lambda_n')) = (yx, 0)$ are elements of ker $\hat{t}$. Now $\widehat{t|_E}$ is the extension of $t|_E$ by continuity and so it is a topological isomorphism of $\hat{E}$ onto $\hat{t}(\hat{E})$ ([5], Chapter VI, Proposition 6, Corollary 1). It follows that $xy=yx=0$ for all $y \in \hat{E}$ and so by Lemma 2 we have that $x \in E$. Consequently, $(x, (\lambda_n)) \in$ ker t.

Now let $y \in \hat{t}(\hat{E}) \cap F$. There are $\hat{x} \in \hat{E}$ and $(x, (\lambda_n)) \in E \times \Phi$ such that $t(x, (\lambda_n)) = y = \hat{t}(\hat{x}, 0)$. Hence $\hat{t}(x-\hat{x}, (\lambda_n)) = 0$ and so by the previous paragraph $x-\hat{x}$ and hence $\hat{x}$ are elements of E. Then $y \in t(E)$ and since $t(E) \subseteq \hat{t}(\hat{E}) \cap F$ we have $t(E) = \hat{t}(\hat{E}) \cap F$. Since $\hat{t}(\hat{E})$ is complete and so closed in $\hat{F}$, this shows that t(E) is closed in F.

Lemma 4. Let E be a B-complete algebra and let t be a continuous nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F. Then t(E) is closed in F.

Proof. Put $J = \{(x, 0): t(x, 0) = 0\}$ and $I = \{x: (x, 0) \in J\}$. It is easily shown that:

(i) J and I are closed ideals in $E \times \Phi$ and E, respectively.

(ii) $(x, (\lambda_n)) + J \to (x + I, (\lambda_n))$ $(x \in E, (\lambda_n) \in \Phi)$ is a topological isomorphism of $(E \times \Phi) / J$ onto $(E/I) \times \Phi$.

(iii) $s(x+I, (\lambda_n)) = t(x, (\lambda_n))$ $(x \in E, (\lambda_n) \in \Phi)$ defines a continuous nearly open algebra homomorphism s of $(E/I) \times \Phi$ onto F which is one-to-one on E/I.

Since E/I is B_r-complete (in fact B-complete) ([6], Lemma 2.2), we deduce from Lemma 3 that t(E) = s(E/I) is closed in F.

To complete the proof of Theorem 1, we now let t be a continuous (continuous and one-to-one) nearly open algebra homomorphism of $E \times \Phi$ onto a locally convex algebra F. Under the hypotheses of Theorem 1, we have from Lemma 4 (Lemma 3) that t(E) is closed in F and then from Lemma 1(b) that t is open as required.

4. Unitization

The unitization E_1 of a locally convex algebra E is the locally convex algebra obtained by giving the product $E \times IK$ the multiplication defined by

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu) \quad (x, y \in E, \lambda, \mu \in IK)$$

Note that we do not exclude the case where E already has an identity. We are concerned with B-completeness and B_r-completeness for E_1. First we establish a necessary condition.

Theorem 2. If E is a locally convex algebra such that E_1 is B-complete (Br-complete) then E is also B-complete (B_r-complete).

Proof. Let t be a continuous (continuous and one-to-one) nearly open algebra homomorphism of E onto a locally convex algebra F. If we define $t_1: E_1 \to F_1$ by $t_1(x, \lambda) = (t(x), \lambda)$ then it is easily shown that t_1 is a continuous nearly open algebra homomorphism of E_1 onto F_1 which is one-to-one if t is one-to-one. Consequently t_1 is open, from which it follows that t is open as required.

The converse of Theorem 2 is false in general. Indeed the example of Section 6(ii) shows that the unitization of a B-complete algebra need not be even B_r-complete. However, we can characterize those algebras for which the respective cases of the converse hold.

Theorem 3. Let E be a B_r-complete algebra. Then E_1 is B_r-complete if and only if either $\hat{E}$ has no identity or E has an identity.

Proof. Suppose that $\hat{E}$ has an identity e which is not in E. The set $\{\lambda(e, -1): \lambda \in IK\}$ is a closed ideal in the completion $\hat{E} \times IK$ of E_1 whose intersection with E_1 consists only of the zero element. It now follows from Theorem 0 that E_1 is not B_r-complete. Thus if E_1 is B_r-complete, then $\hat{E}$ has no identity unless it is already in E.

For the converse, let t be a continuous one-to-one nearly open algebra homomorphism of E_1 onto a locally convex algebra F. By Lemma 1(a), $t|_E$ is a continuous one-to-one nearly open algebra homomorphism of E onto the subalgebra G=t(E) of F. Since E is B_r-complete, $t|_E$ is therefore a topological isomorphism of E onto G. If we show that G is closed in F, it will follow by Lemma 1(b) that t is open as required.

Suppose that G is not closed in F. Since $\hat{G} \cap F$ is closed in F, G must be strictly contained in $\hat{G} \cap F$, and since G has codimension 1 in F it then follows that $\hat{G} \cap F = F$. Now if $s: \hat{E} \to \hat{G}$ is the extension of $t|_E$ by continuity, s is a topological isomorphism of $\hat{E}$ onto $\hat{G}$ ([2], Chapter VI, Proposition 6, Corollary 1), in particular s^{-1} is a homomorphism of $\hat{G}$ onto $\hat{E}$. Then if f=(0, 1) is the identity of E_1, it follows that $s^{-1}(t(f))$ is an identity in $\hat{E}$.

If $\hat{E}$ has no identity, we clearly have a contradiction. Suppose E has an identity e. We must then have $s^{-1}(t(f))=e$ and consequently $t(f)=s(e) = t(e, 0)$. This is again a contradiction since t is one-to-one.

As an immediate consequence we have

Corollary. If E is a complete B_r-complete algebra then E_1 is B_r-complete.

Remark. If E is a B_r-complete algebra without an identity, we can always adjoin an identity to E in a natural way so that the resulting locally convex algebra G is B_r-complete. If $\hat{E}$ does not have an identity we may take E_1 for G by Theorem 3. If $\hat{E}$ has an identity e we take for G the subalgebra of $\hat{E}$ generated by E and e with the topology induced by $\hat{E}$. B_r-completeness follows from Theorem 0.

Theorem 4. Let E be a B-complete algebra. Then E_1 is B-complete if and only if for every closed ideal I in E either $\widehat{E/I}$ has no identity or E/I has an identity.

Proof. Suppose that E_1 is B-complete. Let I be a closed ideal in E and put $J=\{(x, 0): x \in I\}$. Then J is a closed ideal in E_1 and it is easily seen that E_1/J is topologically isomorphic with $(E/I)_1$. Since E_1/J is B_r-complete ([6], Lemma 2.2), we see from Theorem 3 that the condition is necessary.

For the converse, let t be a continuous nearly open algebra homomorphism of E_1 onto a locally convex algebra F. Suppose there exists an element $(x_0, \lambda_0) \in J = \ker t$ with $\lambda_0 \neq 0$. Since

$$t(x_0, \lambda_0) = t(x_0, 0) + \lambda_0 t(0, 1)$$

we have

$$t(0, 1) = -\lambda_0^{-1} t(x_0, 0) \in t(E).$$

Therefore, for all $(x, \lambda) \in E_1$,

$$t(x, \lambda) - t(x, 0) + \lambda t(0, 1) \in t(E),$$

which implies that t(E) = F. It now follows from Lemma 1(b) that t is open. If $\lambda = 0$ for all $(x, \lambda) \in J$, then $J = I \times \{0\}$ for some closed ideal I in E. As before E_1/J is topologically isomorphic with $(E/I)_1$ which is B_r-complete by Theorem 3. Since $t = s^0 q$ where q: $E_1 \to E_1/J$ is the quotient map and s is a continuous one-to-one nearly open algebra homomorphism, we deduce that t is open.

5. Another Multiplication on $E \times \phi$

Throughout this section, the multiplication on ϕ is convolution, i. e.

$$(\lambda_n)(\mu_n) = \left(\sum_{r=1}^{n} \lambda_r \mu_{n-r+1}\right) \quad ((\lambda_n), (\mu_n) \in \phi).$$

and if E is a locally convex algebra, $E \times \phi$ has the jointly continuous multiplication defined by

$$(x, (\lambda_n))(y, (\mu_n)) = (xy + x \sum_{n=1}^{\infty} \mu_n + y \sum_{n=1}^{\infty} \lambda_n, (\lambda_n)(\mu_n))$$

(For the associative law we have to note that if $(\gamma_n) = (\lambda_n)(\mu_n)$ then $\sum_{n=1}^{\infty} \gamma_n = \sum_{n=1}^{\infty} \lambda_n \sum_{n=1}^{\infty} \mu_n \ (*)$, continuity follows from the continuity of the algebraic operations on E and ϕ and the continuity of the linear functional $(\lambda_n) \to \sum_{n=1}^{\infty} \lambda_n$ on ϕ.) Clearly this multiplication is an extension to $E \times \phi$ of the unitization multiplication on $E \times \mathbb{K}$. Moreover

Theorem 5. If E is a locally convex algebra, then E_1 is topologically isomorphic to the quotient of $E \times \phi$ by the closed ideal $\{(0, (\lambda_n)) : \sum_{n=1}^{\infty} \lambda_n = 0\}$.

Proof. Let $L = \{(0, (\lambda_n)) : \sum_{n=1}^{\infty} \lambda_n = 0\}$. It is easy to show that L is a closed ideal of $E \times \phi$ (again use (*)) and that $(x, (\mu_n)) + L \to (x, \sum_{n=1}^{\infty} \mu_n)$ is a topological isomorphism of $(E \times \phi)/L$ onto E_1.

We now consider B-completeness of $E \times \phi$. Since a quotient of a B-complete algebra by a closed ideal is again B-complete, it follows from Theorems 5 and 2 that if $E \times \phi$ is B-complete then E_1 and E must be B-complete. However, if E is B-complete, E_1 and therefore $E \times \phi$ may fail to be B-complete (Section 6(ii)) and so the Savgulidze-Smoljanov theorem will not hold in general for B-completeness. It is perhaps surprising that B-completeness of E_1 is the crucial factor.

Theorem 6. Let E be a B-complete algebra. Then $E \times \phi$ is B-complete if and only if E_1 is B-complete.

Proof. We have already shown that the condition is necessary.

Suppose that E_1 is B-complete and let t be a continuous nearly open algebra homomorphism of $E \times \phi$ onto a locally convex algebra F. Although we have a different multiplication on $E \times \phi$ the same construction as in Lemma 4 allows us to restrict attention to the situation where $t|_E$ is one-to-one. We show that t(E) is closed in F as in Lemma 3 except that the proof of ker $t=\ker \hat{t}$ (which follows) is rather more complicated.

Let $(x,(\lambda_n)) \in \ker \hat{t}$. For all $y \in \hat{E}$,

$$(x,(\lambda_n))(y,0) = (xy + y\sum_{n=1}^{\infty}\lambda_n, 0) \text{ and}$$

$$(y,0)(x,(\lambda_n)) = (yx + y\sum_{n=1}^{\infty}\lambda_n, 0)$$

are elements of ker $\hat{t}$. Since $\hat{t}|_{\hat{E}}$ is one-to-one we then have that

$$xy + y\sum_{n=1}^{\infty}\lambda_n = yx + y\sum_{n=1}^{\infty}\lambda_n = 0 \text{ for all } y \in \hat{E}.$$

If $\sum_{n=1}^{\infty}\lambda_n \neq 0$ this says that $-\left[\sum_{n=1}^{\infty}\lambda_n\right]^{-1} x$ is an identity in $\hat{E}$ which, by Theorem 3, implies that $x \in E$ and $(x, (\lambda_n)) \in \ker t$. If $\sum_{n=1}^{\infty}\lambda_n = 0$ we have $xy=yx=0$ for all $y \in \hat{E}$ and we reach the same conclusion by Lemma 2.

Finally we apply Lemma 1(b) to deduce that t is open. The situation with regard to B_r-completeness is as before.

Theorem 7. If E is a B_r-complete algebra so also is $E \times \phi$.

Proof. Suppose first that E is complete and let t be a continuous one-to-one nearly open algebra homomorphism of $E \times \phi$ onto a locally convex algebra F. It follows from Lemma 1(a) that $t|_E$ is a topological isomorphism of E onto t(E). Thus t(E) is complete and therefore closed in F. We then have by Lemma 1(b) that t is open as required.

For the general case we have by the first part and Theorem 0 that $\hat{E} \times \phi$ is B_r-complete and we have to show that if I is a non-zero closed ideal of $\hat{E} \times \phi$ then $I \cap (E \times \phi) \neq \{0\}$.

Suppose there is $(x, (\lambda_n)) \in I$ with $(\lambda_n) \neq 0$. If k is the largest suffix n such that $\lambda_n \neq 0$, let (μ_n) be the element of ϕ with $\mu_1 = 1$, $\mu_{k+1} = -1$ and $\mu_n=0$ otherwise. Then the kth component of $(\lambda_n)(\mu_n)$ is λ_k and so $(x, (\lambda_n))(0, (\mu_n)) = (0, (\lambda_n)(\mu_n))$ is a non-zero element of $I \cap (E \times \phi)$. If $(\lambda_n)=0$ for all $(x, (\lambda_n)) \in I$, then $J = \{x: (x, 0) \in I\}$ is a non-zero closed ideal of $\hat{E}$. As before $J \cap E \neq \{0\}$ and therefore $I \cap (E \times \phi) \neq \{0\}$.

Remarks. (i) If E is B_r-complete, E_1 may fail to be B_r-complete (Section 6(ii)), although $E \times \phi$ is always B_r-complete. Using Theorem 5 we see that a quotient of a B_r-complete algebra by a closed ideal need not be B_r-complete. (See also ([6], Example 4.5).

(ii) Let $m: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any function with the property $m(m(i, j), k) = m(i, m(j, k))$ for all $i, j, k \in \mathbb{N}$. Given $(\lambda_n), (\mu_n) \in \phi$, put $\gamma_n = \sum\{\lambda_i\mu_j : m(i,j) = n\}$ $(n \in \mathbb{N})$, with the usual convention that empty sums are 0. Then $(\lambda_n)(\mu_n) = (\gamma_n)$ defines a multiplication on ϕ such that $\sum_{n=1}^{\infty}\gamma_n = \sum_{n=1}^{\infty}\lambda_n\sum_{n=1}^{\infty}\mu_n$. Convolution is just the particular case $m(i, j) = i + j - 1$. In fact, convolution can be replaced by any such multiplication in the formula at the beginning of this section to give a jointly continuous multiplication on $E \times \phi$. However, Theorem 7 may be false for a more general multiplication of this type (Section 6(v)).

6. Some Examples

Let C(IR) be the algebra of continuous complex-valued functions on IR with the usual pointwise operations and the topology of uniform convergence on compact subsets of IR. Since C(IR) is a Frechet space, it is B-complete as a locally convex space and therefore also as a locally convex algebra. (See also ([6], Theorem 3.2))

(i) We show that the dense subalgebra E, composed of those elements of C(IR) which have compact support, is also B-complete. Let I be a closed ideal of C(IR) and let K be any non-empty compact subset of IR. If K_0 is any compact subset of IR which contains K in its interior, we can find $g \in E$ such that $g(x) = 1$ for all $x \in K$ and the support of g is contained in K_0. Then if $f \in I$ we have $fg \in I \cap E$ and $f(x)\,g(x) - f(x) = 0$ for all $x \in K$. This shows that $I \cap E$ is dense in I and our assertion now follows from Theorem 0.

(ii) E_1 is not even B_r-complete by Theorem 3, since $\hat{E}$=C(IR) which has an identity not in E. Consequently, for the multiplication of Section 5, $E \times \phi$ is not B-complete (Theorem 6), although it is B_r-complete (Theorem 7).

(iii) Let ϕ have componentwise multiplication. Then if $F = E \times \phi$ with the multiplication of Section 3, F is B-complete (Theorem 1). Now $\hat{F}$= C(IR) $\times \phi$ which has no identity. Thus, by Theorem 3, F_1 is B_r-complete. However ϕ is a closed ideal in F and $F / \phi = E$. We deduce by Theorem 4 that F_1 is not B-complete.

(iv) If G is the subalgebra of C(IR) generated by E and the unit function, G is B-complete by Theorem 0(cf. (i)). Let ϕ have componentwise multiplication and put $H = G \times \phi$ with the multiplication of Section 3.

Again by Theorem 1, H is B-complete. Now it is easily shown that if I is a closed ideal in H, there are closed ideals J, L in G, ϕ respectively such that $I = J \times L$ and H/I is topologically isomorphic with $(G/J) \times (\phi / L)$. Since G/J has an identity and since ϕ/L is complete, being a quotient of a B-complete locally convex space, it follows that $\widehat{H/I}$ has an identity if and only if H/I has an identity. Thus, by Theorem 4, H_1 is B-complete. In fact both alternatives of Theorem 4 can occur in H, e.g. with $I = \{0\}$, $H/I = C(\mathbb{R}) \times \phi$ which has no identity and with $I = \phi$, $H/I = G$ which has an identity.

(v) We let $E \times \phi$ have multiplication defined as in Section 5 except that convolution on ϕ is replaced by the multiplication derived from $m(i, j) = i \wedge j$ (Section 5, Remarks (ii)). If e is the unit function on $\mathbb{R}$ and f is the element (-1, 0, 0,...) of ϕ, then $I = \{\lambda (e, f) : \lambda \in \mathbb{K}\}$ is a closed ideal in $\hat{E} \times \phi$ having zero intersection with $E \times \phi$. It follows from Theorem 0 that $E \times \phi$ is not B_r-complete.

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