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QUASI-PERMUTATION REPRESENTATIONS OF METACYCLIC 2-GROUPS

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Abstract

By a quasi-permutation matrix we mean a square matrix over the complex field **C** with non-negative integral trace. Thus, every permutation matrix over **C** is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field **Q**, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. In this paper, we will calculate the irreducible modules and characters of metacyclic 2-groups and we also find $c(G)$, $q(G)$ and $p(G)$ for these groups.

Introduction

If G is a finite linear group of degree n, that is, a finite group of automorphisms of an n-dimensional complex vector space (or, equivalently, a finite group of non-singular matrices of order n with complex coefficients), we shall say that G is a quasi-permutation group if the trace of every element of G is a non-negative rational integer. The reason for this terminology is that, if G is a permutation group of degree n, its elements, considered as acting on the elements of a basis of an n-dimensional complex vector space V, induce automorphisms of V forming a group isomorphic to G. The trace of the automorphism corresponding to an element x of G is equal to the number of letters left fixed by x, and so is a non-negative integer. Thus, a permutation group of degree n has a representation as a quasi-permutation group of degree n. See [9].

Keywords: Metacyclic 2-groups; Quasi-permutation; Quasi-permutation representation; Representation theory

By a quasi-permutation matrix we mean a square matrix over the complex field **C** with non-negative integral trace. Thus, every permutation matrix over **C** is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field **Q**, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [1].

By a rational valued character we mean a character χ corresponding to a complex representation of G such that $\chi(g) \in \mathbf{Q}$ for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from G to $GL(n, \mathbf{Q})$ a rational representation of G and its corresponding character will be called a rational character of G. It is easy to see that

$$c(G) \leq q(G) \leq p(G)$$

where G is a finite group.

We state algorithms obtained elsewhere [1] for calculating $p(G)$, $q(G)$ and $c(G)$ where G is a finite group with a unique minimal normal subgroup. Then we will calculate irreducible modules and irreducible characters of metacyclic 2-groups and we will apply the algorithms to the metacyclic 2-groups. We will show that

$$c(G)= q(G)= |Z(G)|\,|G: Z(G)|^{1/2}$$

if G is a finite metacyclic 2-group with cyclic center.

Algorithm for $p(G)$, $c(G)$ and $q(G)$

Lemma 2.1. Let G be a finite group with a unique minimal normal subgroup. Then $p(G)$ is the smallest index of a subgroup with trivial core (that is, containing no non-trivial normal subgroup).

Proof. See [[1], Corollary 2.4].

Definition 2.2. Let χ be a character of G such that, for all $g \in G$, $\chi(g) \in \mathbf{Q}$ and $\chi(g) \geq 0$. Then we say that χ is a non-negative rational valued character.

Notation. Let $\Gamma(\chi)$ be the Galois of $\mathbf{Q}(\chi)$ over $\mathbf{Q}$.

Definition 2.3. Let G be a finite group. Let χ be an irreducible complex character of G. Then define

(1) $d(\chi) = |\Gamma(\chi)|\,\chi(1)$

(2) $m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G \\ \left|\min \{ \sum_{\alpha\in\Gamma(\chi)} \chi^{\alpha}(g) : g \in G\}\right| & \text{otherwise} \end{cases}$

(3) $c(\chi) = \sum_{\alpha\in\Gamma(\chi)} \chi^{\alpha} + m(\chi)\, 1_G$

Corollary 2.4. Let $\chi \in \mathrm{Irr}(G)$. Then $\sum_{\alpha\in\Gamma(\chi)} \chi^{\alpha}$ is a rational valued character of G. Moreover $c(\chi)$ is a non-negative rational valued character of G and $c(\chi)(1)= d(\chi) + m(\chi)$.

Proof. See [[1], Corollary 3.7].

Now we will give algorithms for calculating $c(G)$ and $q(G)$ where G is a finite group with a unique minimal normal subgroup.

Lemma 2.5. Let G be a finite group with a unique minimal normal subgroup. Then
(1) $c(G) = \min\{c(\chi)(1)$: χ is a faithful irreducible complex character of $G\}$,.
(2) $q(G) = \min\{m_Q(\chi)c(\chi)(1)$: χ is a faithful irreducible complex character of $G\}$.

Proof. See [[1], Corollary 3.11].

Lemma 2.6. Let G be a finite group. If the Schur index of each non-principal irreducible character is equal to m, then $q(G)= mc(G)$.

Proof. See [[1], Corollary 3.15].

Some Facts on p-Groups

Lemma 3.1. Let G be a p-group and $H \leq G$. Let H_G denote the core of H in G. Then $H_G= 1$ if and only if $Z(G) \cap H= 1$. Furthermore, if G has nilpotency class 2 and $H_G = 1$ then H is an Abelian group.

Proof. See [[1], Lemma 4.2].

Corollary 3.2. Let G be a finite p-group and $p \neq 2$. Then $m_Q(\chi) = 1$ for all $\chi \in \mathrm{Irr}(G)$ and $c(G)= q(G)$.

Proof. This follows from [[6], Corollary 10.14] and Lemma 2.6.

Lemma 3.3. Let $A = \langle a\rangle$ be cyclic of order p^s. Let χ_{p^s} be the character of the $\mathbf{Q}A$-module $\mathbf{Q}(w)$ where w is a primitive p^s-th root of unity and a acts by multiplication by w. Then χ_{p^s} is faithful and

$$\chi_{p^s}(a^i) = \begin{cases} -p^{s-1} & \text{if } (i, p^s) = p^{s-1} \\ p^{s-1}(p-1) & i=0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. This follows from [[2], Lemma 3.4].

Theorem 3.4. Let G be a finite p-group with a unique minimal normal subgroup. Then there exists a faithful irreducible character χ. Suppose that all faithful irreducible characters of G have degree $\chi(1)$ and $\chi^2(1) = |G: Z(G)|$. Then $c(G) = \chi(1)\,|Z(G)| = |Z(G)|\,|G: Z(G)|^{1/2}$.

Proof. See [[1], Theorem 4.6].

Metacyclic 2-Groups

Let G be a non-exceptional and non-cyclic metacyclic 2-group. In [[7], Theorem 3.2] it is proved that G has a uniquely reduced presentation as $<a,b: a^{2^m}= 1, b^{2^n}=a^{2^{m-s}}, a^b= a^{1+2^r}>$ for certain integers m, n, r, s. A presentation is uniquely reduced if and only if the parameters satisfy the following conditions:

(a) split: $0= s \le m-r<\min\{n+1,m\}$;
(b) non-split: $\max\{1, m-n+1\} \le s <\min\{m-r, r+1\}$.
It is known that G is an extension

$$1 \to C_{2^m} \to G \to C_{2^n} \to 1$$

in which $A=<a>$ is the normal subgroup of order 2^m and G/A is generated by the image of b. In particular, if $B=<b>$, then G is of order 2^{m+n}, $G=AB=\{a^ib^j: 0\le i\le 2^m-1, 0\le j\le 2^n-1\}$ and $A\cap B=<b^{2^n}=a^{2^{m-s}}>$. It is the case that G is split only in case (a), that is, only when $A\cap B=1$. In discussing a generic metacyclic 2-group G in what follows, we will assume the notation above.

Lemma 4.1. Let $i=2^lk$, where $(k,2)=1$. Then $2^{r+j+l} | (1+2^r)^{i2^j}-1$, but $2^{r+j+l+1}$ does not divide $(1+2^r)^{i2^j}-1$.

Proof. It is easy to prove.

Lemma 4.2. Let G be a metacyclic 2-group. Then
(a) $G'=<a^{2^r}>$;
(b) if $a^\alpha b^\beta \in Z(G)$ then a^α and b^β are in $Z(G)$;
(c) $Z(G)=<a^{2^{m-r}}, b^{2^{m-r}}>$ and $|Z(G)|=2^{n-m+2r}$.

Moreover if $Z(G)$ is cyclic then
(d) $Z(G)=<a^{2^{m-r}}>$ and $n=m-r$ if G splits;
(e) $Z(G)=<b^{2^{m-r}}>$ and $s=r$ if G does not split.

Proof. (a): See [[3], 47.10].

(b), (c): As $G=<a,b>$, $a^\alpha\in Z(G)$ if and only if $a^\alpha= b^{-1}a^\alpha b=a^{\alpha(1+2^r)}$. This happens precisely when $o(a)=2^m | \alpha 2^r$, that is, when $2^{m-r}|\alpha$.

Similarly, $b^\beta\in Z(G)$ if and only if $b^\beta=a^{-1}b^\beta a=b^\beta b^{-\beta} a^{-1}b^\beta a=b^\beta a^{-(1+2^r)^\beta}a$. This happens precisely when $2^m | 1-(1+2^r)^\beta$. Using Lemma 4.1 we conclude that $b^\beta\in Z(G)$ if and only if $2^{m-r}|\beta$.

Thus, $<a^{m-r}, b^{m-r}>\le Z(G)$. If $a^\alpha b^\beta\in Z(G)$, then $a^\alpha b^\beta= b^{-1}a^\alpha b^\beta b=a^{\alpha(1+2^r)}b^\beta$ so that $a^{\alpha 2^r}=1$ and so $2^{m-r}|\alpha$. But then $a^\alpha\in Z(G)$ and so $b^\beta\in Z(G)$; this proves (b).

As $Z(G)/A\cap Z(G)$ is generated by the image of $b^{2^{m-r}}$, it is of order 2^{n-m+r} by the introductory material of this section. But $|A\cap Z(G)|=o(a^{2^{m-r}})=2^r$. It follows that $|Z(G)|=2^{n-m+2r}$

(d), (e): Since $Z(G)$ is cyclic by hypothesis, it is generated by $a^{2^{m-r}}$ or by $b^{2^{m-r}}$, as follows from (c) and the fact that $Z(G)$ is a 2-group.

If G is split, then $A\cap B=1$. Thus, if $Z(G)=<b^{m-r}>$, then $Z(G)\le B$ and $A\cap Z(G)=1$. But $A\lhd G$ so that this contradicts; consequently $Z(G)=<a^{2^{m-r}}>$.
But then $2^{n-m+2r}=|Z(G)|=o(a^{2^{m-r}})=2^r$ so that $n-m+2r=r$ and $n=m-r$ as stated.

Let G be non-split. The order of $b^{2^{m-r}}$ is $2^{n+s+r-m}$ and the order of $a^{2^{m-r}}$ is 2^r, but we know that $\max\{1,m-n+1\}\le s$. So $m-n<s$. Hence $s-m+n>0$, and $s-m+n+r>r$. So $Z(G)=<b^{2^{m-r}}>$. From (c) we have $|Z(G)|=2^{n-m+2r}$, so $2^{n-m+2r}=2^{n+s+r-m}$. Therefore $s=r$.

Corollary 4.3. Let G be a metacyclic 2-group and let $Z(G)$ be cyclic. Then, in the standard notation.
(a) if G is split, $G=<a,b: a^{2^m}=b^{2^{m-r}}=1, a^b=a^{1+2^r}>$;
(b) if G is non-split, $G=<a,b: a^{2^m}=1, b^{2^n}=a^{2^{m-r}}, a^b=a^{1+2^r}>$.

Lemma 4.4. Let ξ be a primitive 2^m-*th* root of unity, and $m>r\ge 0$. Then

$$\sum_{i=0}^{2^{m-r}-1}\xi^{k(1+2^r)^i}=0$$

where k is an integer, $(k,2)=1$.

Proof. As ξ^k is also a primitive 2^m-*th* root of unity, we may assume that $k=1$. Since $m>r$, ξ^{2^r} is a primitive 2^{m-r}-*th* root of unity so that

$$1+\xi^{2^r}+\xi^{2^{r+1}}+\ldots+\xi^{(2^{m-r}-1)2^r}=0.$$

Multiplying by ξ, we have

$$\xi+\xi^{1+2^r}+\xi^{1+2^{r+1}}+\ldots+\xi^{1+(2^{m-r}-1)2^r}=0 \qquad (1)$$

We know from Lemma 4.1 that the order of $1+2^r$ mod 2^m is 2^{m-r}. Thus, the residues mod 2^m of the integers $(1+2^r)^i$, $0\le i\le 2^{m-r}-1$, are distinct. It follows that, for each i, $0\le i\le 2^{m-r}-1$, there is a unique i', $0\le i'\le 2^{m-r}-1$, such that $(1+2^r)^i\equiv 1+i'2^r \bmod 2^m$. we can rewrite (1) as

$$\xi+\xi^{1+2^r}+\xi^{(1+2^r)^2}+\ldots+\xi^{(1+2^r)^{2^{m-r}-1}}=0,$$

the required identity.

Corollary 4.5. Let k be an integer such that $(k, 2^m)=2^\alpha$,

and let $m-\alpha>r\geq 0$.
Then

$$\sum_{i=0}^{2^{m-r}-1} \xi^{k(1+2^r)^i} = 0.$$

Proof. If $(k, 2^m)= 2^\alpha$, then ξ^k is a primitive $2^{m-\alpha}$-*th* root of unity. So by Lemma 4.4 we have $\sum_{i=0}^{2^{m-r-\alpha}-1} \xi^{k(1+2^r)^i} = 0$
By Lemma 4.1, $(1+2^r)^{2^{m-\alpha-r}} \equiv 1 \bmod 2^{m-\alpha}$. Thus, if $0 \leq i \leq 2^{m-r}-1$, $(1+2^r)^i \equiv (1+2^r)^j \bmod 2^{m-\alpha}$ when $i \equiv j \bmod 2^{m-\alpha-r}$. It follows that $\xi^{k(1+2^r)^i} = \xi^{k(1+2^r)^j}$ and so

$$\sum_{i=0}^{2^{m-r}-1} \xi^{k(1+2^r)^i} = 2^\alpha \sum_{i=0}^{2^{m-r-\alpha}-1} \xi^{k(1+2^r)^i} = 0.$$

Lemma 4.6. Let ξ be a primitive 2^m-*th* root of unity, and let k be an integer such that $(k, 2^m)= 2^\alpha$ and let $m-\alpha>r\geq 0$. Let $1\leq j\leq 2^m-1$, and let $S= \sum_{i=0}^{2^{m-r-\alpha}-1} \xi^{jk(1+2^r)^i}$. Then
(a) $\chi^{jk} = \chi^{jk(1+2^r)} = \ldots = \chi^{jk(1+2^r)^{2^{m-r-\alpha}-1}}$ if and only if $2^{m-r-\alpha} \mid j$;
(b)

$$S=\begin{cases} 2^{m-r-\alpha}\xi^{jk} & \text{if } 2^{m-r-\alpha} \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (a). We know that $jk(1+2^r)^i= jk(1+2^r k_1)$ for some k_1. If $j=j_1 2^{m-r-\alpha}$, then $jk2^r\equiv 0 \bmod 2^m$. So $jk(1+2^r)^i\equiv jk \bmod 2^m$. Hence $\xi^{jk} = \xi^{jk(1+2^r)} = \ldots = \xi^{jk(1+2^r)^{2^{m-r-\alpha}-1}}$.

Now let $\xi^{jk} = \xi^{jk(1+2^r)} = \ldots = \xi^{jk(1+2^r)^{2^{m-r-\alpha}-1}}$. So $jk \equiv jk(1+2^r) \bmod 2^m$.
Hence $jk \equiv 0 \bmod 2^{m-r}$. So $2^{m-r-\alpha} \mid j$.

(b) If $2^{m-r-\alpha}$ does not divide j, then $(j, 2^m)<2^{m-r-\alpha}$ and $(jk, 2^m)<2^{m-r}$. So by Corollary 4.5 we have $S= 0$. Hence (b) follows.

Now we want to calculate the irreducible representations and irreducible characters of metacyclic 2-groups. In [8], P.A. Tucker gives a method based on the reduction of induced representations of finite groups. Let G be a split extension of A by G/A and let L be an irreducible representation of $K[A]$, where K is a field. Then by the method given in [8] we can calculate the irreducible components of the representations of G induced from L.

In [5], Y. Iida and T. Yamada studied the faithful irreducible characters of a metacyclic 2-group.

Let $G = \langle a,b : a^{2^m} = 1, b^{2^n} = a^{2^{m-s}}, a^b = a^{1+2^r}\rangle$ as earlier. Let α be integer, $0 \leq\alpha<m-r$, and let k be a positive integer such that $(k, 2^m)= 2^\alpha$ and let ξ be a primitive 2^m-*th* root of unity. Let $\sigma= \min\{t: tm-n-s>0, t\in \mathbf{Z}\}$. Note that $\sigma\geq 1$. Let $\sigma_1= \min\{t: tm-n-r-\alpha>0, t\in \mathbf{Z}\}$ and $\eta = \xi^{2^{\sigma_1 m-n-r-\alpha}}$ (from the conditions (a) and (b) for the uniquely reduced representation of G it follows that $n-m+r\geq 1$ so η is of order $2^{n-m+r+\alpha}$). Let $1\leq l\leq 2^{n-m+r+\alpha}$. Define $y_{l,k}^{\alpha}: G \rightarrow GL(2^{m-r-\alpha}, \mathbf{C})$ by

$$y_{l,k}^{\alpha}(a) = \text{diag}\,(\xi^k, \xi^{k(1+2^r)}, \ldots, \xi^{k(1+2^r)^{2^{m-r-\alpha}-1}})$$

$$y_{l,k}^{\alpha}(b) = \begin{pmatrix} 0 & . & 0 & \eta^l \xi^{k2^{(\sigma+1)m-n-r-s-\alpha}} \\ 1 & 0 & . & 0 \\ . & . & . & . \\ 0 & . & 1 & 0 \end{pmatrix}$$

where $m-r-\alpha>1$ and

$$y_{l,k}^{\alpha}(a) = \text{diag}\,(\xi^k, \xi^{k(1+2^r)})$$

$$y_{l,k}^{\alpha}(b) = \begin{pmatrix} 0 & \eta^l \xi^{k2^{\sigma m-n-s+1}} \\ 1 & 0 \end{pmatrix}$$

where $m-r-\alpha= 1$.

We want to show that this is a representation of G. In order to do this we need to prove that:

(a) $y_{l,k}^{\alpha}(a^{2^m}) = I_{2^{m-r-\alpha}}$
(b) $y_{l,k}^{\alpha}(b^{2^n}) = y_{l,k}^{\alpha}(a^{2^{m-s}})$;
(c) $y_{l,k}^{\alpha}(a^b) = y_{l,k}^{\alpha}(a^{1+2^r})$.

Since $(\text{diag}\,(\xi^k, \xi^{k(1+2^r)}, \ldots, \xi^{k(1+2^r)^{2^{m-r-\alpha}-1}}))^{2^m} = I_{2^{m-r-\alpha}}$, so (a) follows. Let $U= \text{diag}\,(u_1,u_2,\ldots,u_d)$ and V be a $d\times d$ matrix as follows:

$$V = \begin{pmatrix} 0 & . & 0 & v \\ 1 & 0 & . & 0 \\ . & . & . & . \\ 0 & . & 1 & 0 \end{pmatrix},$$

For $1\leq j\leq d$, let $C_j= vI_j$. By induction we can prove that

$$V^j = \begin{pmatrix} 0 & C_j \\ I_{d-j} & 0 \end{pmatrix}$$

for $1\leq j<d$ and $V^d= C_d=vI_d$.

If r is a non-negative integer, then $r= kd+j$ for some non-negative k and $0\leq j<d$ and $V^r=(C_d)^k V^j$. Hence any non-negative power of V is either a diagonal matrix or its diagonal entries are zero. In particular, for $r= kd$, we have $V^r= (C_d)^k= v^k I_d$.

We know that when G is non-split then $s<min\{m-r, r+1\}\leq r+1$, so $s\leq r$. If G splits then $s=0$. Hence $r-s\geq 0$. Therefore $2^m \mid 2^{m+r-s}$. So, for $f\geq 0$, $k2^{m-s}(1+2^r)^f \equiv k2^{m-s}$ mod 2^m. Hence $y^{\alpha}_{l,k}(a^{2^{m-s}}) = \xi^{k2^{m-s}} I_{2^{m-r-\alpha}}$. Also letting E= $y^{\alpha}_{l,k}(b^{2^{m-r-\alpha}})$ we see that $E = \eta^l \xi^{k2^{(\sigma+1)m-n-r-s-\alpha}} I_{2^{m-r-\alpha}}$ Since η has order $2^{n-m+r+\alpha}$ so $y^{\alpha}_{l,k}(b^{2^n}) = E^{2^{n-m+r+\alpha}} = \xi^{k2^{\sigma m-s}} I_{2^{m-r-\alpha}} = \xi^{k2^{m-s}} I_{2^{m-r-\alpha}}$. So (b)follows.

Now

$$V^{-1} = \begin{pmatrix} 0 & 1 & . & 0 \\ . & 0 & 1 & 0 \\ . & . & . & . \\ v^{-1} & 0 & . & 0 \end{pmatrix}$$

and $V^{-1}UV=$ diag $(u_2,u_3,\ldots,u_n,u_1)$. This shows that

$$y^{\alpha}_{l,k}(a^b) = \mathrm{diag}(\xi^{k(1+2^r)}, \xi^{k(1+2^r)^2}, \ldots, \xi^{k(1+2^r)^{2^{m-r-\alpha}-1}}, \xi^k)$$

and

$$y^{\alpha}_{l,k}(a^{1+2^r}) = \mathrm{diag}(\xi^{k(1+2^r)}, \xi^{k(1+2^r)^2}, \ldots, \xi^{k(1+2^r)^{2^{m-r-\alpha}-1}}, \xi^{k(1+2^r)^{2^{m-r-\alpha}}}).$$

But since $(k, 2^m)= 2^{\alpha}$ and since $2^{m-\alpha} \mid (1+2^r)^{2^{m-r-\alpha}}$ by Lemma 4.1, so $\xi^{k(1+2^r)^{2^{m-r-\alpha}}} = \xi^k$. Therefore (c) follows.

We know that, for $j\geq 0$, either V^j is a diagonal matrix or all of its diagonal entries are zero. Since U is diagonal, so, for $i,j\geq 0$, either U^iV^j is a diagonal matrix or its diagonal entries are zero. It is a diagonal matrix whenever d | j; otherwise its diagonal entries are zero.

We want to show that the representations $y^{\alpha}_{l,k}$ are irreducible. By the above, for $i,j\geq 0$, $y^{\alpha}_{l,k}(a^ib^j)$ is either a diagonal matrix or all its diagonal entries are zero, and it is diagonal precisely when $2^{m-r-\alpha}/j$. Let S be the sum of the diagonal in this case. Thus $S = \eta^{j_1 l} \xi^{j_1 k 2^{(\sigma+1)m-n-r-s-\alpha}} \sum_{f=0}^{2^{m-r-\alpha}-1} \xi^{ik(1+2^r)^f}$ where $j= j_1 2^{m-r-\alpha}$. Furthermore, by Lemma 4.6, if $2^{m-r-\alpha} \mid i$, then $S = \eta^{j_1 l} \xi^{ik+j_1 k 2^{(\sigma+1)m-n-r-s-\alpha}} 2^{m-r-\alpha}$, while, if $2^{m-r-\alpha}$ does not divide i, then $S = 0$.

Let $\chi^{\alpha}_{l,k}$ denote the character of $\chi^{\alpha}_{l,k}$. Then, from the above, for $i, j\geq 0$, we have:

$$\chi^{\alpha}_{l,k}(a^i b^j) = \begin{cases} 2^{m-r-\alpha}\eta^{j_1 l} \xi^{ik+j_1 k 2^{(\sigma+1)m-n-r-s-\alpha}} & \text{if } 2^{m-r-\alpha} \mid i, j, \\ & \text{and } j=j_1 2^{m-r-\alpha} \\ 0 & \text{otherwise} \end{cases}$$

So $\chi^{\alpha}_{l,k}$ has exactly $(2^{r+\alpha})(2^{n-m+r+\alpha}) = 2^{n-m+2r+2\alpha}$ non-zero values, each with norm $2^{m-r-\alpha}$.

To show that $y^{\alpha}_{l,k}$ is irreducible, it suffices to show that $[\chi^{\alpha}_{l,k}, \chi^{\alpha}_{l,k}]= 1$; but

$$[\chi^{\alpha}_{l,k}, \chi^{\alpha}_{l,k}] = \frac{1}{|G|} \sum_{g\in G} \chi^{\alpha}_{l,k}(g)\overline{\chi^{\alpha}_{l,k}(g)}$$
$$= \frac{1}{2^{n+m}} (2^{2(m-r-\alpha)} 2^{n-m+2r+2\alpha} + 0) = 1.$$

Let us consider $\chi^{\alpha}_{l,k}$ for different values of α, l and k. Let $\chi^{\alpha}_{l,k} = \chi^{\alpha'}_{l',k'}$ where $0\leq \alpha, \alpha' < m-r$ and $1\leq k, k'$, $(k, 2^m)= 2^{\alpha}= (k', 2^m)$. Since $\chi^{\alpha}_{l,k}(1) = \chi^{\alpha'}_{l',k'}(1)$ so $\alpha = \alpha'$. Now consider $\chi^{\alpha}_{l,k}(a^{2^{m-r-\alpha}}) = \chi^{\alpha'}_{l',k'}(a^{2^{m-r-\alpha}})$. Then $\xi^{k2^{m-r-\alpha}} = \xi^{k'2^{m-r-\alpha}}$. Hence $k2^{m-r-\alpha} \equiv k'2^{m-r-\alpha}$ mod 2^m. Therefore $k\equiv k'$ mod $2^{r+\alpha}$.

Finally, as $\chi^{\alpha}_{l,k}(b^{2^{m-r-\alpha}}) = \chi^{\alpha}_{l',k'}(b^{2^{m-r-\alpha}})$, then $\xi^{k2^{(\sigma+1)m-n-r-s-\alpha}} \eta^l = \xi^{k'2^{(\sigma+1)m-n-r-s-\alpha}} \eta^{l'}$. As $k2^{m-r-\alpha} \equiv k'2^{m-r-\alpha}$ mod 2^m and $\sigma m-n-s>0$, so $\eta^l= \eta^{l'}$.

As the order of η is equal to $2^{n-m+r+\alpha}$ and as $1\leq l$, $l'\leq 2^{n-m+r+\alpha}$, we conclude that $l= l'$.

Thus, for each α, $0\leq \alpha<m-r$, the characters $\chi^{\alpha}_{l,k}$, $1\leq l\leq 2^{n-m+r+\alpha}$, $k= k_1 2^{\alpha}$, $1\leq k<2^r$, $(k_1, 2)= 1$, are distinct; there are $2^{n-m+r+\alpha}2^{r-1}(2-1)$ such characters.

To show that these, together with the $2^{r+n}= |G:G'|$ linear characters, are the only irreducible characters of G, it suffices to show that $|G|= 2^{r+n} + \sum \chi^{\alpha}_{l,k}(1)^2$, where the sum is over all α, $0\leq \alpha<m-r$, all l, $1\leq l\leq 2^{n-m+r+\alpha}$, and all k, $1\leq k\leq 2^{r+\alpha}$, $(k, 2^m) = 2^{\alpha}$.

We know that

$$2^{m-r} -1= (2-1)(2^{m-r-1} +2^{m-r-2} + \ldots +1).$$

Therefore

$$2^{m+n}-2^{r+n}= 2^{r+n}(2^{m-r}-1) = 2^{r+n}(2-1)(2^{m-r-1}+2^{m-r-2}+\ldots+1)$$
$$= 2^{n-m+2r-1}(2-1)2^{2(m-r)}+2^{n-m+2r}(2-1)2^{2(m-r-1)}+\ldots+2^{n+r-2}(2-1)2^2.$$

It follows that

$$2^{r+n}+2^{n-m+2r+1}(2-1)2^{2(m-r)}+2^{n-m+2r}(2-1)2^{2(m-1)}+\ldots+2^{n+r-2}(2-1)2^2 = 2^{r+n}+2^{m+n}-2^{n+r}= 2^{n+m}= |G|,$$

as required.

From the above discussions we have the following theorem.

Theorem 4.7. Let $G = \langle a, b: a^{2^m} = 1, b^{2^n} = a^{2^{m-s}}, a^b = a^{1+2^r}\rangle$ as above. Let $0 \le \alpha < m-r$. Let k be a positive integer such that $(k, 2^m) = 2^\alpha$ and let ξ be a primitive 2^m-*th* root of unity. Let $\sigma = \min\{t: tm-n-s > 0, t \in \mathbf{Z}\}$. Let $1 \le l \le 2^{n-m+r+\alpha}$, and let $\eta = \xi^{2^{\sigma_1 m-n-r-\alpha}}$ where $\sigma_1 = \min\{t: tm-n-r-\alpha > 0, t \in \mathbf{Z}\}$. Define $y^{\alpha}_{l,k}: G \to GL(2^{m-r-\alpha}, \mathrm{C})$ by

$$y^{\alpha}_{l,k}(a) = \mathrm{diag}\,(\xi^k, \xi^{k(1+2^r)}, \ldots, \xi^{k(1+2^r)^{2m-r-\alpha-1}})$$

$$y^{\alpha}_{l,k}(b) = \begin{pmatrix} 0 & . & 0 & \eta^l \xi^{k2^{(\sigma+1)m-n-r-s-\alpha}} \\ 1 & 0 & . & 0 \\ . & . & . & . \\ 0 & . & 1 & 0 \end{pmatrix}$$

where $m-r-\alpha > 1$ and

$$y^{\alpha}_{l,k}(a) = \mathrm{diag}\,(\xi^k, \xi^{k(1+2^r)})$$

$$y^{\alpha}_{l,k}(b) = \begin{pmatrix} 0 & \eta^l \xi^{k2^{\sigma m-n-s+1}} \\ 1 & 0 \end{pmatrix}$$

where $m-r-\alpha = 1$. Then each non-linear irreducible representation of G is equivalent to one of the form $y^{\alpha}_{l,k}$ for some l, k and α. Let $\chi^{\alpha}_{l,k}$ denote the character of this representation. Then

$$\chi^{\alpha}_{l,k}(a^i b^j) = \begin{cases} 2^{m-r-\alpha}\eta^{j_1 l}\,\xi^{ik+j_1k2^{(\sigma+1)m-n-r-s-\alpha}} & \text{if } 2^{m-r-\alpha} \mid i, j, \\ & \text{and } j = j_1 2^{m-r-\alpha} \\ 0 & \text{otherwise} \end{cases}$$

Now we want to calculate $c(G)= q(G)$ of a metacyclic 2-group when $Z(G)$ is a cyclic subgroup. In order to do this we will consider two different cases.

Lemma 4.8. Let χ be a faithful irreducible character of a faithful metacyclic 2-group G. Then $m_{\mathrm{Q}}(\chi)= 1$ except when G is a generalized quaternion group of order 2^m for some m.

Proof. See [[5], Corollary 4.7]

Corollary 4.9. Let G be a non-exceptional metacyclic 2-group. Then $c(G)= q(G)$.

Proof. This follows from Lemma 2.6 and Lemma 4.8.

Theorem 4.10. Let G be a non-cyclic metacyclic 2-group with cyclic centre. Let G be split so that $G = \langle a,b: a^{2^m} = b^{2^{m-r}} = 1, a^b = a^{1+2^r}\rangle$ as earlier. Then $c(G)= q(G)= p(G)= 2^m$.

Proof. Suppose that $(k, 2^m) = 1$, that is, $\alpha = 0$; we then have $\eta = 1$ and $l = 1$ since $n= m-r$ by Lemma 4.2(d). Hence

$$\chi^{0}_{1,k}(a^i b^j) = \begin{cases} 2^{m-r}\xi^{ik} & \text{if } 2^{m-r} \mid i \\ 0 & \text{otherwise.} \end{cases}$$

Since $\xi^{ik} \neq 1$ for $1 \le i \le 2^m$ and since $b^{2^{m-r}} = 1$, $\chi^{o}_{l,k}$ is faithful and $\chi^{0}_{1,k}(g) \neq 0$ for all $g \in Z(G)$ and equal to zero otherwise. In the other hand, when $\alpha \neq 0$, the kernel has more than one element as $a^{2^{m-1}}$ is in the kernel, that is, $\chi^{\alpha}_{l,k}$ is not faithful. So G satisfies the conditions of Theorem 3.4. Since $|Z(G)| = 2^r$ by Lemma 4.3, so $c(Z(G)) = 2^r$ and we have $c(G) = 2^{m-r} c(Z(G)) = 2^{m-r}2^r = 2^m$.

Since $Z(G) \cap B = 1$, so $B_G = 1$ and $p(G) \le |G: B| = 2^m$. But $q(G) \le p(G)$. This implies that $c(G) = q(G) = p(G) = 2^m$.

Now let G be a metacyclic 2-group and let G be non-split with cyclic centre. By Corollary 4.3 we have $G = \{a,b: a^{2^m} = 1, b^{2^n} = a^{2^{m-r}}, a^b = a^{1+2^r}\}$ in the earlier notation. Let $\alpha > 0$ and k be such that $(k, 2^m) = 2^\alpha$. Then $a^{2^{m-1}}$ is in the kernel of $\chi^{\alpha}_{l,k}$ for all l, so when $\alpha > 0$, the characters $\chi^{\alpha}_{l,k}$ are not faithful. Since the centre is cyclic so there exists a faithful character. Moreover, any faithful character must have degree 2^{m-r} and in this case $\alpha = 0$. Since the degree of each faithful irreducible character of G is 2^{m-r} and $(\chi^{0}_{l,k}(1))^2 = |G: Z(G)|$ and the value of this character is zero in $G \backslash Z(G)$, so G satisfies the conditions of Theorem 3.4 and $c(G) = q(G) = 2^{m-r}2^{n-m+2r} = 2^{n+r}$. Therefore we have the following lemma.

Lemma 4.11. Let G be a metacyclic 2-group with cyclic centre and let G be non-split. Suppose that $G= \langle$a,b :

$a^{2^m} = 1,\ b^{2^n} = a^{2^{m-r}},\ a^b = a^{1+2^r} >$ as earlier. Then $c(G) = q(G) = 2^{n+r}$.

Lemma 4.12. Let G be a metacyclic 2-group. Then, in the standard notation, there exists an i such that $H = <b^{2^{n-m+s}} a^i>$ has order 2^{m-s} and $H \cap B = 1$.

Proof. By induction on d it is possible to prove that

$$(b^j a^i)^d = b^{jd} a^{i((1+2^r)^{(d-1)j} + \ldots + (1+2^r)^j + 1)} = b^{jd} a^{i\frac{(1+2^r)^{dj}-1}{(1+2^r)^j-1}}.$$

Let $d = 2^l$. Then $\frac{(1+2^r)^{jd}-1}{(1+2^r)^j-1} = 2^l k$, where $(k, 2) = 1$, by Lemma 4.1.

Let $1 \le i < 2^{m-s}$ be an integer such that $ik \equiv -1 \bmod 2^{m-s}$; since $(k,2) = 1$, such an i exists. Now let $H = < b^{2^{n-m+s}} a^i >$. Then $(b^{2^{n-m+s}} a^i)^{2^{m-s}} = 1$ and, for $0 \le j < m-s$, $(b^{2^{n-m+s}} a^i)^{2^j} = b^{2^{n-m+s+j}} a^{ik_1 2^j} \neq 1$ for some k_1 such that $(k_1, 2) = 1$. This shows that the order of H is 2^{m-s}.

Theorem 4.13. Let G be a non-cyclic metacyclic 2-group with cyclic centre. Let G be non-split. Suppose that $G = <a, b: a^{2^m} = 1,\ b^{2^m} = a^{2^{m-r}},\ a^b = a^{1+2^r} >$ as earlier. Then $c(G) = q(G) = p(G) = 2^{n+r}$.

Proof. Since $Z(G)$ is cyclic, so $s = r$. Then by Lemma 4.12 there exists i such that $H = <b^{2^{n-m+r}} a^i>$ has order 2^{m-r}. But $H \cap Z(G) = 1$, so by Lemma 3.1, $H_G = 1$. Therefore $p(G) \le 2^{n+r}$. Hence by using Lemma 4.11 and the fact that $c(G) \le p(G)$, the result follows.

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