FOCAL POINT AND FOCAL K-PLANE

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Abstract
This paper deals with the basic notions of k-taut immersions. These notions come from two special cases; that is, tight and taut immersions. Tight and taut based on height and distance functions respectively and their basic notions are normal bundle, end-point map, focal point, critical normal. We generalize height and distance functions to cylindrical function and define basic notions of k-taut immersions such as k-plane normal bundle, end k-plane map, focal k-plane, and critical k-plane normal. Then we prove index theorems for cylindrical function similar to the standard index theorems of distance function. In this way, the key point is the relation between focal point and focal k-plane.

Introduction

Let \( f: M \rightarrow \mathbb{R}^n \) be an immersion. \( f \) is said to be tight (convex, minimal) if every non-degenerate height function has the minimal number of critical points. This idea was introduced by Chern and Lashof [1] and studied by Kučjak and many others. A good reference is [2]. The immersions for which every non-degenerate distance function has the minimal number of critical points have been studied in [3]. Such immersions are called taut immersions.

The notion of tauteness has been generalized to k-taut immersion [4] by taking the distance from k-planes (rather than points) in \( \mathbb{R}^n \). The main results of [4] have been published in [5], and the rest of [5] deals with the generalization of "spherical two-piece property" introduced by Banchoff [6] for k-taut immersions. Only [4] and [5] are compact manifolds considered. In [7], the general properties of k-taut immersions on compact and non-compact manifolds were investigated.

The study of k-taut immersions needs theorems for cylindrical functions similar to the standard theorems for distance functions. This generalization has not been done in [4], or elsewhere, thoroughly, and is so important that this paper is devoted to it. Notions such as normal bundle, end-point map, focal point, critical normal etc. need to be generalized. The key notions are focal point and focal k-plane.

Notation and Definition

We work throughout in the category of smooth \((C^\infty)\) manifolds and smooth maps. \( M \) is always a connected \( m \)-manifold without boundary which is second countable. For convenience, when we say "the point \( \vec{e}^+ \in \mathbb{R}^n \) we mean the end point of the vector \( \vec{e}^+ \).

Definition 1.1.

Let \( f: M \rightarrow \mathbb{R}^n \) be an immersion. For each k-plane \( \Pi \) in \( \mathbb{R}^n \), we define the k-cylindrical function \( C_\Pi: M \rightarrow \mathbb{R} \) by

\[
C_\Pi(p) = \inf \{ ||f(p) - x|| : x \in \Pi \}.
\]

When \( k = 0 \); that is, when \( \Pi \) is a point, \( C_\Pi \) is a distance function.

Definition 1.2.

Let \( f: M \rightarrow \mathbb{R}^n \) be an immersion. We can define the normal bundle of \( f \) in the usual way, having as total space the subset \( N(M) \subset M \times \mathbb{R}^n \) of pairs \( (p, \vec{n}) \) for which \( p \in M \)
and $<df(q), n^*> = 0$ for all $q \in T_p M$. Let $\eta: N(M) \to R^n$ be the end-point map given by $\eta(p, n^*) = f^*(p) + n^*$. The point $e^* \in R^n$ is a focal point of $(M, p)$ with multiplicity $\mu > 0$ if $e^* = f(p^*) + n^*$, where $(p, n^*) \in N(M)$ and the Jacobian of $\eta$ at $(p, n^*)$ has nullity $\mu$. The point $e^*$ is a focal point of $M$ if $e^*$ is a focal point of $(M, p)$ for some $p \in M$. The critical points of $\eta$ are called critical normals.

These notions are essential in the study of distance functions. Now we generalize these to use them for cylindrical functions.

**Definition 1.3.**

Let $f: M \to R^n$ be an immersion and $N(M)$ be the total space of the normal bundle of $f$. Let $G(k, R^n)$ be the Grassmann manifold of $k$-dimensional vector subspaces in $R^n$. We define $L_N(M) \subset N(M) \times G(k, R^n)$ as follows: $(p, n^*, \lambda) \in L_N(M)$ if, and only if, $(p, n^*) \in N(M)$ and $n^*$ is orthogonal to $\lambda$. We observe that $L_N(M)$ is locally the same as $R^n \times G(k, R^n)$. Therefore, $L_N(M)$ is a manifold of dimension $n+k(n-1)$. It is also worth noting that if $L_k(R^n)$ is the set of all $k$-planes in $R^n$, then dim $L_k(R^n) = n + k(n-1)$. The map $\eta: L_N(M) \to L_k(R^n)$ is defined as follows: $\eta((p, n^*, \lambda))$ is the $k$-plane parallel to $\lambda$ which passes through the point $f^*(p) + n^*$ in $R^n$. Of course, $\eta_{\lambda}$ is the same as $\eta$.

We call $L_N(M)$ the $k$-plane normal bundle and $\eta_{\lambda}$ the end $k$-plane map. The $k$-plane $\Pi \in L_k(R^n)$ is a focal $k$-plane of $(M, p)$ with multiplicity $\mu > 0$ if $\Pi = \eta_{\lambda}(p, n^*, \lambda)$ for some $(p, n^*, \lambda) \in L_N(M)$ and the Jacobian of $\eta$ at $(p, n^*, \lambda)$ has nullity $\mu$. $\Pi \in L_k(R^n)$ will be called a focal $k$-plane of $M$ if $\Pi$ is a focal $k$-plane of $(M, p)$ for some $p \in M$. We call the critical points of $\eta_{\lambda}$ the critical $k$-plane normals.

We make use of the following theorem (see [8]):

**Theorem 1.4.** (Sard). If $f_1$ and $f_2$ are smooth manifolds of the same dimension and $f: M_1 \to M_2$ is a smooth map, then the set of critical values of $f$ has measure 0 in $M_2$.

**Corollary 1.5.** For almost all $\Pi \in L_k(R^n)$, $\Pi$ is not a focal $k$-plane of $M$.

**Proof.** Obvious.

**First and Second Fundamental Forms**

The main concepts in this paper are focal point and focal $k$-plane. For a better understanding of these notions, it is necessary to introduce "the first and the second fundamental forms" of a manifold in Euclidean space. We will not attempt to give an invariant definition, but will make use of a fixed local coordinate system: Let $x = (x_1, x_2, \ldots, x_n)$ be a chart of the manifold $M$, and $y = (y_1, y_2, \ldots, y_n)$ be a chart of $R^n$. Then $f: M \to R^n$ determines $n$ smooth functions:

$$y_1 = y_1(x), y_2 = y_2(x), \ldots, y_n = y_n(x).$$

These functions will be written briefly as $Y^i(x) = (y_1(x), y_2(x), \ldots, y_n(x))$, where $Y = (Y_1, Y_2, \ldots, Y_n)$. The first fundamental form associated with the coordinate systems is defined to be the symmetric matrix of real valued functions

$$(g_{ij}) = \left( \frac{\partial Y^i}{\partial x_j}, \frac{\partial Y^j}{\partial x_i} \right).$$

The second fundamental form, on the other hand, is a symmetric matrix $(l^i_\nu)$ of vector valued functions and is defined as follows: the vector $\frac{\partial^2 Y^i}{\partial x_j \partial x_j}$ at a point of $M$ can be expressed as the sum of a vector tangent to $M$ and a vector normal to $M$. Define $l^\nu_i$ to be the normal component of $\frac{\partial^2 Y^i}{\partial x_j \partial x_j}$. Given any unit vector $v^\nu$ which is normal to $M$ at $p$, the matrix

$$\left( v^\nu, \frac{\partial^2 Y^i}{\partial x_j \partial x_j} \right) = (v^\nu, l^\nu_i)$$

is called the "second fundamental form of $M$ at $p$ in the direction $v^\nu".

We can choose coordinates such that $(g_{ij})$, evaluated at $p$, be the identity matrix. Then the eigenvalues of $(v^\nu, l^\nu_i)$ are called the principal curvatures $k_1, \ldots, k_m$ of $M$ at $p$ in the normal direction $v^\nu$. The reciprocals $k^{-1}_1, \ldots, k^{-1}_m$ of these principal curvatures are called principal radii of curvature.

Of course, it may happen that the matrix $(v^\nu, l^\nu_i)$ is singular, in which case, one or more of the $k^{-1}_i$ will not be defined.

Now consider the normal line $l$ consisting of all $f(p) + tv^\nu$ ($t$ is real), where $v^\nu$ is a fixed unit vector normal to $M$ at $p$.

**Lemma 2.1.** The focal points of $(M, p)$ along $l$ are precisely the points $f(p) + tk^{-1}_i v^\nu$, where $1 \leq i \leq m, k_i \neq 0$. Thus, there are at most $m$ focal points of $(M, p)$ along $l$, each being counted with its proper multiplicity.
Proof. See Milnor's Morse theory ([9]).

Distance Functions and Cylindrical Functions
Now we consider critical points of distance and cylindrical functions. For a fixed $Y_o^+ \in \mathbb{R}^n$, distance function $L_{r_0}^+ : M \rightarrow \mathbb{R}$ is defined as follows:

$$L_{r_0}(Y^+(x_1, ..., x_n)) = \|Y^+(x_1, ..., x_n) - Y_o^+\|^2.$$ 

Thus

$$\frac{\partial L_{r_0}}{\partial x_i} = 2 \frac{\partial Y^+}{\partial x_i} \cdot (Y^+ - Y_o^+).$$

Since $f$ is an immersion, $L_{r_0}^+$ has a critical point at $p$ if, and only if, either $f(p)^+ = Y_o^+$ or $f(p)^+ - Y_o^+$ is normal to $M$ at $p$. Therefore:

**Lemma 3.1.** If $\eta(p, n^+) = f(p)^+ + n^+ = Y_o^+$, then $p$ is a critical point of $L_{r_0}^+$. Conversely, if $p$ is a critical point of $L_{r_0}^+$, then there exists an $n^+$ such that $(p, n^+) \in N(M)$ and we have $\eta(n^+) = Y_o^+$. In particular, $p$ is a critical point of $L_{r_0}^+$ for every $p \in M$.

We have a similar result for end $k$-plane maps and cylindrical functions:

**Theorem 3.2.** If $\eta(p, n^+, \lambda) = \Pi$, then $p$ is a critical point of $C_{r_0}^+$. Conversely, if $p$ is a critical point of $C_{r_0}^+$, then there exists $(p, n^+, \lambda) \in L_{r_0}N(M)$ with $\eta(n^+) = \Pi$. In particular, $\Pi$ is a $k$-plane passing through $f(p)^+$, then $p$ is a critical point of $C_{r_0}^+$.

Proof. Let $Y_o^+(x_1, ..., x_n) = \Pi$ of the end point of $Y^+(x_1, ..., x_n)$ on $\Pi$. Then we have

$$C_{r_0}(Y^+(x_1, ..., x_n)) = \|Y^+(x_1, ..., x_n) - Y_o^+(x_1, ..., x_n)\|^2.$$ 

Thus,

$$\frac{\partial C_{r_0}}{\partial x_i} = 2 \frac{\partial Y^+}{\partial x_i} \cdot (Y^+ - Y_o^+).$$

Since $\frac{\partial Y^+}{\partial x_i}$ is parallel to $\Pi$ and $Y^+ - Y_o^+$ is perpendicular to $\Pi$, we have

$$\frac{\partial C_{r_0}}{\partial x_i} = 2 \frac{\partial Y^+}{\partial x_i} \cdot (Y^+ - Y_o^+).$$

First let $\eta(p, n^+, \lambda) = \Pi$. We have $(Y^+ - Y_o^+)^p = n^+$. Hence, the right hand side of (*) at $p$ is equal to zero for every $i$; i.e. for each $i$, $\frac{\partial C_{r_0}}{\partial x_i} p = 0$. This means that $p$ is a critical point of $C_{r_0}^+$. Conversely, let $p$ be critical point of $C_{r_0}^+$. We put $(Y^+ - Y_o^+)^p = n^+$. Let $\lambda$ be the $k$-plane parallel to $\Pi$ passing through the origin. Then in (*), since $\frac{\partial C_{r_0}}{\partial x_i} p = 0$ for every $i$, we deduce that $n^+ \perp \frac{\partial Y^+}{\partial x_i} p$ for every $i$.

Therefore, $(p, n^+, \lambda) \in L_{r_0}N(M)$ and $\eta(n^+) = \Pi$.

Degeneracy of Distance Function and Cylindrical Function
There is an intimate relation between degenerate critical points of distance functions and focal points. We have a similar relation for cylindrical functions. First we consider the distance function $L_{r_0}^+$:

$$L_{r_0}(Y^+(x_1, ..., x_n)) = \|Y^+(x_1, ..., x_n) - Y_o^+\|.$$ 

The second partial derivatives are

$$\frac{\partial^2 L_{r_0}}{\partial x_i \partial x_j} = 2 \left( \frac{\partial Y^+}{\partial x_i} \cdot \frac{\partial Y^+}{\partial x_j} + \frac{\partial^2 Y^+}{\partial x_i \partial x_j} (Y^+ - Y_o^+) \right).$$

At a critical point, if we assume $Y^+ = Y^+ + n^+$, this becomes

$$\frac{\partial^2 L_{r_0}}{\partial x_i \partial x_j} = 2 (g_{ij} - n^+ n_j).$$

Therefore, if we choose $x_1, ..., x_n$ around $p \in M$ such that $(g_{ij})$ becomes the identity matrix, we have:

**Lemma 4.1.** The point $p \in M$ is a degenerate critical point of $L_{r_0}^+$ if, and only if, $Y_o^+$ is a focal point of $(M, p)$.

We have a similar result for cylindrical functions (see [4]):

**Theorem 4.2.** The point $p \in M$ is a degenerate critical point of $C_{r_0}^+$ if, and only if, $\Pi$ is a focal $k$-plane of $(M, p)$.

Now we get other criterions for degeneracy of cylindrical functions. First we look at the second partial derivatives of $C_{r_0}^+$:

$$\frac{\partial^2 C_{r_0}}{\partial x_i \partial x_j} = 2 \left( \frac{\partial^2 Y^+}{\partial x_i \partial x_j} (Y^+ - Y_o^+) + \frac{\partial Y^+}{\partial x_i} (\frac{\partial Y^+}{\partial x_j} - \frac{\partial Y^+}{\partial x_j}) \right).$$
If we assume \( Y_\ast = Y^\ast + iv^\ast \), this, at a critical point, becomes

\[
\frac{\partial^2 C_n}{\partial x_i \partial x_j} = 2 \left( g_{\bar{y}} \frac{\partial Y_\ast}{\partial x_i} \frac{\partial Y_\ast}{\partial x_j} - iv^\ast \cdot l_y^\ast \right).
\]

Thus:

**Theorem 4.3.** The point \( p \in M \) is a degenerate critical point of \( C_n \) if, and only if,

\[
(g_{\bar{y}} \frac{\partial Y_\ast}{\partial x_i} \frac{\partial Y_\ast}{\partial x_j} - iv^\ast \cdot l_y^\ast)
\]

is singular at that point.

If we put

\[
H_y(t) = \frac{1}{t} \frac{\partial Y_\ast}{\partial x_i} \frac{\partial Y_\ast}{\partial x_j} + v^\ast \cdot l_y^\ast,
\]

then

\[
\frac{\partial^2 C_n}{\partial x_i \partial x_j} = 2t \left( g_{\bar{y}} - H_y(0) \right).
\]

Therefore:

**Theorem 4.4.** \( p \) is a degenerate critical point of \( C_n \) if, and only if, \( \frac{1}{t} \) is an eigenvalue of \( (H_y(t)) \).

Since the eigenvalue of \( (H_y(t)) \) and \( (v^\ast \cdot l_y^\ast) \) are generally different, if \( \Pi \) is a focal \( k \)-plane, \( f(p)^\ast + iv^\ast \) may not be a focal point. In this case, we have the following result:

**Theorem 4.5.** If \( f : M \to R^n \) is not substantial (i.e., \( f(M) \subset R^n \)) such that \( s < n \), then any \( k \)-plane \( (k < n - s) \) through a focal point in \( R^n \) perpendicular to \( R^n \) and to the normal ray passing through that point is a focal \( k \)-plane.

**Proof.** Corresponding to this plane, \( \frac{\partial Y_\ast}{\partial x_i} \frac{\partial Y_\ast}{\partial x_j} \) for all \( i, j \).

Therefore, \( \frac{\partial Y_\ast}{\partial x_i} \frac{\partial Y_\ast}{\partial x_j} \) is the zero matrix and \( (H_y(t)) = (v^\ast) \cdot (l_y^\ast) \).

Thus, since \( \frac{1}{t} \) is an eigenvalue of \( (v^\ast \cdot l_y^\ast) \), it is also an eigenvalue of \( (H_y(t)) \).

Although distance function is a special case of cylindrical function, the type of their critical points can be different. For example, \( p \) is a non-degenerate critical point of \( L_x^\ast \). But we show that, for some \( k \)-planes passing through \( f(p)^\ast \), \( p \) is a degenerate critical point of \( C_n \); if \( \Pi \) is transversal to \( f \), then \( M \) is of dimension \( m + k - n \). If \( m + k - n \geq 1 \), then \( \Pi \) is a non-degenerate critical point of \( C_n \); therefore, it is degenerate. Hence, we have the following result:

**Theorem 4.6.** If \( f : M \to R^n \) is an immersion, \( \Pi \in L_x(R^n) \) is passing through \( f(p) \) and is transversal to \( f \), and \( k \geq n - m + 1 \), then \( p \) is a degenerate critical point of \( C_n \).

We also observe that if \( f : R^n \to R^n \) is defined by \( f(x) = (x, x')(n \geq 2) \) and \( \Pi \) is the \( x \)-axis, then \( x \neq 0 \) is a non-degenerate critical point of \( L_x^\ast \), but it is a degenerate critical point of \( C_n \). This inspires the following:

**Theorem 4.7.** If \( f : M \to R^n \) is an immersion and \( \Pi = df_p(T_pM) \), then \( p \) is a degenerate critical point of \( C_n \) for every \( p \in M \).

**Proof.** We choose coordinates \( x_1, \ldots, x_n, x_\ast \) at \( f(p) \) such that \( x_1, \ldots, x_n \) vary in \( \Pi \), and (by the help of \( df \) and exponential map) \( x_1, \ldots, x_\ast \) be coordinates for \( M \) in a neighbourhood of \( p \). Then

\[
C_n(x_1, \ldots, x_\ast) = x_{m+1}^2 + \ldots + x_n^2,
\]

Thus,

\[
\frac{\partial C_n}{\partial x_i} = \frac{\partial x_{m+1}}{\partial x_i} + \ldots + 2x_n \frac{\partial x_n}{\partial x_i} = 0 \text{ at } (0, \ldots, 0) \in R^n.
\]

and

\[
\frac{\partial^2 C_n}{\partial x_i \partial x_j} = \frac{\partial^2 x_{m+1}}{\partial x_i \partial x_j} + \ldots + 2 \frac{\partial x_{m+1}}{\partial x_j} \frac{\partial x_{m+1}}{\partial x_i} = 0 \text{ at } (0, \ldots, 0) \in R^n.
\]

**Index Theorems for Cylindrical Functions**

Now we study non-degenerate critical points of cylindrical functions. In this way, we give several index theorems for cylindrical functions similar to the one for distance functions. (Index of a map at a non-degenerate critical point is equal to the number of negative eigenvalues of its Hessian at that point; see [9]). For distance functions we have:

**Theorem 5.1.** (Index theorem for \( L_{x_\ast}^\ast \)). The index of \( L_{x_\ast}^\ast \) at a non-degenerate critical point \( p \in M \) is equal to the number of focal points of \( (M, p) \) which lie on the segment \( Y_{x_\ast}^\ast \cap f(p)^\ast \); each focal point being counted with its multiplicity.
Proof. see [9].

For index theorems of cylindrical functions we need the following two lemmas:

Lemma 5.2. \((G_\gamma(t)) = (g_\gamma^i \cdot H_\gamma(t))\) is symmetric.

Proof. We have \((Y_\gamma^+ \cdot Y_\gamma^-) \cdot \frac{\partial Y_\gamma^+}{\partial x_i} = 0\) and \((Y_\gamma^- \cdot Y_\gamma^+) \cdot \frac{\partial Y_\gamma^+}{\partial x_j} = 0\).

Therefore,

\[
\begin{align*}
\frac{\partial Y_\gamma^+}{\partial x_j} \cdot \frac{\partial Y_\gamma^+}{\partial x_i} - \frac{\partial Y_\gamma^+}{\partial x_i} \cdot \frac{\partial Y_\gamma^+}{\partial x_j} + t v^+ \cdot \frac{\partial Y_\gamma^+}{\partial x_j} \cdot \frac{\partial Y_\gamma^+}{\partial x_i} = 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial Y_\gamma^-}{\partial x_j} \cdot \frac{\partial Y_\gamma^+}{\partial x_i} - \frac{\partial Y_\gamma^-}{\partial x_i} \cdot \frac{\partial Y_\gamma^+}{\partial x_j} + t v^- \cdot \frac{\partial Y_\gamma^+}{\partial x_j} \cdot \frac{\partial Y_\gamma^+}{\partial x_i} = 0
\end{align*}
\]

By subtracting these relations, we deduce that

\[
\begin{align*}
\frac{\partial Y_\gamma^+}{\partial x_i} \cdot \frac{\partial Y_\gamma^-}{\partial x_j} = \frac{\partial Y_\gamma^+}{\partial x_j} \cdot \frac{\partial Y_\gamma^-}{\partial x_i}
\end{align*}
\]

that is, the matrix \(\left(\frac{\partial Y_\gamma^+}{\partial x_i} \cdot \frac{\partial Y_\gamma^+}{\partial x_j}\right)\) is symmetric. Since \((g_\gamma) = \frac{\partial Y_\gamma^+}{\partial x_i} \cdot \frac{\partial Y_\gamma^+}{\partial x_j}\) and \((v^+ \cdot I_\gamma)\) are also symmetric, we deduce that \((G_\gamma(t))\) is symmetric.

Lemma 5.3. We can choose coordinates \(x_1, \ldots, x_n\) such that, at \(p\),

(a) \(g_\gamma = I\) and

(b) \(G(t) = (G_\gamma(t))\) is a diagonal matrix.

Proof. Let \(x_1, \ldots, x_n\) be any coordinates centered at \(p\). Let \(\tilde{x}_1, \ldots, \tilde{x}_n\) be some other coordinates which we are going to choose. We have

\[
\begin{align*}
\frac{\partial Y_\gamma^+}{\partial x_i} \cdot \frac{\partial Y_\gamma^+}{\partial x_j} = v^+ \cdot \frac{\partial Y_\gamma^+}{\partial x_i} \cdot \frac{\partial Y_\gamma^+}{\partial x_j}
\end{align*}
\]

If we write \(J = (J_{ab}) = \left(\frac{\partial x_i}{\partial x_a}\right)\), then we have

\[
\begin{align*}
\frac{\partial Y_\gamma^+}{\partial x_i} \cdot \frac{\partial Y_\gamma^+}{\partial x_j} = J_{ab} G_\gamma(t) J_{ij}
\end{align*}
\]

that is, \(\tilde{G}(t) = J^T G(t) J\). Also, \(\tilde{g}_\gamma = J^T g_\gamma J\); i.e. \(\tilde{g} = J^T g J\).

It is obvious that \((g_\gamma)\) can be chosen to be positive definite. Therefore, by simultaneous diagonalization theorem (110), there is an orthogonal matrix \(W\) such that \(W^T (g_\gamma) W = I\) and \(W^T (G(t)) W\) is a diagonal matrix. Now (by the help of the exponential map) we choose \(\tilde{x}_1, \ldots, \tilde{x}_n\) such that \((J) = (\frac{\partial x_i}{\partial x_a}) = W\). Then, with these new coordinates, \((g_\gamma) = I\) and \((G_\gamma(t))\) is a diagonal matrix.

Theorem 5.4. (Index theorem for \(C_n\)). The index of \(C_n\) at a non-degenerate critical point \(p \in M\) is equal to the number of eigenvalues of \((H_\gamma(t))\) which are larger than \(\frac{1}{t}\).

Proof. By Lemma 5.3, we may assume that, at \(p\), \((g_\gamma) = I\) and \((G_\gamma(t))\), and therefore, \((H_\gamma(t))\) is diagonal. Let

\[
\begin{pmatrix}
\begin{array}{cc}
a_{11} & 0 \\
0 & a_{nm}
\end{array}
\end{pmatrix}
\]

We form the characteristic polynomial of \((I - H_\gamma(t))\):

\[
(x - \frac{1}{t} + a_{11})(x - \frac{1}{t} + a_{22}) \cdots (x - \frac{1}{t} + a_{mm}) = 0.
\]

\(1 - a_{11} < 0 \Rightarrow a_{11} > 1\) and the theorem is proved.

We prove the following lemma which is useful in our subsequent discussion.

Lemma 5.5. If \(\tilde{\Pi}\) is the plane parallel to \(\Pi\) passing through the origin, then \(\frac{\partial Y_\gamma^+}{\partial x_i} = \text{projection of } \frac{\partial Y_\gamma^+}{\partial x_i} \text{ onto } \tilde{\Pi}\).

Proof. Let \(e_1^+, \ldots, e_k^+\) be an orthonormal basis for \(\tilde{\Pi}\) and \(\Pi = t v^+ + \tilde{\Pi}\). For simplicity, we take \(f(0)^+\) as the origin. Let \(Y_\gamma^+ = t v^+ + \lambda_1 e_1^+ + \ldots + \lambda_k e_k^+\). We know for every \(j\), \((Y_\gamma^- \cdot Y_\gamma^+)^e_j = 0\). Hence, for each \(j\),

\[
0 = (Y_\gamma^- \cdot Y_\gamma^+)^e_j = Y_\gamma^- e_j^+ - t v^+ e_j^+ - \lambda_j^+
\]

Therefore, \(\gamma_j = Y_\gamma^- e_j^+\). Thus,

\[
Y_\gamma^- = t v^+ + \sum_{j=1}^{k} (Y_\gamma^- e_j^+) e_j^+.
\]
From this we get
\[ \frac{\partial Y}{\partial x_i} = \sum_{j=1}^t \left( \frac{\partial Y}{\partial x_i} \cdot e_j^* \right) e_j^* , \]
that is,
\[ \frac{\partial Y}{\partial x_i} = \text{projection onto } \tilde{\Pi} \text{ of } \frac{\partial Y}{\partial x_i} . \]
Now we consider the matrix
\[ (Q_y) = \begin{pmatrix} \frac{\partial Y}{\partial x_i} & \frac{\partial Y}{\partial x_j} \end{pmatrix} . \]

We choose coordinates at a critical point \( p \) such that
\( (Q_y) \) has a useful form: Let \( T = df(T_x) \). Suppose \( \tilde{\Pi} \) and \( \tilde{T} \) be the planes passing through the origin parallel to \( \Pi \) and \( T \) respectively. We write \( \tilde{\Pi} = (\Pi \cap \tilde{T}) \oplus \tilde{T} \) and \( \tilde{\Pi} = (\Pi' \cap \tilde{T}) \oplus \tilde{T} \). Let \( r = \dim (\Pi' \cap \tilde{T}) \). We choose orthonormal basis \( a_1^* \ldots a_r^* \) for \( \Pi \cap \tilde{T} \) and extend it to the orthonormal basis \( a_1^* \ldots a_r^* , e_1^* \ldots e_{m-r}^* \) for \( \tilde{T} \). We choose coordinates \( x_i \) at \( p \) such that \( e_i = \frac{\partial Y}{\partial x_i} \). We also extend \( a_1^* \ldots a_r^* \) to \( a_1^* \ldots a_r^* , e_1^* \ldots e_{m-r}^* \) to be an orthonormal basis for \( \tilde{\Pi} \).
Then, at \( p \),
\[ (Q_y) = \begin{pmatrix} \frac{\partial Y}{\partial x_i} \end{pmatrix} = I \text{ and } \frac{\partial Y}{\partial x_i} = a_i^* , i=1, \ldots , m . \]
\[ \frac{\partial Y}{\partial x_i} = \text{projection of } \frac{\partial Y}{\partial x_i} . \]
\[ a_j^* = \begin{pmatrix} a_j^* , 1 \leq j \leq r \\sum_{h=1}^{r} (a_j^* \cdot e_h^*) e_h^* , r+1 \leq j \leq m \end{pmatrix} . \]

Hence, at \( p \),
\[ (B_y) = \begin{pmatrix} \frac{\partial Y}{\partial x_i} & \frac{\partial Y}{\partial x_j} \end{pmatrix} = \begin{pmatrix} \frac{\partial Y}{\partial x_i} \end{pmatrix} = \begin{pmatrix} r \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \end{pmatrix} . \]
where
\[ C = \left( \sum_{h=1}^{r} (a_j^* \cdot e_h^*) (a_j^* \cdot e_h^*) \right) , r+1 \leq i,j \leq m . \]

Then \( (Q_y) = I - \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \), which gives the following corollaries:

**Theorem 5.6.** (Index theorem for \( C_\Pi \) when \( k=n-1 \).) The index of \( C_\Pi \) at a non-degenerate critical point \( p \in M \) is equal to the number of focal points which are on the positive direction of \( v^* \).

**Proof.** In this case we have \( r=m \), where \( r=\dim (\Pi' \cap \tilde{T}) \).

Therefore, \( (B_y) = \begin{pmatrix} \frac{\partial Y}{\partial x_i} & \frac{\partial Y}{\partial x_j} \end{pmatrix} = I \). Thus, \( (H_y) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \).

By simultaneous diagonalization, we may assume that \( (v^* , l_v^*) \) is diagonal. Now we observe that the eigenvalues of \( (H_y) \) are in the form of \( 1+a \). Therefore, the number of eigenvalues of \( (H_y) \) which are larger than \( 1 \) is equal to the number of positive eigenvalues of \( (v^* , l_v^*) \), and we are done.

**Theorem 5.7.** If \( \Pi \) is a focal \((n-1)\)-plane, then any \( \Pi' \) parallel to \( \Pi \) is also a focal \((n-1)\)-plane.

**Proof.** Let \( \Pi \) be a focal \((n-1)\)-plane passing through the point \( f(p)^* + \lambda v^* \), and \( \Pi' \) be an arbitrary \((n-1)\)-plane intersecting the direction of \( v^* \). Let \( \lambda \) be the eigenvalue of \( (1+\lambda)X \). Thus, there is a vector \( X_i \) such that
\[ \frac{1}{\lambda} (1+\lambda)X_i = 1 \text{ and } X_i = \frac{1}{\lambda} X_i . \]

We write
\[ \frac{1}{\lambda} (1+\lambda)X_i = \frac{1}{\lambda} \left( \frac{1}{\lambda} (1+\lambda)X_i \right) = \frac{1}{\lambda} \left( \frac{1}{\lambda} (1+\lambda)X_i \right) = \frac{1}{\lambda} X_i \]
that is, $\lambda$ is an eigenvalue of $(\lambda I + v^* l_0^-)$. Therefore, $\Pi'$ is a focal $(n-1)$-plane.

Now let $(H_\nu(t)) = (1 - B_\nu + D_\nu)$, where $B_\nu = \sum_{i=1}^{m} \frac{\partial \Pi'}{\partial x_i} \frac{\partial v^*}{\partial x_i}$ and $D_\nu = v^* l_0^-$. We evaluate $(D_\nu)$ at $p$ and $(H_\nu(t))$ for $\Pi' = \eta_\nu(p, v^*, \lambda) = at p$:

Let $\lambda_1(t), \ldots, \lambda_m(t)$ be the eigenvalues of $(H_\nu(t))$ and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $(D_\nu)$. Also there is a $t_0 > 0$ such that all functions $\lambda_i(t)$ ($1 \leq i \leq m$) are non-decreasing on $[t_0, \infty)$. If $\lambda_i(t_0) > \frac{1}{t_0} (1 \leq i \leq p)$, then $\forall t \geq t_0$, $\lambda_i(t) > \lambda_i(t_0)$.

Therefore:

Theorem 5.9. There is a $t_0$ such that for every $t > t_0$, in which $\Pi' = \eta_\nu(p, v^*, \lambda)$ is not a focal k-plane, the index of $C_{\mu}$ at $p$ is at most equal to the number of focal points on $r(v^*)$ (including the focal point at infinity which is counted with its multiplicity).

Theorem 5.10. In theorem 5.9, if $(D_\nu) = (v^* l_0^-)$ has no zero eigenvalue at $p$, then the index $C_{\mu}$ at $p$ is exactly equal to the number of focal points on $r(v^*)$.

References: