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FOCAL POINT AND FOCAL *K*-PLANE

A.A. Alemzadeh

Department of Mathematics, Teacher Training University, Tehran, Islamic Republic of Iran

Abstract

This paper deals with the basic notions of *k-taut* immersions. These notions come from two special cases; that is, tight and taut immersions. Tight and taut based on *hight* and *distance* functions respectively and their basic notions are normal bundle, end-point map, focal point, critical normal. We generalize hight and distance functions to cylindrical function and define basic notions of k-taut immersions such as *k-plane normal bundle, end k-plane map, focal k-plane*, and *critical k-plane normal.* Then we prove index theorems for cylindrical function similar to the standard index theorems of distance function. In this way, the key point is the relation between focal point and focal k-plane.

Introduction

Let f: $M \to R^n$ be an immersion. f is said to be *tight* (convex, minimal) if every non-degenerate *hight* function has the minimal number of critical points. This idea was introduced by Chern and Lashof [1] and studied by Kuijec and many others. A good reference is [2]. The immersions for which every non-degenerate *distance* function has the minimal number of critical points have been studied in [3]. Such immersions are called *taut* immersions.

The notion of tautness has been generalized to *k*- taut immersion in [4] by taking the distance from *k*-planes (rather than points) in R^n. The main results of [4] have been published in [5], and the rest of [5] deals with the generalization of "spherical two-piece property" introduced by Banchoff [6] for *k*-taut immersions. Only [4] and [5] are compact manifolds considered. In [7], the general properties of *k*-taut immersions on compact and non-compact manifolds were investigated.

The study of *k*-taut immersions needs theorems for cylindrical functions similar to the standard theorems for distance functions. This generalization has not been done in [4], or elsewhere, thoroughly, and is so important that this paper is devoted to it. Notions such as normal bundle, end-point map, focal point, critical normal etc. need to be generalized. The key notions are focal point and focal *k*-plane.

Keywords: Cylindrical function; Focal *K*-plane; Focal point; *K*-Plane map; *K*-plane normal bundle

Notation and Definition

We work throughout in the category of smooth (C^∞) manifolds and smooth maps. *M* is always a connected *m*-manifold without boundary which is second countable. For convenience, when we say " the point $\vec{e} \in R^n$" we mean the end point of the vector $\vec{e}$.

Definition 1.1.

Let $f: M \to R^n$ be an immersion. For each *k*-plane Π in R^n, we define the *k-cylindrical function* C_Π: $M \to R$ by

$$C_\Pi(p) = inf\{\|f(p) - x\|^2 : x \in \Pi\}.$$

When $k = 0$; that is, when Π is a point, C_Π is a distance function.

Definition 1.2.

Let $f: M \to R^n$ be an immersion. We can define the *normal bundle* of f in the usual way, having as total space the subset $N(M) \subset M \times R^n$ of pairs $(p, \vec{n})$ for which $p \in M$

and $<df_p(q).\vec{n}>=0$ for all $q \in T_pM$. Let $\eta: N(M) \to R^n$ be the *end - point map* given by $\eta(p, \vec{n}) = \vec{f}(p) + \vec{n}$. The point $\vec{e} \in R^n$ is a *focal point* of (M, p) with multiplicity $\mu > 0$ if $\vec{e} = \vec{f(p)} + \vec{n}$, where $(p, \vec{n}) \in N(M)$ and the Jacobian of η at $(p, \vec{n})$ has nullity μ. The point $\vec{e}$ is a focal point of M if $\vec{e}$ is a focal point of (M, p) for some $p \in M$. The critical points of η are called *critical normals.*

These notions are essential in the study of distance functions. Now we generalize these to use them for cylindrical functions.

Definition 1.3.

Let $f: M \to R^n$ be an immersion and $N(M)$ be the total space of the normal bundle of f. Let $G(k, R^n)$ be the Grassmann manifold of k-dimensional vector subspaces in R^n. We define $L_kN(M) \subset N(M) \times G(k, R^n)$ as follows: $(p, \vec{n}, \wedge) \in L_kN(M)$ if, and only if, $(p, \vec{n}) \in N(M)$ and $\vec{n}$ is orthogonal to $\wedge$. We observe that $L_kN(M)$ is locally the same as $R^n \times G(k, R^{n-1})$. Therefore, $L_kN(M)$ is a manifold of dimension $n+k(n-1-k)$. It is also worth noting that if $L_k(R^n)$ is the set of all k-planes is R^n, then $\dim L_k(R^n) = n + k(n-1-k)$. The map $\eta_k: L_kN(M) \to L_k(R^n)$ is defined as follows: $\eta_k(p, \vec{n}, \wedge)$ is the k-plane parallel to $\wedge$ which passes through the point $\vec{f}(p)+\vec{n}$ in R^n. Of course, η_0 is the same as η.

We call $L_kN(M)$ the *k-plane normal bundle* and η_k the *end k-plane map*. The k-plane $\Pi \in L_k(R^n)$ is a *focal k-plane* of (M,p) with multiplicity $\mu > 0$ if $\Pi = \eta_k(p, \vec{n}, \wedge)$ for some $(p, \vec{n}, \wedge) \in L_kN(M)$ and the Jacobian of η_k at $(p, \vec{n}, \wedge)$ has nullity μ. $\Pi \in L_k(R^n)$ will be called a *focal k-plane* of M if Π is a focal k-plane of (M,p) for some $p \in M$. We call the critical points of η_k the *critical k-plane normals.*

We make use of the following theorem (see [8]):

Theorem 1.4. (Sard). If M_1 *and* M_2 *are smooth manifolds of the same dimension and* $f: M_1 \to M_2$ *is a smooth map, then the set of critical values of f has measure o in* M_2.

Corollary 1.5. *For almost all* $\Pi \in L_k(R^n)$, Π *is not a focal k-plane of M.*

Proof. Obvious.

First and Second Fundamental Forms

The main concepts in this paper are focal point and focal k-plane. For a better understanding of these notions, it is necessary to introduce "the first and the second fundamental forms" of a manifold in Euclidean space. We will not attempt to give an invariant definition, but will make use of a fixed local coordinate system: Let $x = (x_1, x_2, \ldots, x_m)$ be a chart of the manifold M, and $Y = (y_1, y_2, \ldots, y_n)$ be a chart of R^n. Then $f: M \to R^n$ determines n smooth functions:

$$Y_1 = y_1 fx^{-1}, Y_2 = y_2 fx^{-1}, \ldots, Y_n = y_n fx^{-1}.$$

These functions will be written briefly as $\vec{Y}(x_1, \ldots, x_m)$, where $\vec{Y} = (Y_1, \ldots, Y_n)$.

The *first fundamental form* associated with the coordinate systems is defined to be the symmetric matrix of real valued functions

$$(g_{ij}) = \left(\frac{\partial \vec{Y}}{\partial x_i} . \frac{\partial \vec{Y}}{\partial x_j}\right).$$

The *second fundamental form*, on the other hand, is a symmetric matrix $(\vec{l}_{ij})$ of vector valued functions and is defined as follows: the vector $\frac{\partial^2 \vec{Y}}{\partial x_i \partial x_j}$ at a point of M can be expressed as the sum of a vector tangent to M and a vector normal to M. Define $\vec{l}_{ij}$ to be the normal component of $\frac{\partial^2 \vec{Y}}{\partial x_i \partial x_j}$. Given any unit vector $\vec{v}$ which is normal to M at p, the matrix

$$(\vec{v} . \frac{\partial^2 \vec{Y}}{\partial x_i \partial x_j}) = (\vec{v} . \vec{l}_{ij})$$

is called the "second fundamental form of M at p in the direction $\vec{v}$."

We can choose coordinates such that (g_{ij}), evaluated at p, be the identity matrix. Then the eigenvalues of $(\vec{v} . \vec{l}_{ij})$ are called the *principal curvatures* $k_1, \ldots, k_m$ of M at p in the normal direction $\vec{v}$. The reciprocals $k_1^{-1}, \ldots, k_m^{-1}$ of these principal curvatures are called *principal radii of curvature*. Of course, it may happen that the matrix $(\vec{v} . \vec{l}_{ij})$ is singular, in which case, one or more of the k_i^{-1} will not be defined.

Now consider the normal line l consisting of all $\vec{f(p)} + t\vec{v}$ (t is real), where $\vec{v}$ is a fixed unit vector normal to M at p.

Lemma 2.1. *The focal points of* (M,p) *along l are precisely the points* $\vec{f(p)} + k_i^{-1}\vec{v}$, *where* $1 \le i \le m$, $k_i \neq 0$. *Thus, there are at most m focal points of* (M,p) *along l, each being counted with its proper multiplicity.*

Proof. See Milnor's Morse theory ([9]).

Distance Functions and Cylindrical Functions

Now we consider critical points of distance and cylindrical functions. For a fixed $\vec{Y_0} \in R^n$, distance function $\vec{L_{Y_0}}: M \to R$ is defined as follows:

$$\vec{L_{Y_0}}(\vec{Y}(x_1, \ldots, x_m)) = \|\vec{Y}(x_1, \ldots, x_m) - \vec{Y_0}\|^2.$$

Thus

$$\frac{\partial \vec{L_{Y_0}}}{\partial x_i} = 2\frac{\partial \vec{Y}}{\partial x_i}.(\vec{Y} - \vec{Y_0}).$$

Since f is an immersion, $\vec{L_{Y_0}}$ has a critical point at p if, and only if, either $\vec{f(p)} = \vec{Y_0}$ or $\vec{f(p)} - \vec{Y_0}$ is normal to M at p. Therefore:

Lemma 3.1. *If $\eta(p,\vec{n}) = \vec{f(p)} + \vec{n} = \vec{Y_0}$, then p is a critical point of $\vec{L_{Y_0}}$. Conversely, if p is a critical point of $\vec{L_{Y_0}}$, then there exists an $\vec{n}$ such that $(p,\vec{n}) \in N(M)$ and we have $\eta(p,\vec{n}) = \vec{Y_0}$. In particular, p is a critical point of $\vec{L_{f(p)}}$ for every $p \in M$.*

We have a similar result for end k-plane maps and cylindrical functions:

Theorem 3.2. *If $\eta_k(p, \vec{n}, \wedge) = \Pi$, then p is a critical point of C_Π. Conversely, if p is a critical point of C_Π, then there exists $(p, \vec{n}, \wedge) \in L_kN(M)$ with $\eta_k(p, \vec{n}, \wedge) = \Pi$. In particular, Π is a k-plane passing through $\vec{f(p)}$, then p is a critical point of C_Π.*

Proof. Let $\vec{Y_*}(x_1,\ldots,x_m)$ be the projection of the end point of $\vec{Y}(x_1, \ldots, x_m)$ on Π. Then we have

$$C_\Pi(\vec{Y}(x_1, \ldots, x_m)) = \|\vec{Y}(x_1, \ldots, x_m) - \vec{Y_*}(x_1, \ldots, x_m)\|^2.$$

Thus,

$$\frac{\partial C_\Pi}{\partial x_i} = 2(\vec{Y} - \vec{Y_*})(\frac{\partial \vec{Y}}{\partial x_i} - \frac{\partial \vec{Y_*}}{\partial x_i}).$$

Since $\frac{\partial \vec{Y_*}}{\partial x_i}$ is parallel to Π and $\vec{Y} - \vec{Y_*}$ is perpendicular to Π, we have

$$\frac{\partial C_\Pi}{\partial x_i} = 2\frac{\partial \vec{Y}}{\partial x_i}(\vec{Y} - \vec{Y_*}). \quad (*)$$

First let $\eta_k(p, \vec{n}, \wedge) = \Pi$. We have $(\vec{Y} - \vec{Y_*})p = \vec{n}$. Hence, the right hand side of (*) at p is equal to zero for every i; i.e. for each i, $(\frac{\partial C_\Pi}{\partial x_i})p = 0$. This means that p is a critical point of C_Π. Conversely, let p be critical point of C_Π. We put $(\vec{Y} - \vec{Y_*})p = \vec{n}$. Let $\wedge$ be the k-plane parallel to Π passing through the origin. Then in (*), since $(\frac{\partial C_\Pi}{\partial x_i})p = 0$ for every i, we deduce that $\vec{n} \perp (\frac{\partial \vec{Y}}{\partial x_i})_p$ for every i. Therefore, $(p, \vec{n}, \wedge) \in L_kN(M)$ and $\eta_k(p, \vec{n}, \wedge) = \Pi$.

Degeneracy of Distance Function and Cylindrical Function

There is an intimate relation between degenerate critical points of distance functions and focal points. We have a similar relation for cylindrical functions. First we consider the distance function $\vec{L_{Y_0}}$:

$$\vec{L_{Y_0}}(\vec{Y}(x_1, \ldots, x_m)) = \|\vec{Y}(x_1, \ldots, x_m) - \vec{Y_*}\|^2.$$

The second partial derivatives are

$$\frac{\partial^2 \vec{L_{Y_0}}}{\partial x_i \partial x_j} = 2\left(\frac{\partial \vec{Y}}{\partial x_i}.\frac{\partial \vec{Y}}{\partial x_j} + \frac{\partial^2 \vec{Y}}{\partial x_i \partial x_j}(\vec{Y} - \vec{Y_*})\right).$$

At a critical point, if we assume $\vec{Y_*} = \vec{Y} + t\vec{v}$, this becomes

$$\frac{\partial^2 \vec{L_{Y_0}}}{\partial x_i \partial x_j} = 2(g_{ij} - t\vec{v}.\vec{l_{ij}}).$$

Therefore, if we choose $x_1,\ldots,x_m$ around $p \in M$ such that (g_{ij}) becomes the identity matrix, we have:

Lemma 4.1. *The point $p \in M$ is a degenerate critical point of $\vec{L_{Y_0}}$ if, and only if, $\vec{Y_0}$ is a focal point of (M,p).*

We have a similar result for cylindrical functions (see [4]):

Theorem 4.2. *The point $p \in M$ is a degenerate critical point of C_Π if, and only if, Π is a focal k-plane of (M,p).*

Now we get other criterions for degeneracy of cylindrical functions. First we look at the second partial derivatives of C_Π:

$$\frac{\partial^2 C_\Pi}{\partial x_i \partial x_j} = 2\left[\frac{\partial^2 \vec{Y}}{\partial x_i \partial x_j}(\vec{Y} - \vec{Y_*}) + \frac{\partial \vec{Y}}{\partial x_i}\left(\frac{\partial \vec{Y}}{\partial x_j} - \frac{\partial \vec{Y_*}}{\partial x_j}\right)\right].$$

If we assume $\vec{Y_*} = \vec{Y} + t\vec{v}$, this, at a critical point, becomes

$$\frac{\partial^2 C_\Pi}{\partial x_i \partial x_j} = 2\left(g_{ij} - \frac{\partial \vec{Y}}{\partial x_i} \cdot \frac{\partial \vec{Y_*}}{\partial x_j} - t\vec{v} \cdot \vec{l_{ij}}\right).$$

Thus:

Theorem 4.3. *The point $p \in M$ is a degenerate critical point of C_Π if, and only if,*

$$\left(g_{ij} - \frac{\partial \vec{Y}}{\partial x_i} \cdot \frac{\partial \vec{Y_*}}{\partial x_j} - t\vec{v} \cdot \vec{l_{ij}}\right)$$

is singular at that point.

If we put

$$H_{ij}(t) = \frac{1}{t}\frac{\partial \vec{Y}}{\partial x_i} \cdot \frac{\partial \vec{Y_*}}{\partial x_j} + \vec{v} \cdot \vec{l_{ij}},$$

then

$$\frac{\partial^2 C_\Pi}{\partial x_i \partial x_j} = 2t\left[\frac{1}{t} g_{ij} - H_{ij}(t)\right].$$

Therefore:

Theorem 4.4. *p is a degenerate critical point of C_Π if, and only if, $\frac{1}{t}$ is an eigenvalue of $(H_{ij}(t))$.*

Since the eigenvalue of $(H_{ij}(t))$ and $(\vec{v} \cdot \vec{l_{ij}})$ are generally different, if Π is a focal k-plane, $\vec{f(p)} + t\vec{v}$ may not be a focal point. In this case, we have the following result:

Theorem 4.5. *If $f: M \to R^n$ is not substantial* (i.e. $f(M) \subset R^s$) *such that $s<n$, then any k-plane ($k<n-s$) through a focal point in R^n perpendicular to R^s and to the normal ray passing through that point is a focal k-plane.*

Proof. Corresponding to this plane, $\frac{\partial \vec{Y}}{\partial x_i} \perp \frac{\partial \vec{Y_*}}{\partial x_j}$ for all i, j.

Therefore, $\left(\frac{\partial \vec{Y}}{\partial x_i} \cdot \frac{\partial \vec{Y_*}}{\partial x_j}\right)$ is the zero matrix and $(H_{ij}(t)) = (\vec{v} \cdot \vec{l_{ij}})$. Thus, since $\frac{1}{t}$ is an eigenvalue of $(\vec{v} \cdot \vec{l_{ij}})$, it is also an eigenvalue of $(H_{ij}(t))$.

Although distance function is a special case of cylindrical function, the type of their critical points can be different. For example, p is a non-degenerate critical point of $L_{\vec{f(p)}}$. But we show that, for some k-planes passing through $\vec{f(p)}$, p is a degenerate critical point of C_Π: if Π is transversal to f, then

$$M_0(C_\Pi) = \{p \in M: C_\Pi(p) = 0\}$$

is of dimension $m+k-n$. If $m+k-n \geq 1$; that is, $k \geq n-m+1$, then p is not an isolated critical point of C_Π; therefore, it is degenerate. Hence, we have the following result:

Theorem 4.6. *If $f: M \to R^n$ is an immersion, $\Pi \in L_k(R^n)$ is passing through $f(p)$ and is transversal to f, and $k \geq n-m+1$, then p is a degenerate critical point of C_Π.*

We also observe that if $f: R \to R^2$ is defined by $f(x) = (x, x^n)(n \geq 2)$ and Π is the x-axis, then $x=0$ is a non-degenerate critical point of $L_{\vec{(0,0)}}$, but it is a degenerate critical point of C_Π. This inspires the following:

Theorem 4.7. *If $f: M \to R^n$ is an immersion and $\Pi_p = df_p(T_pM)$, then p is a degenerate critical point of C_{Π_p} for every $p \in M$.*

Proof. We choose coordinates $x_1, \ldots, x_m, \ldots, x_n$ at $f(p)$ such that $x_1, \ldots, x_m$ vary in Π_p and (by the help of df and exponential map) $x_1, \ldots, x_m$ be coordinates for M in a neighbourhood of p. Then

$$C_{\Pi_p}(x_1, \ldots, x_m) = x^2_{m+1} + \ldots + x^2_n.$$

Thus,

$$\frac{\partial C_{\Pi_p}}{\partial x_i} = 2x_{m+1}\frac{\partial x_{m+1}}{\partial x_i} + \ldots + 2x_n\frac{\partial x_n}{\partial x_i} = 0 \text{ at } (0, \ldots, 0) \in R^m$$

and

$$\frac{\partial^2 C_{\Pi_p}}{\partial x_i \partial x_j} = 2x_{m+1}\frac{\partial^2 x_{m+1}}{\partial x_i \partial x_j} + \ldots + 2\frac{\partial x_{m+1}}{\partial x_j}\frac{\partial x_{m+1}}{\partial x_i} + \ldots = 0$$

at $(0, \ldots, 0) \in R^m$.

Index Theorems for Cylindrical Functions

Now we study non-degenerate critical points of cylindrical functions. In this way, we give several index theorems for cylindrical functions similar to the one for distance fundtions. (Index of a map at a non-degenerate critical point is equal to the number of negative eigenvalues of its Hessian at that point; see [9]). For distance functions we have:

Theorem 5.1. (Index theorem for $L_{\vec{Y_0}}$). *The index of $L_{\vec{Y_0}}$ at a non-degenerate critical point $p \in M$ is equal to the number of focal points of (M,p) which lie on the segment $\vec{Y_0} - \vec{f(p)}$; each focal point being counted with its multiplicity.*

Proof. see [9].

For index theorems of cylindrical functions we need the following two lemmas:

Lemma 5.2. *$(G_{ij}(t)) = (g_{ij} - t H_{ij}(t))$ is symmetric.*

Proof. We have $(\vec{Y_*} - \vec{Y}) . \frac{\partial \vec{Y_*}}{\partial x_i} = 0$ and $(\vec{Y_0} - \vec{Y}) . \frac{\partial \vec{Y_*}}{\partial x_j} = 0$. Therefore,

$$\frac{\partial \vec{Y_*}}{\partial x_j} . \frac{\partial \vec{Y_*}}{\partial x_i} - \frac{\partial \vec{Y}}{\partial x_j} . \frac{\partial \vec{Y_*}}{\partial x_i} + t\vec{v} . \frac{\partial^2 \vec{Y_*}}{\partial x_j \partial x_i} = 0$$

$$\frac{\partial \vec{Y_*}}{\partial x_i} . \frac{\partial \vec{Y_*}}{\partial x_j} - \frac{\partial \vec{Y}}{\partial x_i} . \frac{\partial \vec{Y_*}}{\partial x_j} + t\vec{v} . \frac{\partial^2 \vec{Y_*}}{\partial x_i \partial x_j} = 0$$

By subtracting these relations, we deduce that

$$\frac{\partial \vec{Y}}{\partial x_i} . \frac{\partial \vec{Y_*}}{\partial x_j} = \frac{\partial \vec{Y_*}}{\partial x_j} . \frac{\partial \vec{Y_*}}{\partial x_i};$$

that is, the matrix $(\frac{\partial \vec{Y}}{\partial x_i} . \frac{\partial \vec{Y_*}}{\partial x_j})$ is symmetric. Since $(g_{ij}) = (\frac{\partial \vec{Y}}{\partial x_i} . \frac{\partial \vec{Y}}{\partial x_j})$ and $(\vec{v} . \vec{l_{ij}})$ are also symmetric, we deduce that $(G_{ij}(t))$ is symmetric.

Lemma 5.3. *We can choose coordinates $x_1, \ldots, x_m$ such that, at p,*
(a) $g = (g_{ij}) = I$ and
(b) $G(t) = (G_{ij}(t)$ is a diagonal matrix.

Proof. Let $x_1, \ldots, x_m$ be any coordinates centered at p. Let $\tilde{x}_1, \ldots, \tilde{x}_m$ be some other coordinates which we are going to choose We have

$$G_{ij}(t) = \frac{\partial \vec{Y}}{\partial x_i} . \frac{\partial}{\partial x_j} (\vec{Y} - \vec{Y_*}) - t\vec{v} . \frac{\partial}{\partial x_i} (\frac{\partial \vec{Y}}{\partial x_j}),$$

$$\tilde{G}_{kl}(t) = \frac{\partial \vec{Y}}{\partial \tilde{x}_k} . \frac{\partial}{\partial \tilde{x}_l} (\vec{Y} - \vec{Y_*}) - t\vec{v} . \frac{\partial}{\partial \tilde{x}_k} (\frac{\partial \vec{Y}}{\partial \tilde{x}_l}).$$

If we write $J = (J_{ik}) = (\frac{\partial x_i}{\partial \tilde{x}_k})$, then we have

$$\tilde{G}_{kl}(t) = \frac{\partial x_i}{\partial \tilde{x}_k} G_{ij}(t) \frac{\partial x_j}{\partial \tilde{x}_l} = J_{ik} G_{ij}(t) J_{jl};$$

that is, $\tilde{G}(t) = J^T G(t) J$. Also, $\tilde{g}_{kl} = J_{ik} g_{ij} J_{jl}$; i.e. $\tilde{g} = J^T g J$.

It is obvious that $(g_{ij})_p$ can be chosen to be positive definite. Therefore, by simultaneous diagonalization theorem ([10]), there is an orthogonal matrix W such that $W^T(g)_p W = I$ and $W^T(G(t))_p W$ is a diagonal matrix. Now (by the help of the exponential map) we choose $\tilde{x}_1, \ldots, \tilde{x}_m$ such that $(J)_p = (\frac{\partial x_i}{\partial \tilde{x}_k})_p = W$. Then, with these new coordinates, $(g)_p = I$ and $(G(g))_p$ is a diagonal matrix.

Theorem 5.4. (Index theorem for C_Π). *The index of C_Π at a non-degenerate critical point $p \in M$ is equal to the number of eigenvalues of $(H_{ij}(t))$ which are larger than $\frac{1}{t}$.*

Proof. By Lemma 5.3, we may assume that, at p, $(g_{ij}) = I$ and $(G_{ij}(t))$, and therefore, $(H_{ij}(t))$ is diagonal. Let

$$(H_{ij}(t)) = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{mm} \end{bmatrix}.$$

We form the characteristic polynomial of $(\frac{1}{t} I - H_{ij}(t))$:

$$(x - \frac{1}{t} + a_{11})(x - \frac{1}{t} + a_{22}) \ldots (x - \frac{1}{t} + a_{mm}) = 0.$$

$\frac{1}{t} - a_{ii} < 0 \Rightarrow a_{ii} > \frac{1}{t}$ and the theorem is proved.

We prove the following lemma which is useful in our subsequent discussion.

Lemma 5.5. *If $\tilde{\Pi}$ is the plane parallel to Π passing through the origin, then $\frac{\partial \vec{Y_*}}{\partial x_i}$ = projection of $\frac{\partial \vec{Y}}{\partial x_i}$ onto $\tilde{\Pi}$.*

Proof. Let $\vec{e_1}, \ldots, \vec{e_k}$ be an orthonormal basis for $\tilde{\Pi}$ and $\Pi = t\vec{v} + \tilde{\Pi}$. For simplicity, we take $\vec{f(p)}$ as the origin. Let $\vec{Y_*} = t\vec{v} + \lambda_1 \vec{e_1} + \ldots + \lambda_k \vec{e_k}$. We know for every j, $(\vec{Y} - \vec{Y_*}) . \vec{e_j} = 0$. Hence, for each j,

$$0 = (\vec{Y} - \vec{Y_*}) . \vec{e_j} = \vec{Y} . \vec{e_j} - t\vec{v} . \vec{e_j} - \lambda_j.$$

Therefore, $\lambda_j = \vec{Y} . \vec{e_j}$. Thus,

$$\vec{Y_*} = t\vec{v} + \sum_{j=1}^{k} (\vec{Y} . \vec{e_j}) . \vec{e_j}.$$

From this we get

$$\frac{\partial \vec{Y_*}}{\partial x_i} = \sum_{j=1}^{k} \left(\frac{\partial \vec{Y}}{\partial x_i}.\vec{e_j}\right).\vec{e_j};$$

that is,

$\frac{\partial \vec{Y_*}}{\partial x_i}$ = projection onto $\tilde{\Pi}$ of $\frac{\partial \vec{Y}}{\partial x_i}$.

Now we consider the matrix

$$(Q_{ij}) = \left(g_{ij} - \frac{\partial \vec{Y}}{\partial x_i}.\frac{\partial \vec{Y_*}}{\partial x_i}\right).$$

We choose coordinates at a critical point p such that (Q_{ij}) has a useful form: Let $T = df_p(T_pM)$. Suppose $\tilde{\Pi}$ and $\tilde{T}$ be the planes passing through the origin parallel to Π and T respectively. We write $\tilde{T} = (\tilde{\Pi} \cap \tilde{T}) \oplus^{\perp} \tilde{T}_1$ and $\tilde{\Pi} = (\tilde{\Pi} \cap \tilde{T}) \oplus^{\perp} \tilde{\Pi}_1$. Let $r = dim\ (\tilde{\Pi} \cap \tilde{T})$. We choose orthonormal basis $\vec{a_1}, \ldots, \vec{a_r}$ for $\tilde{\Pi} \cap \tilde{T}$ and extend it to the orthonormal basis $\vec{a_1}, \ldots, \vec{a_r}, \vec{a_{r+1}}, \ldots, \vec{a_m}$ *for* $\tilde{T}$. We choose coordinates x_i at p such that $\vec{a_i} = \frac{\partial \vec{Y}}{\partial x_i}$. We also extend $\vec{a_1}, \ldots, \vec{a_r}$ to $\vec{a_1}, \ldots, \vec{a_r}, \vec{e_1}, \ldots, \vec{e_{k-r}}$ to be an orthonormal basis for $\tilde{\Pi}$. Then, at p,

$$(g_{ij}) = \left(\frac{\partial \vec{Y}}{\partial x_i}.\frac{\partial \vec{Y}}{\partial x_j}\right) = I \text{ and } \frac{\partial \vec{Y}}{\partial x_i} = \vec{a_i}, i = 1, \ldots, m,$$

$\frac{\partial \vec{Y_*}}{\partial x_i}$ = projection of

$$\vec{a_j} = \begin{cases} \vec{a_j}, 1 \le j \le r \\ \sum_{h=1}^{k-r} (\vec{a_j}.\vec{e_h})\vec{e_h}, r+1 \le j \le m \end{cases}$$

Hence, at p,

$$B_{ij} = \frac{\partial \vec{Y}}{\partial x_i}.\frac{\partial \vec{Y_*}}{\partial x_j} = \begin{cases} \vec{a_i}.\vec{a_j} \\ \sum_{h=1}^{k-r} (\vec{a_i}.\vec{e_h})(\vec{a_j}.\vec{e_h}) = C_{ij} \end{cases}$$

where in the first row $1 \le i \le m$, $1 \le j \le r$ and in the second row $1 \le i \le m$, $r+1 \le j \le m$. Therefore, at p,

$$(B_{ij}) = \left(\frac{\partial \vec{Y}}{\partial x_i}.\frac{\partial \vec{Y_*}}{\partial x_j}\right) = \begin{matrix} \updownarrow r \\ \updownarrow m-r \end{matrix}\begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}$$

where

$C = (C_{ij}) = (\sum_{h=1}^{k-r} (\vec{a_i}.\vec{e_h})(\vec{a_j}.\vec{e_h})), r+1 \le i,j \le m.$

Then $(Q_{ij}) = I - \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}$, which gives the following corollaries:

Theorem 5.6. (Index theorem for C_{Π} when $k=n-1$). *The index of C_{Π} at a non-degenerate critical point $p \in M$ is equal to the number of focal points which are on the positive direction of $\vec{v}$.*

Proof. In this case we have $r=m$, where $r = dim\ (\tilde{\Pi} \cap \tilde{T})$. Therefore, $(B_{ij}) = \left(\frac{\partial \vec{Y}}{\partial x_i}.\frac{\partial \vec{Y_*}}{\partial x_i}\right) = I$. Thus, $(H_{ij}(t)) = (\frac{1}{t} I + \vec{v}.\vec{l_{ij}})$. By simultaneous diagonalization, we may assume that $(\vec{v}.\vec{l_{ij}})$ is diagonal. Now we observe that the eigenvalues of $(H_{ij}(t))$ are in the form of $\frac{1}{t} + a_i$. Therefore, the number of eigenvalues of $(H_{ij}(t))$ which are larger than $\frac{1}{t}$ is equal to the number of positive eigenvalues of $(\vec{v}.\vec{l_{ij}})$, and we are done.

Theorem 5.7. *If Π is a focal $(n-1)$-plane, then any Π' parallel to Π is also a focal $(n-1)$-plane.*

Proof. Let Π be a focal $(n-1)$- plane passing through the point $\vec{f(p)} + t\vec{v}$, and Π' be an arbitrary $(n-1)$- plane intersecting the direction of $\vec{v}$ *at* $\vec{f(p)} + \lambda\vec{v}$. We know that $\frac{1}{t}$ is an eigenvalue of $(\frac{1}{t} I + \vec{v}.\vec{l_{ij}})$. Thus, there is a vector X_1 such that

$$(\frac{1}{t} I + \vec{v}.\vec{l_{ij}})(X_1) = \frac{1}{t} X_1.$$

We write

$$(\frac{1}{\lambda} I + \vec{v}.\vec{l_{ij}})(X_1) = (\frac{1}{t} I + \vec{v}.\vec{l_{ij}} + (\frac{1}{\lambda} - \frac{1}{t}) I)(X_1) = \frac{1}{t} X_1 + (\frac{1}{\lambda} - \frac{1}{t})(X_1) = \frac{1}{\lambda} X_1;$$

that is, $\frac{1}{\lambda}$ is an eigenvalue of $(\frac{1}{\lambda} I + v^{*}. l_{ij}^{*})$. Therefore, Π' is a focal (n-1)-plane.

Now let $(H_{ij}(t)) = (\frac{1}{t} B_{ij} + D_{ij})$, where $B_{ij} = \frac{\partial Y^{*}}{\partial x_i} \cdot \frac{\partial Y^{*}}{\partial x_i}$ and $D_{ij} = v^{*}. l_{ij}^{*}$. We evaluate (D_{ij}) at p and $(H_{ij}(t))$ for $\Pi_t = \eta_k(p, n^{*}, \wedge) =$ at p:

Let $\lambda_1(t), \ldots \lambda_m(t)$ be the eigenvalues of $(H_{ij}(t))$ and $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of (D_{ij}). Also there is a $t_0 > 0$ such that all functions $\lambda_i(t)$ $(1 \le i \le m)$ are non-decreasing on $[t_0, \infty)$, If $\lambda_i(t_0) > \frac{1}{t_0} (1 \le i \le p)$, then $\forall t \ge t_0$, $\lambda_i(t) \ge \lambda_i(t_0) > \frac{1}{t_0} \ge \frac{1}{t}$ $(1 \le i \le p)$. Hence $\lambda_i \ge 0$ $(1 \le i \le p)$; that is, the number of eigenvalues of $(H_{ij}(t))$ which are greater than $\frac{1}{t}$ $\forall t \ge t_0$, is at most the number of non-negative eigenvalues of (D_{ij}).

Now suppose $\lambda_1, \ldots, \lambda_q > 0, \lambda_{q+1}, \ldots, \lambda_m \le 0$. By continuity of eigenvalues, there exists a t'_0 such that $\forall t \ge t'_0$, $\lambda_1(t), \ldots, \lambda_q(t) > \frac{1}{t}$, $\lambda_{q+1}, \ldots, \lambda_m(t) \le \frac{1}{t}$; that is, the number of eigenvalues of $(H_{ij}(t))$ which are greater than $\frac{1}{t}$ $\forall t \ge t'_0$, is at least the number of positive eigenvalues of (D_{ij}). If $t_1 = \max\{t_0, t'_0\}$, then:

Lemma 5.8. For every $t > t_1$,

[*The number of positive eigenvalues of* (D_{ij})] $\le$

[*The number of eigenvalues of* $(H_{ij}(t))$ *which are larger than* $\frac{1}{t}$] $\le$

[*the number of non-negative eigenvalues of* (D_{ij})]

From the above lemma we see that for every $t > t_1$, the number of eigenvalues of $(H_{ij}(t))$ which are larger than $\frac{1}{t}$ is at most equal to the number of focal points (finite and infinite) on the normal ray $r(v^{*}) = \{f(p)^{*} + tv^{*} | t \ge 0\}$ (which are being counted with their multiplicities); also, if $(D_{ij}) = (v^{*}. l_{ij}^{*})$ has no zero eigenvalue at p, then this number is exactly equal to the number of focal points on $r(v^{*})$. In the special case when $\Pi_t = \eta_k(p, tv^{*}, \wedge)$ $(t > t_1)$ is not a focal k-plane, the index of $C_{\Pi t}$ at p is at most equal to the number of focal points (finite and infinite) on $r(v^{*})$; also, if $(D_{ij}) = (v^{*}.l_{ij}^{*})$ has no zero eigenvalue at p, this index is exactly equal to the number of focal points on $r(v^{*})$.
Therefore:

Theorem 5.9. *There is a* t_1 *such that for every* $t > t_1$ *in which* $\Pi = \eta_k(p, tv^{*}, \wedge)$ *is not a focal k-plane, the index of* C_Π *at p is at most equal to the number of focal points on* $r(v^{*})$ *(including the focal point at infinity which is counted with its multiplicity).*

Theorem 5.10. In theorem 5.9, if $(D_{ij}) = (v^{*}.l_{ij}^{*})$ *has no zero eigenvalue at p, then the index* C_Π *at p is exactly equal to the number of focal points on* $r(v^{*})$.

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