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FLOWS AND UNIVERSAL COMPACTIFICATIONS*

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Abstract

The main purpose of this paper is to establish a relation between universality of certain P-compactifications of a semitopological semigroup and their corresponding enveloping semigroups. In particular, we show that if we take P to be the property that the enveloping semigroup of a compactification of a semitopological semigroup S is left simple, a group, or the trivial singleton semigroup, then the universal P-compactification of S would be the distal, the right simple, or the right zero compactification, respectively.

Preliminaries

Throughout this section, unless stated otherwise, S will be a semitopological semigroup, written multiplicatively. For our notation, we shall follow Berglund *et al.* [1] as far as possible. In particular, we will frequently be referring to the notions of semigroup compactifications, universal P-compactifications, and their homomorphisms.

For a topological space Y, $C(Y)$ denotes the C^*-algebra of all bounded complex-valued continuous functions on Y. A subspace F of $C(S)$ is translation invariant if $L_s F \cup R_s F \subseteq F$, where

$$L_s F = \{L_s f : s \in S, f \in F\},$$
$$R_s F = \{R_s f : s \in S, f \in F\}$$

and

$$(L_s f)(t) = f(st) = (R_t f)(s) \qquad (s, t \in S, f \in C(S)).$$

A translation invariant closed subspace F of $C(S)$ containing the constant functions is left introverted if $T_\mu^F F \subseteq F$ for each $\mu \in M(F)$: = the set of all means on F, where

$$(T_\mu^F f)(s) = \mu(L_s f) \qquad (s \in S, \mu \in F^*, f \in F)$$

We remind the reader that a mean on F is a bounded linear functional μ on F which satisfies $\|\mu\| = \mu(1_s) = 1$, where 1_s denotes the constant function of value 1 on S.

If, in addition, F is a subalgebra, then F is called left *m-introverted* if $T_\mu^F F \subseteq F$ for each $\mu \in MM(F)$:= the set of all multiplicative means on F(i.e. the spectrum of F) which will hereafter be denoted by S^F. An admissible subspace of $C(S)$ is a norm closed, conjugate closed, left introverted subspace of $C(S)$, containing the constant functions. An m-admissible subalgebra of $C(S)$ is a left m-introverted unital C^*-subalgebra of $C(S)$.

The mapping $T^F: \mu \to T_\mu^F : MM(F) \to L(F, C(S))$ is called the introversion operator determined by F, where F is an m-admissible subalgebra of $C(S)$ and $L(F, C(S))$ is the space of all bounded linear operators from F into $C(S)$.

The reader is directed to Berglund *et al.* [1], Theorem 3.1.7, for the correspondence between compactifications

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of S and m-admissible subalgebras of $C(S)$.

Let F be an m-admissible subalgebra of $C(S)$, then S^F under the multiplication $\mu\nu := \mu \circ T_\nu^F$ $(\mu, \nu \in S^F)$ furnished with the Gelfand topology, is a compact Hausdorff right topological semigroup and hence we have a compactification (ε, S^F) where $\varepsilon : S \to S^F$ is the evaluation mapping. This compactification is called the canonical F-compactification of S. Conversely, if (ψ, X) is a compactification of S, then $\psi^*(C(X))$ is the corresponding m-admissible subalgebra of $C(S)$, where ψ^* is the dual mapping of ψ.

Some of the m-admissible subalgebras of C(S), that will be needed in the sequel, are the following:

SAP(S):= Closed linear span in $C(S)$ of the coefficients of continuous finite dimensional unitary representations of S,

LMC(S):= $\{f \in C(S): R_s f$ is relatively compact in $C(S)$ in the topology of pointwise convergence on $S\}$,

$D(S) := \{f \in LMC(S): (\mu\eta\nu)(f) = (\mu\nu)(f)$ for $\mu, \nu, \eta \in S^{LMC}$ with $\eta^2 = \eta\}$,

MD(S):= $\{f \in D(S): (\eta\mu)(f) = \mu(f)$ for $\eta, \mu \in S^{LMC}$ with $\eta^2 = \eta\}$,

SD(S):= $\{f \in D(S): (\mu\eta)(f) = \mu(f)$ for $\eta, \mu \in S^{LMC}$ with $\eta^2 = \eta\}$,

GP(S): MD(S) $\cap$ SD(S),

LZ(S):= $\{f \in D(S): f(st) = f(s)$ for $s, t \in S\}$,

RZ(S):= $\{f \in D(S): f(st) = f(t)$ for $s, t \in S\}$,

We shall occasionally suppress the letter S from the notations for these algebras. For a description of the above algebras, the reader may consult Chapter 4 of Berglund *et al.* [1], and also Junghenn [2].

Remark. Supoose that F_1 and F_2 are m-admissible subalgebras of C(S) with $F_1 \subseteq F_2$, and T^{F_1}, T^{F_2} are the introversion operators determined by F_1, F_2, respectively. Also suppose that (ε_i, S^{F_i}) is the canonical F_i-compactification, where $\varepsilon_i: S \to S^{F_i}$ is the evaluation mapping, $i = 1,2$. By proposition 1.2, there exists a continuous homomorphism $\pi: S^{F_1} \to S^{F_2}$ such that $\pi \circ \varepsilon_1 = \varepsilon_2$; (notice that the homomorphism π is just the restriction map of the functionals in S^{F_2} to the subalgebra F_1). Thus we have

$$(T_\mu^{F_1} f)(s) = \mu(L_s f) = \pi(\mu)(L_s f) = (T_\mu^{F_2} f)(s) \quad (s \in S,\ f \in F,\ \mu \in S^{F_2})$$

Therefore

$$T_\mu^{F_1} f = T_{\pi(\mu)}^{F_2} f \qquad (f \in F_2,\ \mu \in S^{F_1}) \quad (*).$$

Now let $\mu \in S^{F_2}$, since π is onto, there is a $\tilde{\mu} \in S^{F_1}$ such that $\pi(\tilde{\mu}) = \mu$, and

$$\pi(\tilde{\mu})(f) = \mu(f) \quad (f \in F_2).$$

By (*) we have

$$T_{\tilde{\mu}}^{F_1} f = T_{\pi(\tilde{\mu})}^{F_2} f = T_\mu^{F_2} f \quad (f \in F_2) \quad (**).$$

Thus we can suppress the letters F_1 and F_2 from the notation of introversion operators and we can always assume $\mu \in S^{F_1}$.

Flows and Compactifications

A *flow* is a triple (S,X,π), such that S is a semitopological semigroup, X is a compact Hausdorff space and $\pi: S \times X \to X$ is an action of S on X, meaning that $\pi(st, x) = \pi(s, \pi(t, x))$, where $\pi(s,.): X \to X$ is continuous for each $s \in S$. We often write (S, X) for (S,X,π), sx for $\pi(s, x)$ and π_s for $\pi(s,.)$. The *enveloping* semigroup of a flow (S,X), denoted by $\Sigma(S,X)$, is the closure of the semigroup $\{\pi_s: s \in S\}$ in X^X. A flow (S, X) is called separately continuous if the mapping $s \to sx: S \to X$ is also continuous for each $x \in X$..

A flow (S,X) is distal if $\lim_\alpha s_\alpha x_1 \neq \lim_\alpha s_\alpha x_2$ for each distinct pair x_1, x_2 in X and all nets $\{s_\alpha\}$ in S for which both limits exist. By a famous theorem of Ellis (see Berglund *et al.* [1], Theorem 1.6.9), (S,X) is distal if and only if $\Sigma(S,X)$ is a group.

Remark 2.1. (a) (See Berglund *et al.* [1], Proposition 1.6.5) $\Sigma(S,X)$ is a compact right topological subsemigroup of X^X, for the topology of pointwise convergence on X. The mapping $\sigma: S \to \Sigma(S,X)$, defined by $\sigma(s) = \pi_s$, is a homomorphism of S onto a dense subsemigroup of $\Sigma(S,X)$ and for each $s \in S$, the mapping $\zeta \to \pi_s\zeta: \Sigma(S,X) \to \Sigma(S,X)$ is continuous, where

$$(\pi_s\zeta)(x) = s(\zeta(x)) \quad (x \in X).$$

(b) Suppose that F is an m-admissible subalgebra of $LMC(S)$. If we define $\pi: S \times S^F \to S^F$, by $\pi(s, \mu)(f) = \mu(L_s f)$, then (S, S^F, π) is a flow. For each $\mu \in S^F$, there exists a $\tilde{\mu} \in MM(C(S))$ such that

$$\tilde{\mu}(f) = \mu(f) \quad (f \in F)$$

Now if $\{s_\alpha\}$ is a net in S such that $\lim_\alpha s_\alpha = s$, for some $s \in S$, then we have

$$\lim_\alpha \mu(L_{s_\alpha} f) = \lim_\alpha \widetilde{\mu}(L_{s_\alpha} f) = \widetilde{\mu}(L_s f) = \mu(L_s f) \quad (f \in F).$$

Thus the mapping $s \to s\mu$, where $s\mu = \pi(s, \mu)$, is continuous for each $\mu \in S^F$ therefore, (S, S^F, π) is a separately continuous flow (see also Lau [3], Lemma 4.3).

(c) Let (ψ, X) be a compactification of S. Since $\psi^*(C(S)) \subseteq LMC(S)$, by (b), (S, X) is a separately continuous flow and therefore by (a), $(\sigma, \Sigma(S,X))$ is a compactification of S. We call $\Sigma(S,X)$ the *enveloping semigroup* of the compactification (ψ, X); obviously this will cause no confusion. If, in addition, X has a right identity, then the mapping $\theta: x \to \zeta_x: X \to \Sigma(S,X)$ is one-to-one, where $\zeta_x(y) = xy$ for each $y \in X$. Thus by Lemma 2.4 infra, θ is an isomorphism. Therefore $(\sigma, \Sigma(S,X)) \cong (\psi, X)$.

In Theorems 2.6 and 2.7, we will give the relation between universality of certain P-compactifications of S and their corresponding enveloping semigroups. To prove these theorems we need the following statements:

Lemma 2.2. Let S be a semitopological semigroup. Then

(a) if $f \in D(S)$ and $\mu \in S^D$, then $T_\mu f \in MD(S)$,
(b) if $f \in SD(S)$ and $\mu \in S^{SD}$, then $T_\mu f \in GP(S)$,
(c) if $f \in RZ(S)$ and $\mu \in S^{RZ}$, then $T_\mu f$ is a constant function.

Proof. For an m-admissible subalgebra F of $LMC(S)$, if $\mu \in S^F$, then there exists $\widetilde{\mu} \in S^{LMC}$ such that $\widetilde{\mu}(f) = \mu(f)$ for each $f \in F$.
(a) Suppose that $f \in D(S)$ and $\mu \in S^D$, then we have

$$\nu\eta(T_{\widetilde{\mu}} f) = \nu\eta\widetilde{\mu}(f) = \nu\widetilde{\mu}(f) = \nu(T_{\widetilde{\mu}} f) \quad (f \in F, \nu, \eta \in S^{LMC}, \eta^2 = \eta)$$

Thus $T_{\widetilde{\mu}} f \in MD(S)$. Since

$$(T_\mu f)(s) = \mu(L_s f) = \widetilde{\mu}(L_s f) = (T_{\widetilde{\mu}} f)(s) \quad (s \in S, f \in F)$$

so $T_\mu f \in MD(S)$.

(b) By (a), if $f \in SD(S)$ and $\mu \in S^{SD}$, then $T_\mu f \in MD(S)$. Since $SD(S)$ is left introverted, $T_\mu f \in SD(S)$, thus $T_\mu f \in MD(S) \cap SD(S) = GP(S)$.

(c) For $f \in RZ(S)$ we have

$$(L_s f)(t) = f(st) = f(t) \quad (s, t \in S)$$

thus $L_s f = f$, and consequently $(T_\mu f)(s) = \mu(f)$ for all $s \in S$. Therefore, $T_\mu f$ is a constant function.□

Remark 2.3. Suppose that F and G are m-admissible subalgebras of $LMC(S)$ and $\{T_\mu f: \mu \in S^F, f \in F\}$ is a subset of G. Then the mapping $*: S^G \times S^F \to S^F$, defined by $(\mu * \nu)(f) = \mu(T_\nu f)$, is well-defined, and we also have.

$$(\mu_1\mu_2) * \nu(f) = \mu_1\mu_2(T_\nu f) = \mu_1(T_{\mu_2 * \nu} f) = \mu_1 * (\mu_2 * \nu)(f) \quad (f \in F, \mu_1, \mu_2 \in S^F),$$

so that

$$\mu_1 * (\mu_2 * \nu) = (\mu_1\mu_2) * \nu \quad (\mu_1, \mu_2 \in S^G \text{ and } \nu \in S^F).$$

Lemma 2.4. Let F and G be m-admissible subalgebras of $LMC(S)$ and let $\{T_\mu f: f \in F, \mu \in S^F\}$ be a subset of G. Then there exists a continuous homomorphism θ of S^G onto $\Sigma(S, S^F)$.

Proof. By Remark 2.1(b) (S, S^F, π) is a flow and $\pi: S \times S^F \to S^F$, defined by $\pi(s, \nu) = s\nu$, is separately continuous. Suppose that $\mu \in S^G$ and $\nu \in S^F$, then by Remark 2.3, $\mu * \nu \in S^F$. Let $\{s_\alpha\}$ be a net in S such that $\lim_\alpha \epsilon(s_\alpha) = \mu$ and $\lim_\alpha \pi_{s_\alpha} = \zeta$, for some $\zeta \in \Sigma(S, S^F)$. Note that the limits are taken in w^*-topology of S^G and pointwise topology of $\Sigma(S, S^F)$, respectively. Then we have

$$\mu * \nu(f) = \mu(T_\nu f) = \lim_\alpha \epsilon(s_\alpha)(T_\nu f) = \lim_\alpha \nu(L_{s_\alpha} f)$$
$$= \lim_\alpha \pi_{s_\alpha}(\nu)(f) = \zeta(\nu)(f) \quad (f \in F, \nu \in S^F).$$

Thus the mapping $\zeta_\mu: S^F \to S^F$, defined by $\zeta_\mu(\nu) = \mu * \nu$, belongs to $\Sigma(S, S^F)$ for each $\mu \in S^G$. Now if $\mu_1, \mu_2 \in S^G$, by Remark 2.3,

$$\zeta_{\mu_1\mu_2}(\nu)(f) = (\mu_1\mu_2) * \nu(f) = \mu_1\mu_2(T_\nu f) = \mu_1(T_{\mu_2 * \nu} f)$$
$$= \zeta_{\mu_1}(\mu_2 * \nu)(f) = \zeta_{\mu_1}(\zeta_{\mu_2}(\nu))(f) \quad (f \in F, \mu_1, \mu_2 \in S^G, \nu \in S^F).$$

Thus the mapping $\theta: S_G \to \Sigma(S, S^F)$, defined by $\theta(\mu) = \zeta_\mu$, is a homomorphism.

If $\zeta \in \Sigma(S, S^F)$, then there is a net $\{s_\alpha\}$ in S such that

$\lim_\alpha \pi_{s_\alpha} = \zeta$ and $\lim_\alpha \epsilon(s_\alpha) = \mu$ for some $\mu \in S^G$. Thus we have

$$\zeta(v)(f) = \lim_\alpha \pi_{s_\alpha}(f) = \lim_\alpha \mu(L_{s_\alpha} f) = \lim_\alpha \epsilon(s_\alpha)(T_v f) = \mu(T_v f) = \mu * v(f) \quad (f \in F,\ v \in S^F)$$

Therefore the mapping θ is onto. Continuity of θ is obvious.□

Proposition 2.5. Let S be a semitopological semigroup and let F be any one of D(S), SD(S) or RZ(S), then $\sigma^*(C(\Sigma(S,S^F))$ is MD(S), GP(S), or the set of all constant functions, respectively.

Proof. By Remarks 2.1 (b) and (c), (S, S^F, π) is a separately continuous flow and $(\sigma, \Sigma(S,S^F))$ is a compactification of S, where $\sigma: S \to \Sigma(S, S^F)$ is defined by $\sigma(s) = \pi_s$. If F is $RZ(S)$, then by Lemma 2.2 (c), $\sigma^*(C(\Sigma(S,S^F)))$ is the set of all constant functions.

Now let the pair (F, G) be $(D(S), MD(S))$ or $(SD(S), GP(S))$. By Lemmas 2.2 and 2.4, $\theta: S^G \to \Sigma(S, S^F)$, defined by $\theta(\mu) = \zeta\mu$, is a continuous homomorphism of S^G onto $\Sigma(S, S^F)$. Since S^G is left simple or a group, so is $\Sigma(S, S^F)$. Thus $\sigma^*(C(\Sigma(S, S^F)))$ is a subalgebra of $MD(S)$ or $GP(S)$, respectively.

Now it is enough to show that θ is one-to-one. We consider two cases.

Case 1. $F = D(S)$. Let $\mu_1, \mu_2 \in S^D$ with $\zeta_{\mu_1} = \zeta_{\mu_2}$. For $f \in MD(S)$ and $\eta \in S^D$ with $\eta^2 = \eta$, we have $T_\eta f = f$. Thus

$$\mu_1(f) = \mu_1(T_\eta f) = \mu_1 * \eta(f) = \zeta_{\mu_1}(\eta)(f) = \zeta_{\mu_2}(\eta)(f) = \mu_2 * \eta(f) = \mu_2(T_\eta f) = \mu_2(f) \quad (f \in MD(S)),$$

thus $\mu_1 = \mu_2$.

Case 2. $F = SD(S)$. Let $\mu_1, \mu_2 \in S^{GP}$ with $\zeta_{\mu_1} = \zeta_{\mu_2}$. Suppose that η is the identity of S^{GP}, then there exists $\tilde{\eta} \in S^{LMC}$ such that $T_{\tilde{\eta}} f = T_\eta$ for each $f \in GP(S)$. Thus we have

$$\mu_1(f) = \mu_1\eta(f) = \mu_1 * \tilde{\eta}(f) = \mu_1(T_{\tilde{\eta}} f) = \zeta_{\mu_1}(\tilde{\eta})(f) = \zeta_{\mu_2}(\tilde{\eta})(f) = \mu_2 * \tilde{\eta}(f) = \mu_2\eta(f) = \mu_2(f) \quad (f \in GP(S)),$$

thus $\mu_1 = \mu_2$. So, in any case, θ is one-to-one. □

Now consider the following properties of compactifications (ψ, X) of a semitopological semigroup S:

(P_1) $\Sigma(S, X)$ is a left simple semigroup.
(P_2) $\Sigma(S, X)$ is a group (or (S, X) is distal),
(P_3) $\Sigma(S, X)$ is the trivial singleton semigroup.

By proposition 2.5, (ε, S^D), (ε, S^{SD}) and (ε, S^{RZ}) have the properties P_1, P_2 and P_3, respectively.

Theorem 2.6. Let S be a semitopological semigroup, then
(a) (ε, S^D) is the universal P_1- compactification of S.
(b) (ε, S^{SD}) is the universal P_2-compactification of S.
(c) (ε, S^{RZ}) is the universal P_3-compactification of S.

Proof. Let (ψ, X) be a P_i-compactification of S, $i = 1,2,3$. By Remark 2.1 (b) and Lemma 2.3, $\Sigma(S, S^F)$, where $F := \psi^*(C(S))$, is a left simple semigroup, a group or the trivial singleton semigroup for $i = 1,2,3$, respectively.

Let $\bar{v}$ be the restriction of v to F for $v \in S^{LMC}$, $i = 1,2,3$.
(a) Suppose that (ψ, X) is a P_1-compactification of S. For $\mu, v, \eta \in S^{LMC}$ with $\eta^2 = \eta$, we have

$$\mu\eta v(f) = (\mu\eta) * \bar{v}(f) = \mu\eta(T_{\bar{v}} f) = \zeta_{\mu\eta}(\bar{v})(f)$$

$$= \zeta_\mu(\zeta_\eta(\bar{v})(f) = \zeta_\mu(\bar{v})(f) = \mu v(f) \quad (f \in F).$$

Thus $f \in D(S)$ for each $f \in F$.
(b) Suppose that (ψ, X) is a P_2-compactification of S. Since $\Sigma(S, S^F)$ is a group, ζ_η is the identity function on S^F for each $\eta \in S^{LMC}$ with $\eta^2 = \eta$. Suppose that $v \in S^{LMC}$, then we have

$$\eta v(f) = \eta\,\bar{v}(f) = \zeta_\eta(\bar{v})(f) = \bar{v}(f) = v(f) \quad (f \in F),$$

thus $F \subseteq SD(S)$.
(c) Suppose that (ψ, X) is a P_3-compactification of S. Since $\Sigma(S, S^F)$ is the trivial singleton semigroup, we have

$$\zeta_{\varepsilon(s)}(v) = v \quad (s \in S,\ v \in S^{RZ})$$

and

$f(st) = \varepsilon(s)\varepsilon(t)(f) = \zeta_{\varepsilon(s)}(\varepsilon(t))(f) = \varepsilon(t)(f) = f(t)$ $(s, t \in s, f \in F)$, thus $F \subseteq RZ(S)$. □

Let C_1 and C_2 be the classes of all compactifications (ψ, X) of S such that X has a right identity and an identity, respectively. It is clear that (ε, S^{MD}) and (ε, S^{LZ}) are in C_1, and (ε, S^{GP}) and (ε, S^{SAP}) are in C_2.
Suppose that (ψ, X) is in C_i, $i = 1,2$. By Remark 2.3 (c), $(\sigma, \Sigma(S, X)) \cong (\psi, X)$; thus by Proposition 1.1, we have the following theorem.

Theorem 2.7. Let S be a semitopological semigroup. Then
(a) (ε, S^{MD}) is universal among P_1-compactifications in C_1.

(b) (ε, S^{GP}) is universal among P_2-compactifications in C_2.
(c) (ε, S^{LZ}) is universal among compactifications in C_1 for which $\Sigma(S, X)$ is a left zero semigroup.

It is an immediate consequence of Theorem 2.7(b) that (ε, S^{SAP}) is universal among P_2-compactifications (ψ, X) in C_2 for which $\Sigma(S, X)$ is a topological semigroup.

We would like to mention that Lawson [4] contains interesting and detailed results concerning the interrelation between flows and compactifications. Our results however, have a different orientation and are of independent interest.

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