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ASSOCIATED PRIME IDEALS IN *C(X)*

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Abstract

The minimal prime decomposition for semiprime ideals is defined and studied on z-ideals of $C(X)$. The necessary and sufficient condition for existence of the minimal prime decomposition of a z-ideal I is given, when I satisfies one of the following conditions: (i) I is an intersection of maximal ideals. (ii) I is an intersection of O^x, s, when X is basically disconnected. (iii) $I=O_x$, when $x \in X$ has a countable base of neighborhoods. (iv) I is finitely generated. (v) I is countably generated, when X is compact and countable of first kind.

1. Preliminaries

The minimal primary decomposition is defined for an ideal in commutative ring R with unit and it is proved that whenever R is Noetherian, then every ideal is decomposable [see 8]. Let I be a z-ideal in $C(X)$. For every $F \in C$, $Ann(f+I)$ is a z-ideal. We also know that every primary z-ideal is prime. Hence for a z-ideal I, we study the prime decomposition instead of the primary decomposition. We know that every z-ideal is an intersection of prime ideals, hence it has a prime decomposition which may not be minimal. The Theorem (2.2) shows that if S is a prime decomposition of I, this decomposition is minimal if and only if $Ass(R/I)=S$. Of course, if $C(X)$ is Noetherian, then every z-ideal is decomposable. But $C(X)$ is Noetherian if and only if X is finite and this is very special. We generalize this concept for general spaces. In this paper, R is assumed to be commutative with an identity. $C(X)=C$ denotes the ring of continuous functions from the completely regular space X into R, the reals. For $f \in C$, $Z(f)$ denotes the zeros of f.

Let I be a (proper) ideal in C. The family $Z[I]=\{Z(f): f \in I\}$ is a z-filter: all finite inter-sections of members and all zero-sets c
ontaining members, are members, and $\emptyset$ is not a member. Every ideal is contained in at least one maximal ideal [5].

If $\cap Z[I]$ is nonempty, I and $Z(I)$ are fixed; else, free.

The fixed maximal ideals are the sets $M_p=\{f: p \in Z(f)\}$ for $p \in X$. More generally, maximal ideals, free or fixed, are the sets

$$M_p = \{f: p \in cl_{\beta X} Z(f)\} \ (p \in \beta X)$$

where βX is the stone- Čech compactification of X, and "cl" denotes colsure in βX. Related to these are the ideals

$$O^p = \{f: cl\ Z(f) \text{ is neighborhood (in } \beta X) \text{ of } p\}.$$

If $p \in X$, $M^p=M_p$; if $p \in \beta X-X$, then M^p and O^p are free. M^p is the unique maximal ideal containing O^p. Every prime ideal $\subset M^p$ contains O^p [5].

The ideal $(f_\alpha, \ldots)$ generated by the functions $f_\alpha, \ldots$ is the smallest ideal containing every f_α; it consists of all finite sums $\sum_k s_k f_{\alpha k}$, where $s_k \in C$. Also for any ideal I, we define

$$\theta(I) = \{p \in \beta X : M^p \supset I\}$$

Let I be an ideal of R. A prime ideal P in R is called an associated prime ideal of R/I, if P is annihilator of some $a + I \in R/I$. The set of associated primes R/I is

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written $Ass(R/I)$ and the set of all minimal prime ideals containing I is written by $Min(R/I)$. In fact

$$Min(R/I) = \{P \in Spec(R): P/I \text{ is minimal in } R/I\}$$
$$Ass(R/I) = \{Ann_R(a+I): a \in R\} \cap Spec(R)$$

A nonzero ideal in R is said to be essential if it intersects every nonzero ideal nontrivially and the intersecction of all essential ideals is called the socle.

2. Minimal Prime Decomposition

2.1. Definition. Suppose I is a semiprime ideal in R(i.e. $I=\sqrt{I}$) and $S \subseteq Min(R/I)$. If $I = \bigcap_{P \in S} P$, then S is called a prime decomposition of I. Also, if for every $P \in S$, $\bigcap_{P' \in S-\{P\}} P' \not\subseteq P$, we shall say that S is minimal prime decomposition of I and I is called decomposable.

Remark. We know that every semiprime ideal is an intersection of prime ideals, hence $Min(R/I)$ is a prime decomposition of a semiprime ideal I.

The following proposition shows the relation between associated prime ideals and the minimal prime decomposition.

2.2. Proposition. Let I be a semiprime ideal and S be a prime decomposition of I. Then

i) $Ass(R/I) \subseteq S$. Furthermore, equality holds if and only if the decomposition is minimal.

ii) $P \in Ass(R/I)$ if and only if $\bigcap_{P' \in S-\{P\}} P' \not\subseteq P$.

Proof. (i) Suppose $P \in Ass(R/I)$, hence there is $a \in R$ such that $P = Ann(a+I)$. Therefore, $P = Ann(a+I) = \bigcap_{P' \in S} Ann(a+P') = \bigcap_{a \notin P'} Ann(a+P') = \bigcap_{a \notin P'} P'$, so $P \subseteq \bigcap_{a \notin P'} P'$. Thus, there is $P' \in S$ such that $P'=P$, hence $Ass(R/I) \subseteq S$. Equality follows from (ii).

(ii) Assume $P \in S$ and $\bigcap_{P' \in S-\{P\}} P' \subseteq P$, so there is $a \in \bigcap_{P' \in S-\{P\}} P' - P$. Hence $P=Ann(a+I)$ implies $P \in Ass(R/I)$ and ($\Leftarrow$) holds. Now suppose $P \in Ass(R/I)$. If $\bigcap_{P' \in S-\{P\}} P' \subseteq P$, then $S' = S-\{P\}$ is prime decomposition of I. Hence by (i), $Ass(R/I) \subseteq S'$ and this is impossible.

2.3. Corollary. The minimal prime decomposition of every semiprime ideal in the case of existance is unique. In fact, if S is the minimal prime decomposition of I, then $S = Ass(R/I)$.

3. The Intersection of Maximal Ideals

In this section we obtain $Ass(C/I)$, where I is an intersection of maximal ideals.

Definition. Let A be a subspace of X and A_0 be the set of isolated points of A and $A_0 = cl_{\beta X} A_0 \cap A$. If $A = A_0$, we say A is an almost discrete space.

The following theorem states a necessary and sufficient condition for existence of minimal prime decomposition for z-ideals which are an intersection of maximal ideals.

Theorem 3.1. Suppose $A \subseteq \beta X$ and $I = \bigcap_{x \in A} M^x$ then $Ass(C/I) = \{M_x : x \in A_0\}$.
Furthermore, I is decomposable if and only if A is almost discrete.

Proof. Suppose $a \in A_0$ and $A_1 = A-\{a\}$, hence there is a closed set F in βX, such that $A_1 \subseteq F$ and $a \notin F$. So there is $f \in C(X)$, such that $f(a)=1$ and $A_1 \subseteq F \subseteq clZ(f)$. Therefore, $f \in \bigcap_{x \in A_1} M^x - M^a$, hence $\bigcap_{x \in A_1} M^x \not\subset M^a$ and (2.2) shows that $M^a \in Ass(C/I)$. So ($\supseteq$) holds. Now suppose $P \in Ass(C/I)$, by (2.2) there is $a \in A$ such that $P \subseteq M^a$. Hence $a \in A_0$ (For if $a \in A_0$, then for every $f \in \bigcap_{x \in A_1} M^x$, $a \in A_1 \subseteq clZ(f)$. So $\bigcap_{x \in A_1} M^x \subseteq M^a$ implies $I = \bigcap_{x \in A_1} M^x$. Thus again by (2.2), $P \subseteq M^x$, for some $x \in A_1$, a contradiction.) Therefore $P = M^a \in Ass(C/I)$. For the second part we suppose I is decomposable, hence by (2.2), $Ass(C/I)$ is the minimal prime decomposition of I and $I = \bigcap_{x \in A_0} M^x$.

Let $a \in A$, if $a \notin A_0$, so there is a function $f \in C(X)$ such that $f(a) = 1$ and $A_0 \subseteq clZ(f)$. Therefore, $f \in I = \bigcap_{x \in A_0} M^x$ and $f \in M^a$, hence $I \not\subseteq M^a$ and this is a contradiction. Hence, $a \in A_0$, i.e., $A = A_0$. Conversely, assume A_0 is dense in A and $f \in \bigcap_{x \in A_0} M^x$. Now $A_0 \subseteq clZ(f)$, hence $A = A_0 \subseteq clz(f)$ so $f \in I$. Therefore, $I = \bigcap_{x \in A_0} M^x$ and this means that

Ass(C/I) is the minimal prime decomposition of *I*.

Example. Let X be a discrete and infinite space and let X^* be the one-point compactification of X, then every ideal of $C(X^*)$ which is an intersection of maximal ideals is decomposable.

3.2. Corollary. Suppose X is a *P-space* and I is an ideal of $C(X)$. Then I is decomposable if and only if $\theta(I)$ is almost discrete.

Proof. The proof is immediate for $I = \bigcap_{x\in\theta(I)} M^x$ by [5].

It is easy to see that every finitely generated z-ideal I is a principal ideal generated by an idempotent. The following theorem characterizes finitely generated z-ideals which has the minimal prime decomposition.

3.3. Theorem. For every finitely generated z-ideal $I=(f)$, we have:

$$Ass(C/I) = \{M^x : x \text{ is isolated in } cl_{\beta_X} Z(f)\}.$$

Furthermore, I is decomposable if and only if $clZ(f)$ is almost discrete.

Proof. We note that $clZ(f) = \theta(I)$. We now prove that $I= \bigcap_{x\in\theta(I)} M^x$. To see this, suppose $g \in \bigcap_{x\in\theta(I)} M^x$. Hence $cl_{\beta_X} Z(f) \subseteq cl_{\beta_X} Z(g)$. So $Z(f) \subseteq Z(g)$ (If $x \in Z(f)$ and $x \notin Z(g)$, there is a neigborhood U of x in βX such that g is nonzero on U, but $x \in clZ(g)$ and this is a contradiction.) Hence $g \in I= (f)$. So $I = \bigcap_{x\in clZ(f)} M^x$ and by Theorem (3.1) the proof is complete.

The following proposition follows from (3.3) and [1]. Next, we give more proof of this fact.

3.4. Proposition. We have

$$Ass(C) = \{M_x : x \in X \text{ and } x \text{ is isolated}\}.$$

Proof. First we suppose that $x \in X$ is an isolated point, so there is a $g \in C$ such that $g(x) = 1$ and $g(\{x\}^C) = 0$, then $M_x \subseteq Ann(g)$. But $1 \notin Ann(g)$ and this means that $M_x = Ann(g)$. So $M_x \in$ Ass(C). Conversely, suppose $P \in Ass(C)$, then there is $0 \neq g \in C(X)$, such that $P = Ann(g)$. Since $X - Z(g) \subseteq Z[P]$ and P is prime, then $X - Z(g) = \{x\}$, for some isolated point $x \in X$. This implies that $P = M_x$ and the theorem is proved.

The equivalent conditions (i) - (iv) of the following proposition is proved in [2] and [7]. We add some nore equivalent conditions.

3.5. Proposition. For a topological space X, the following are equivalent:

(i) The In-topology on C is Hausdorff.

(ii) If S is the Socle of C, then $Ann(S) = 0$.

(iii) Every intersection of essential ideals of C is an essential ideal.

(iv) The set of isolated points X_0 of X is dense in X.

(v) (0) is decomposable.

(vi) E is essential ideal in C if and only if for every $P \in Ass(C)$, $E \subseteq P$.

proof. (*iv*) $\Leftrightarrow$ (*v*) Since $(0) = \bigcap_{x\in X} M_x$, hence by (3.1) the proof is trivial.

4. The intersection of O^x, s

In this section, we study associated prime ideals, decomposablity of O^x and the intersection of O^x, s. In [4] some ideals which are an intersection of O^x, s, have been identified. In particular, if I is countably generated z-ideal and $\theta(I)$ is a zero-set, then $I = \bigcap_{x\in\theta(I)} O^x$. We first give a theorem about the decomposablity of O_x when O_x is countably generated. It is well know that O_x is countably generated if and only if $x \in X$ has a countable base of neighborhoods, see [5].

4.1. Theorem. Suppose $x \in X$ has a countable base of open neighborhoods and $O_x = M_x$, then $Ass(C/O_x) = \phi$. Furthermore O_x is decomposable if and only if $O_x = M_x$.

Proof. Suppose $P \in Ass(C/O_x)$, we show that there is $f \in M_x - O_x$ such that $P = Ann(f+O_x)$. In order to see this, first we suppose O_x is not prime, then there is $f \in C$ such that $P = Ann\ (f+O_x) \not\subseteq O_x$. If $x \notin Z(f)$, there is a open neighborhood U of x such that $U \cap Z(f) = \phi$. on the other hand, there is $g \in Ann(f+O_x) - O_x$ and open neighborgood V of x such that $x \in V \subseteq U$, $V \subseteq Z(fg)$. Since $Z(f) \cap V= \phi$. hence $V \subseteq Z(g)$, therefore $g \in O_x$ which is impossible, so $f \in M_x - O_x$. Also, if O_x is prime, obviously for every $f \in M_x - O_x$, we have $P = Ann(f+O_x)$. Now by our hypothesis there is a countable base of open neighborhoods for x such as $\{U_n\}$ such that for each $n \in N$, we have $U_{n+1} \subseteq U_n$. Suppose $x_1 \in U_1 - Z(f)$ and replacing $k_1=1$, there are open neighborhoods V_1 of x_1 and $U_{k2} \in \{U_n\}$ such that $V_1 \subseteq U_{k1}$ and $V_1 \cap U_{k2} = \emptyset$ and f on V_1 is nonzero. Also there is $x_2 \in U_{k2} - Z(f)$ and open neighborhoods V_2 of x_2 and $U_{k3} \in \{U_n\}$ such that $V_2 \cap U_{k3} = \emptyset$, $V_2 \subseteq U_{k2}$ and f on V_2 is nonzero. Continuing this process, there are sequence

$\{x_n\}$ and increasing sequence $\{k_n\}$ and open neighborhoods V_n of x_n and U_{kn} such that $x_n \in U_{kn} - Z(f)$, $U_{kn+1} \cap V_n = \emptyset$, $V_n \subseteq U_{kn}$ and f on V_n is nonzero. It is evident that for each $m = n$, $V_m \cap V_n = \emptyset$. Therefore, there are functions $\varphi_n, \psi_n \in C$ such that

$$\varphi_n(X - V_{2n}) = 0,\ \varphi_n(x_{2n}) = \frac{1}{2^n},\ 0 \leq \varphi_n \leq \frac{1}{2^n}$$

$$\psi_n(X - V_{2n-1}) = 0,\ \psi_n(x_{2n-1}) = \frac{1}{2^n},\ 0 \leq \psi_n \leq \frac{1}{2^n}$$

Now letting $\varphi = \sum_{n=1}^{\infty} \varphi_n, \psi = \sum_{n=1}^{\infty} \psi_n$, it is apparent that $\varphi, \psi \in C$ and $\psi\varphi = 0$. But $\varphi \notin P$, for if $\varphi \in P$, then there is some n such that $V \subseteq V_{2n} \subseteq Z(\varphi f)$, so $x_{2n} \in Z(\varphi f)$, a contradiction. The same proof shows that $\psi \notin P$, so P is not prime and this is a contradiction. Therefore $Ass(C/O_x) = \emptyset$.

Remark. This condition that $x \in X$ has a countable base of neighborhood in Theorem (4.1) is necessary for a counter example, suppose $X = D \cup \{\infty\}$ where D is a discrete space and ∞ is the only nonisolated point of X, then X is a finite union of closed basically disconnected subspaces if and only if M_∞ contains only finitely many minimal prime ideals of $C(X)$ such as $P_1, P_2, ..., P_n$[9]. In fact, $Ass(C/O_\infty) = \{P_1, P_2, ..., P_n\}$, by Proposition (2.2).

The following result was proved in [5] and [6]. We also obtain this as a consequent of our result.

4.2. Corollary. Suppose $x \in X$ has a countable base of neighborhoods, then O_x is prime if and only if x is isolated.

Proof. (4.1) and [5].

4.3. Corollary. If $x \in X$ has a countable base then every prime ideal in C/O_x is essential.

4.4. Corollary. Suppose $x \in X$ has a countable base of neighborhoods and O_x is not prime, then

(a) O_x can not be a finite intersection of prime ideals.

(b) the number of minimal prime ideals containing O_x is infinite.

Next, we state the necessary and sufficient condition for the existence of the minimal prime decomposition for z-ideal $I = \bigcap_{x \in A} O_x$ $(A \subseteq \beta X)$ when X is basically disconnected. But first we need the following lemmas.

Lemma 4.5. Suppose I, J are z-ideal, $I \subseteq J \subseteq P$ and $P \in Ass(C/I)$, then $P \in Ass(C/J)$.

Proof. There is $f \in C$ such that $P = Ann(f+I)$. Since $f \notin P$, hence $f \notin J$, therefore $P \subseteq Ann(f+J)$. Now if $g \in Ann(f+J)$, then $gf \in J \subseteq P$ implies $g \in P$, hence $P = Ann(f+J) \in Ass(C/J)$.

4.6. Lemma. Suppose $A \subseteq \beta X$ and $I = \bigcap_{x \in A} O^x$, then $Ass(C/I) \subseteq \bigcup_{x \in A} Ass(C/O^x)$.

Proof. Let $P' \in Ass(C/I)$, since $I = \bigcap_{p \in Min(C/O^x), x \in A} P$, hence by (2.2) there is $x \in A$ such that $P' \in Min(C/O^x)$, thus by (4.5) $P' \in Ass(C/O^x)$.

4.7. Theorem. Let X be a basically disconnected space and $A \subseteq \beta X$, then $I = \bigcap_{x \in A} O^x$ is decomposable if and only if $A_0 = A$ and for every $x \in A_0$, O^x is decomposable. Furthermore, in this case

$$Ass(C/I) = \bigcup_{x \in A_0} Ass(C/O^x).$$

Proof. ($\Rightarrow$) First we define $A_1 = \{x \in A : Ass(C/I) \cap Ass(C/O^x) \neq \emptyset\}$. We prove that $A_0 = A_1$. By (4.6) and (2.2), $Ass(C/I) \subseteq \bigcup_{x \in A_1} Ass(C/O^x)$ and $I = \bigcap_{P \in Ass(C/I)} P$, hence $I = \bigcap_{x \in A} O^x$.

Also for every $a \in A_1$, $\bigcap_{x \in A_1 - \{a\}} O^x \not\subset O^a$ (otherwise, there is $P \in Ass(C/I) \cap Ass(C/O^a)$ such that $\bigcup_{x \in A_1 - \{a\}} O^x \subseteq P$, hence $Ass(C/I) - \{P\}$ is the minimal prime decomposition of I, a contradiction.) Hence there is $f \in C^*$ such that $f \in \bigcup_{x \in A_1 - \{a\}} O^x - O^a$, so $A_1 - \{a\} \subseteq Intcl\, Z(f)$ and $a \notin Intcl\, Z(f)$. But X is basically disconnected, hence by [5], $Intcl\, Z(f) = IntZ(f^\beta)$ is closed, therefore a is islated in A_1 and A_1 is discrete.

On the other hand, $A_1 = A$ whenever $A_1 = clA_1 \cap A$. Since, if there is $a \in A - A_1$, then exists $f \in C(X)$ such that $f(a) = 1$ and $A_1 \subseteq Intcl\, Z(f)$, therefore $f \in \bigcap_{x \in A_1} O^x - O^a$, hence

$I = \bigcap_{x\in A_1} O^x \not\subseteq O^a$ and this is a contradictiion. So $\bar{A_1}=A$ which implies the points of A_1 are isolated in A, i.e., $A_1=A_0$. Now for every $a \in A_0$, we show that $O^a = \bigcap_{P\in Ass(C/O^a)} P$, i.e., O^a is decomposable. Let O^a be not decomposable, then there is $P \in Min(C/O^a)\text{-}Ass(C/O^a)$ such that $\bigcap_{P'\in Ass(C/O^a)} P' \not\subseteq P$. Since $\bigcap_{x\in A_0-\{a\}} O^x \subseteq O^a \subseteq P$, then $\bigcap_{P'\in Ass(C/I)} P' \not\subseteq P$, So $P \in Ass(C/I)$, hence $P \in Ass(C/O^a)$ and this is a contradiction. Therefore O^a is decomposable.

($\Leftarrow$) Assume A_0 is dense in A. It is observed that $I = \bigcap_{x\in A_0} O^x (f \in \bigcap_{x\in A_0} O^x$ implies $A_0 \subseteq Intcl\, Z(f)$, hence $A = \bar{A_0} \subseteq Intcl\, Z(f)$, i.e. $f \in \bigcap_{x\in A} O^x)$. we show that $Ass(C/I) = \bigcup_{x\in A_0} Ass(C/O^x)$. By (4.6)($\subseteq$) holds. For ($\supseteq$), Let $S = \bigcup_{x\in A_0} Ass(C/O^x)$ and $P \in Ass(C/O^a)$, hence $\bigcap_{P'\in Ass(C/O^a)-\{P\}} P' \not\subseteq P$. Also $\bigcap_{x\in A_0-\{a\}} O^x \not\subseteq O^a$, therefore $\bigcap_{P'\in S-\{P\}} P' \not\subseteq P$, so $P \in Ass(C/I)(I = \bigcap_{P\in S} P)$. This implies that I is decomposable.

4.8. Proposition. Let X be a countable of the first kind, $A \subseteq X$ and $I = \bigcap_{x\in A} O_x$, then

$$Ass(C/I) = \{M_x : x \in A \text{ and } x \text{ is isolated}\}.$$

Furthermore, if X is basically disconnected, then I is decomposable if and only if $A_0 = A \cap X_0$, $\bar{A_0}=A$.

Proof. Assume $P \in Ass(C/I)$, hence by (4.6), there is $x \in X$ such that $P \in Ass(C/O_x)$. Since X is a countable of the first kind, x is isolated by (4.1), thus ($\subseteq$) holds. conversely, suppose $x \in A$ is isolated. Hence there is $f \in C$ such that $f(x) = 1$ and $f(\{x\}^c) = 0$. Thus, $\bigcap_{x\in A-\{a\}} O_x \not\subseteq M_a$ and (2.2) implies $M_x \in Ass(C/I)$.

4.9. Corollary. Let X be a compact and countable of the first kind space and I be a countably generated z-ideal, then

$$Ass(C/I) = \{M_x : x \in \theta(I) \text{ and } x \text{ is isolated}\}.$$

Proof. [See Ref. 4].

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