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DERIVATIONS OF TENSOR PRODUCT OF SIMPLE C^*-ALGEBRAS

M. Mirzavaziri[1] and A. Niknam[2]

[1]Department of Mathematics, Damghan University, Damghan, Islamic Republic of Iran
[2]Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, P. O. Box 1159-91775, Islamic Republic of Iran

Abstract

In this paper we study the properties of derivations of $A \otimes_\gamma B$, where A and B are simple separable C^*-algebras, and $A \otimes_\gamma B$ is the C^*-completion of $A \otimes B$ with respect to a C^*-norm γ on $A \otimes B$ and we will characterize the derivations of $A \otimes_\gamma B$ in terms of the derivations of A and B.

1. Introduction

A C^*-algebra A is said to be a uniformly hyperfinite C^*-Algebra (UHF algebra) if there is an increasing sequence $\{A_n\}$ of finite type-I subfactors (i. e. finite dimensional full matrix algebras) such that $\iota \in A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$, where ι is the identity of A and the uniform closure of $\bigcup_{n=1}^{\infty} A_n$ is A. Let A be a UHF algebra and $t \mapsto \alpha_t$ be a strongly continuous one-parameter group of *- automorphisms on A (for a full definition in detail, one can see example [4]). The system $\{A,\alpha\}$ is called a C^*-dynamics with A. A C^*-dynamics $\{A,\alpha\}$ is said to be approxiamately inner if there is a sequence $\{h_n\}$ of self-adjoint elements in A such that $(1-\delta_{ih_n})^{-1} \to (1-\delta)^{-1}$ strongly, where $\alpha_t = \exp(t\delta)$ and $\delta_{ih_n}(x) = i[h_n, x]$. Powers and Sakai have conjectured that any C^*-dynamics $\{A,\alpha\}$ with a UHF algebra A is approximately inner (see [4], 4. 5. 9).

If $d: A \otimes_\gamma B \to A \otimes_\gamma B$ is a derivation, where A and B are C^*-algebras and $A \otimes_\gamma B$ is the C^*-completion of $A \otimes B$ with respect to a C^*-norm γ on $A \otimes B$, then we can write $d(x \otimes y) = d((x \otimes \iota)(\iota \otimes y)) = d(x \otimes \iota)(\iota \otimes y) + (x \otimes \iota) d(\iota \otimes y)$, where ι is the identity of A (in $\iota \otimes y$) or the identity of B (in $x \otimes \iota$). This motivates us to consider $d(x \otimes \iota)$ and $d(\iota \otimes y)$ and study the properties of this component restrictions. We add precision to these notions in the following and find the relations between them.

In the following, A and B are simple C^*-algebras with a countable basis. We know that each simple C^*-algebra A has the property $A' = C_\iota$, where $A'$is the center of A and ι is the identity of A (see[2]).

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[1]E-mail: vaziri@science2.um.ac.ir
[2]E-mail:niknam@science2.um.ac.ir.

2. Preliminaries

Definition 2.1. A linear map $\delta: D(\delta) \to A$, where $D(\delta)$ is a dense subalgebra of A, is called a *derivation* if for each $x, y \in D(\delta)$, $\delta(xy) = \delta(x)y + x\delta(y)$. It is called a **-derivation* if it also satisfies $\delta(x^*) = \delta(x)^*$. For a fixed element $a \in A$, we can define $\delta_a: A \to A$ by $\delta_a(x) = [a,x] = ax - xa$. It can be shown that δ_a is a derivation. A derivation δ is called an *inner derivation* if $\delta = \delta_a$ for some $a \in A$. A derivation δ is called *approximately inner* if it is the limit of a sequence of inner derivations.

Definition 2.2. A linear map $\delta_1: D(\delta_1) \to A \otimes_\gamma B$, where $D(\delta_1) = A$, is called an *A-derivation with respect to* $A \otimes_\gamma B$ if it satisfies $\delta_1(xz) = \delta_1(x)(z \otimes \iota) + (x \otimes \iota)\delta_1(z)$. It is called a*-*A-derivation with respect to* $A \otimes_\gamma B$ if it also satisfies $\delta_1(\delta(x^*) = ((x))^*$. Analogously a linear map $\delta_2: D(\delta_2) \to A$

$\otimes_\gamma B$, where $D(\delta_2)=B$, is called a *B-derivation with respect to* $A\otimes_\gamma B$ if it satisfies $\delta_2(yw)=\delta_2(y)(\iota\otimes w)+(\iota\otimes y)\delta_2(w)$. It is called a*-*B-derivation with respect to* $A\otimes_\gamma B$ if it also satisfies $\delta_2(y^*)=(\delta_2(y))^*$.

Example 2.1. Let $c\in A\otimes_\gamma B$ and define $\delta_1: A\to A\otimes_\gamma B$ by $\delta_1(x)=\delta_c(x\otimes\iota)=[c,x\otimes\iota]=c(x\otimes\iota)-(x\otimes\iota)c$. Then δ_1 is an A-derivation with respect to $A\otimes_\gamma B$, which we call it an *inner A-derivation with respect to* $A\otimes_\gamma B$. Moreover, if c is self-adjoint then $i\delta_1$ is a*-A-derivation with respect to $A\otimes_\gamma B$, which we call an *inner*-A-derivation with respect* to $A\otimes_\gamma B$.

Difinition 2.3. Let δ_1 and δ_2 be A-derivation and B-derivations with respect to $A\otimes_\gamma B$, respectively, then δ_1 is said to be compatible with δ_2 if the map $d: A\otimes_\gamma B\to A\otimes_\gamma B$ defined by $d(x\otimes y)=\delta_1(x)(\iota\otimes y)+(x\otimes\iota)\delta_2(y)$ is a derivation of $A\otimes_\gamma B$. In this case we say that δ_1 and δ_2 are the first and the second component restrictions of d, respectively.

Example 2.2. Let $c\in A\otimes_\gamma B$ and $\delta_1(x)=\delta_c(x\otimes\iota)$ and $\delta_2(y)=\delta_c(\iota\otimes y)$. Then δ_1 is compatible with δ_2 and we have $\delta_1(x)(\iota\otimes y)+(x\otimes\iota)\delta_2(y)=\delta_c(x\otimes y)$.

Example 2.3. Let ζ and ξ be two derivations of A and B, respectively. Then $\delta_1(x)=\zeta(x)\otimes\iota$ is compatible with $\delta_2(y)=\iota\otimes\xi(y)$, since $d=\zeta\otimes I_B+I_A\otimes\xi$ is a derivation of $A\otimes_\gamma B$, where I_A and I_B are the identity maps of A and B, respectively (see[3]).

Lemma 2.1. *δ_1 is compatible with δ_2 if and only if $\delta_{(x\otimes\iota)}(\delta_2(\delta_2(y))=\delta_{(\iota\otimes y)}(\delta_1(x))$ for each $x\in D(\delta_1)$ and $y\in D(\delta_2)$.*

Proof. Let δ_1 be compatible with δ_2. Then $d(x\otimes y)=\delta_1(x)(\iota\otimes y)+(x\otimes\iota)\delta_2(y)$ is a derivation of $A\otimes_\gamma B$. Now we have

$$\begin{aligned}d(x\otimes y)&=d((x\otimes\iota)(\iota\otimes y))\\&=(\delta_1(x)(\iota\otimes\iota)+(x\otimes\iota)\delta_2(\iota))(\iota\otimes y)+\\&\quad(x\otimes\iota)(\delta_1(\iota)(\iota\otimes y)+(\iota\otimes\iota)\delta_2(y))\\&=\delta_1(x)(\iota\otimes y)+(x\otimes\iota)\delta_2(y)\end{aligned}$$

on the other hand

$$\begin{aligned}d(x\otimes y)&=d((\iota\otimes y)(x\otimes\iota))\\&=(\delta_1(\iota)(\iota\otimes y)+(\iota\otimes\iota)\delta_2(y))(x\otimes\iota)+\\&\quad(\iota\otimes y)(\delta_1(x)(\iota\otimes\iota)+(x\otimes\iota)\delta_2(\iota))\\&=\delta_2(y)(x\otimes\iota)+(\iota\otimes y)\delta_1(x).\end{aligned}$$

Hence

$$\delta_1(x)(\iota\otimes y)+(x\otimes\iota)\delta_2(y)=\delta_2(y)(x\otimes\iota)+(\iota\otimes y)\delta_1(x).$$

And so we have

$$\delta_{(x\otimes\iota)}(\delta_2(y))=\delta_{(\iota\otimes y)}(\delta_1(x)).$$

Conversely if $\delta_{(x\otimes\iota)}(\delta_2(y))=\delta_{(\iota\otimes y)}(\delta_1(x))$ then $d(x\otimes y)=\delta_1(x)(\iota\otimes y)+(x\otimes\iota)\delta_2(y)$ is a derivation of $A\otimes_\gamma B$, since

$$\begin{aligned}&d((x\otimes y)(z\otimes\omega))=d(xz\otimes y\omega)\\&=\delta_1(x)(z\otimes y\omega)+(x\otimes\iota)(\delta_1(z)\\&(\iota\otimes y)+(z\otimes\iota)\delta_2(y))(\iota\otimes\omega)+(xz\otimes y)\delta_2(\omega).\end{aligned}$$

But $\delta_1(z)(\iota\otimes y)+(z\otimes\iota)\delta_2(y)\ \delta_2(y)(z\otimes\iota)+(\iota\otimes y)\delta_1(z)$. So we have $d((x\otimes y)(z\otimes\omega))=(\delta_1(x)(\iota\otimes y)+(x\otimes\iota)\delta_2(y))(z\otimes\omega)+(x\otimes y)\delta_1(z)(\iota\otimes\omega)+(z\otimes\iota)\delta_2(\omega))$.

And the requirment is met. □

Difinition 2.4. If δ_2 and δ_2' are compatible with δ_1, then we say that δ_2 is δ_1-compatible to δ_2' and we write $\delta_2\equiv\delta_2'$ (mod δ_1). We denote the set of B-derivations compatible with δ_1 by $[\delta_1]_B$.

Lemma 2.2. *Let $c\in A\otimes_\gamma B$ and for each $x\in A$, $\delta_{(x\otimes\iota)}(c)=0$. Then $c=\iota\otimes b$ for some $b\in B$. Moreover $\delta_2\equiv\delta_2'$ (mod δ_1) if and only if for some derivation ξ of B, $(\delta_2-\delta_2')(y)=\iota\otimes\xi(y)$ for each $y\in B$.*

Proof. Let $c=\Sigma_{j=1}^{\infty}aj\otimes bj$, where bj's are linearly independent. Now we have

$$0=\delta_c(x\otimes\iota)=\Sigma_{j=1}^{\infty}\delta_{aj}(x)\otimes bj.$$

This implies $\delta_{aj}(x)=0$ for each $x\in A$. so aj is in the center of A which is equal to C_1. Thus $aj=\alpha j\iota$ for some complex number αj. Hence $c=\Sigma_{j=1}^{\infty}\alpha j\iota\otimes bj=\iota\otimes(\Sigma_{j=1}^{\infty}\alpha jbj)=\iota\otimes b$ where $b=\Sigma_{j=1}^{\infty}\alpha jbj$.

To prove the second assertion of the Lemma, we have $\delta_{(x\otimes\iota)}(\delta_2(y))=\delta_{(\iota\otimes y)}(\delta_1(x))=\delta_{(x\otimes\iota)}(\delta'_2(y))$.

We can therefore deduce that $\delta_{(x\otimes\iota)}(\delta_2(y)-(\delta'_2(y))=0$ for each $x\in A$. Thus $\delta_2(y)-\delta'_2(y)=\iota\otimes b_y$, for some $b_y\in B$. But we have

$$\begin{aligned}\iota\otimes b_{y\omega}&=\delta_2(y\omega)-\delta'_2(y\omega)\\&=(\delta_2(y)-\delta'_2(y))(\iota\otimes\omega)+(\iota\otimes y)(\delta_2(\omega)-\\&\quad(\delta'_2(\omega))=\iota\otimes(b_y\omega+yb\omega).\end{aligned}$$

Hence $b_{y\omega}=b_{y\omega}+yb_\omega$, and we may take $\xi(y)=b_y$. □

Lemma 2.3. *Let δ_1 be an A-derivation with respect to $A\otimes_\gamma B$ and there is an inner B-derivation δ_2 in $[\delta_1]_B$. Suppose that each derivation of B is inner. Then each B-derivation $\delta_2\in[\delta_1]_B$ is inner. Let δ_1 be an inner A-derivation with respect to $A\otimes_\gamma B$ and each derivation of B is inner. Then there is an inner B-derivation with respect to $A\otimes_\gamma B$, say δ_2, such that δ_1 is compatible with δ_2. Moreover, each*

B-derivation $\delta'_2 \in [\delta_1]_B$ is inner.

Proof. We have $(\delta_2-\delta'_2)(y)=\iota\otimes\xi(y)$ for some derivation ξ of B. But $\xi=\delta_b$ for some $b\in B$ and we have $(\delta_2-\delta_2)(y)=\iota\otimes\delta_b(y)=\delta_{(\iota\otimes b)}(\iota\otimes y)=$. Now since δ_2 is inner, there is a $c\in A\otimes_\gamma B$ such that $\delta_2(y)=\delta_c(\iota\otimes y)$. Thus $(\delta_2)(y)=\delta_2(y)-\delta_{(\iota\otimes b)}(\iota\otimes y)=\delta_{(c-(\iota\otimes b))}(\iota\otimes y)$. This proves the first part of the Lemma.

Now if $\delta_1(x)=\delta_c(x\otimes\iota)$ for some $c\in A\otimes_\gamma B$, then we may put $\delta_2(y)=\delta_c(\iota\otimes y)$. □

Remark 2.1. We can easily extend the above Lemma to the case of approximately inner derivations.

Remark 2.2 We may extend the above notion, inductively, to finite tensor product $\otimes_{j=1}^{n}A_j$ and we may define A_j-derivation with respect to $\otimes_{j=1}^{n}A_j$. Moreover we can extend this to infinite tensor product $\otimes_{j=1}^{\infty}A_j$.[1]

We finish this section with the following Theorem of Sakai (see[4]) which helps us prove our main result. Note that we say δ is a derivation on A whenever $D(\delta)=A$.

Theorem 2.1. *Every derivation on a simple C*-algebra is inner.*

3. Results

We now aim to characterize the derivations of $A\otimes_\gamma B$ in terms of the derivations of A and B, but prior to that we characterize the A-derivations with respect to $A\otimes_\gamma B$.

Theorem 3.1. *Let $\{e_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ be two bases for A and B, and δ_1,δ_2 be A-derivation and B-derivations with respect to $A\otimes_\gamma B$, respectively. Then there are sequences $\{\zeta_j\}$ and $\{\xi_j\}$ of derivations of A and B, respectively, such that* $\delta_1(x)=\Sigma_{j=1}^{\infty}\zeta_j\otimes f_j$, and $\delta_2(y)=\Sigma_{j=1}^{\infty}e_j\otimes\xi_j(y)$.

Proof. We write $\delta_1(x)=\Sigma_{j=1}^{\infty}\zeta_j\otimes f_j$, where $\zeta_j(x)\in A$. Now for each $x, z\in D(\delta_1)$ we have

$$\delta_1(xz)=\delta_1(x)(z\otimes\iota)+(x\otimes\iota)\delta_1(z).$$

So

$$\Sigma_{j=1}^{\infty}(\zeta_j(xz)-(\zeta_j(x)z+x\,\zeta_j(z)))\otimes f_j=0.$$

And since f_js are linearly independent, we have

$(\zeta_j(xz)=(\zeta_j(x)z+x\,\zeta_j(z)).$

By the same argument we can show $\delta_2(y)=\Sigma_{j=1}^{\infty}e_j\otimes\xi_j(y)$ for some sequence $\{\zeta_j\}$ of derivations of B. □

Theorem 3.2. *Suppose that $\delta_1(x)=\zeta(x)\otimes b$ is an A-derivation with respect to $A\otimes_\gamma B$, where ζ is a derivation $[\delta_1]_B$ of A and b is an element of B. Let also $[\delta_1]$ is non-void. Then ζ is approximately inner or $b=\beta\iota$ for some $\beta\in C$.*

Proof. Let $\delta_2\in[\delta_1]_B$, and $\{e_j\}$, $\{f_j\}$ be two bases for A and B, respectively. By the above Therorem we can write $\delta_2(y)=\Sigma_{j=1}^{\infty}e_j\otimes\xi_j(y)$ for some sequence $\{\zeta_j\}$ of derivations of B. Now by Lemma 2.1 we must have $\delta_{(x\otimes)}\delta_2(y))=\delta_{(\iota\otimes)}(\delta_1(x))$. Hence

$$\zeta(x)\otimes\delta_b(y)=\Sigma_{j=1}^{\infty}\delta_{e_j}(x)\otimes\xi_j(y).$$

We write $\xi_j(y)\Sigma_{k=1}^{\infty}\alpha_{jk}(y)f_k$, where $\alpha_{jk}(y)\in\mathbb{C}$. Thus

$$\zeta(x)\otimes\delta_b(y)=\Sigma_{j=1}^{\infty}\delta_{e_j}(x)\otimes\Sigma_{k=1}^{\infty}\alpha_{jk}(y)f_k)$$
$$=\Sigma_{k=1}^{\infty}(\Sigma_{j=1}^{\infty}\alpha_{jk}(y)\delta_{e_j}(x))\otimes f_k.$$

We can also write $\delta_b(y)=\Sigma_{k=1}^{\infty}\alpha_k(y)f_k$, where $\alpha_k(y)\in C$. So we must have

$$\Sigma_{k=1}^{\infty}\alpha_k(y)\zeta(x)\otimes f_k=\Sigma_{k=1}^{\infty}(\Sigma_{j=1}^{\infty}\alpha_{jk}(y)\delta_{e_j}(x))\otimes f_k.$$

And since f_k's are linearly independent, we can therefore deduce that

$$\alpha_k(y)\zeta(x)=\Sigma_{j=1}^{\infty}\alpha_{jk}(y)\delta_{e_j}(x))\quad\forall\kappa\in\mathbb{N}\quad(\dagger)$$

Now if $b\notin C_\iota$, then $b\notin B'$ and we can find a $y\in B$ such that $\delta_b(y)\neq 0$. This shows that $\alpha_k(y)\neq 0$ for some natural number k, and so by $(\dagger)$ we can deduce that $\zeta(x)=\Sigma_{j=1}^{\infty}\frac{\alpha_{jk}(y)}{\alpha_k(y)}\delta_{e_j}(x)$, which is approximately inner. □

In the following we suppose that $\{e_j\}_{j=0}^{\infty}$ and $\{f_j\}_{j=0}^{\infty}$ are Schauder bases of A and B, where e_0 and f_0 are the identities of A and B, respectively.

Theorem 3.3. *Let $\delta_1(x)=\Sigma_{j=0}^{\infty}\zeta_j(x)\otimes f_j$ be compatible with $\delta_2(y)=\Sigma_{j=0}^{\infty}e_j\otimes\zeta_j(y)$. Then ζ_j and ξ_j are inner derivations on A and B, respectively, for each $j\in\mathbb{N}$*

Proof. By Lemma 2.1 we must have:

$$\Sigma_{j=0}^{\infty}\zeta_j(x)\otimes\delta_{f_j}(y)=\Sigma_{j=0}^{\infty}\delta_{e_j}(x)\otimes\xi_j(y)\qquad(\dagger)$$

[1] The notation $\otimes$ denotes the C^*-tensor product under a C^*-norm.

for each $x \in D(\delta_1)$ and $y \in D(\delta_2)$. The left-hand side of (†) is defined for each $y \in A$, and hence so is the right-hand side. This shows that $\xi_j(y)$ is defined for each $y \in A$ provided that $\delta_{e_j}(x) \neq 0$ for some $x \in D(\delta_1)$. But for each $j \neq 0$, we can find an $x \in D(\delta_1)$ with $\delta_{e_j}(x) \neq 0$, since otherwise, if $\delta_{e_j}(x)=0$ for each $x \in D(\delta_1)$, then $\delta_{e_j}(x)=0$ for each $x \in A$, and so $e_j \in A'=C\iota$, which is impossible since e_j's are linearly independent. We can therefore deduce that ξ_js are everywhere defined for each $j \in \mathbb{N}$ and Theorem 2.1 implies that ξ_j's are inner. □

For each derivation ζ of A and ξ of B, denoting by $\zeta \otimes \xi$ the derivation mentioned in Example 2.3, i. e. $\zeta \otimes \xi = \zeta \otimes I_B + I_A \otimes \xi$, we have

Theorem 3.4. *Let $d: D(d) \subseteq A \otimes_\gamma B \mapsto A \otimes_\gamma B$ be a derivation. Then $d=\zeta \otimes \xi + \delta$, where δ is an approximately inner derivation of $A \otimes_\gamma B$.*

Proof. Let δ_1 and δ_2 be the first and the second component restrictions of d, respectively. By the above Theorem we can write

$$\delta_1(x)=\zeta(x) \otimes \iota + \Sigma_{j=1}^{\infty} \delta_{a_j}(x) \otimes f_j$$

where $a_j \in A$. Define δ_2 by $\delta_2(y)=\iota \otimes \xi'(y)+\Sigma_{j=1}^{\infty} a_j \otimes \delta_{f_j}(y)$, where ξ' is an arbitrary derivation of B. Then $\delta'_2 \in [\delta_1]_B$, because if $d'(x \otimes y)$
$=\delta_1(x)(\iota \otimes y)+(x \otimes \iota)\delta'_2(y)$, then $d'(x \otimes y)=(\zeta \otimes \xi')$

$$(x \otimes y)+ \Sigma_{j=1}^{\infty} \delta(a_j \otimes f_j)(x \otimes y).$$

Now $=\Sigma_{j=1}^{\infty} \delta(_{a_j} \otimes f_j$ is an approximately inner derivation of $A \otimes_\gamma B$ and we have

$$d'=\zeta \otimes \xi' + \delta.$$

But since $\delta_2=\delta'_2 (\text{mod } \delta_1)$, by Lemma 2.2 there is a derivation ξ'' of B such that $\delta_2=\delta'_2+\iota \otimes \xi''$. so we have

$$\begin{aligned} d(x \otimes y) &= \delta_1(x)(\iota \otimes y)+(x \otimes \iota)\delta_2(y) \\ &= d'(x \otimes y)+\otimes \xi''(y)) \\ &= ((\zeta \otimes (\zeta'+\xi''))+\delta)(x \otimes y) \end{aligned}$$

Putting $\xi=\xi'+\xi''$ completes the proof.□

Corollary 3.1. *If the derivations of A and B are approximately inner, then so are the derivations of $A \otimes_\gamma B$.*

Remark 3.1. Powers and Sakai have conjectured that each strongly continuous one-parameter group of*-automorphisms of a UHF algebra is approximately inner. In [1] it is shown that the conjecture can be reduced to a certain UHF algebra A_∞, namely the Glimm algebra of rank $\{s(n)=p_1 \dots p_n\}$, where p_j is the $_j$-th prime number. The above Corollary helps us to reduce the conjecture again to the UHF algebras A and B with $A \otimes_\gamma B = A_\infty$ But $A_\infty = \otimes_{j=1}^{\infty} M_{p_j}$, where Mp is the matrix algebra of $p \times p$ matrices over $\mathbb{C}$. This motivates us to extend our results to the case of countable tensor product of simple C^*-algebras.

References

1. Mirzavaziri, M. and Niknam, A. *Reducing the conjecture of Sakai to a certain UHF algebra*. Unpublished. Presented in 29th Annual Conference on Mathematics.
2. Murphy, G. J. *C*-algebras and operator theory*. Academic Press, London, New York, (1990).
3. Niknam, A. *Infinitesimal generators of C*-algebrus. Potential Analysis*, **6**, 1-9, (1997).
4. Sakai, S. *Operator algebras in dynamical systems*. Cambridge University Press, (1991).