J. Sci. I. R. Iran

Vol.10 No.4
Autumn 1999

ESTIMATION OF SCALE PARAMETER UNDER A REFLECTED GAMMA LOSS FUNCTION

M. Towhidi and J. Behboodian

Department of Statistics, Shiraz University, Shiraz 71454, Islamic Republic of Iran

Abstract

In this paper, the estimation of a scale parameter τ under a new and bounded loss function, based on a reflection of the gamma density function, is discussed. The best scale-invariant estimator of τ is obtained and the admissibility of all linear functions of the sufficient statistic, for estimating τ in the absence of a nuisance parameter, is investigated.

Introduction

A loss function $L(\delta,\theta)$ represents the amount by which a statistician is penalized when θ is the true state of nature and δ is the statistician's action. In the literature, $L(\delta,\theta)$ is usually taken to be convex in δ and even in $(\delta - \theta)$. For example, let $X_1, ..., X_n$ be a random sample of size n from a density $\frac{1}{\tau}f(\frac{x}{\tau})$, where f is known and τ is an unknown scale parameter. In this case, the commonly used quadratic loss is given by

$$L(\delta, \tau) = (\frac{\delta}{\tau} - 1)^2 \tag{1.1}$$

This loss function has been criticized by some researchers (e.g., Rukhin and Ananda [11], Day, Ghosh and Srinivasan [5], Akaike [1,2]). They motivated an asymmetric loss function for estimating an unknown scale parameter in the form,

$$L_1(\delta,\tau) = \left\{ \frac{\delta}{\tau} - ln\frac{\delta}{\tau} - 1 \right\}, \tag{1.2}$$

which is called the entropy loss function. It has been considered by various authors (e.g., James and Stein [9], Haff [6,7], Ighodaro, Santner and Brown [8], Yang [13]), but this loss function, with its infinite maximum value, is not appropriate in describing, for example, the loss associated with a product. In practice, the maximum loss can be a function of many things (e. g., production resources, scrap or rework) but generally it is finite.

Spiring [12] employed a bounded loss function, by the name of reflected normal loss function, for a location parameter estimation. It is constructed by a normal density and given by

$$L(\delta,\theta) = k\left\{ 1 - exp\left(- \frac{(\delta - \theta)^2}{2\gamma^2}\right)\right\}$$

where $\gamma > 0$ is a shape parameter and $k > 0$ is the maximum loss parameter. Similarly, for a scale parameter estimation, we use a simple transformation of the gamma density to have a desired loss function. The general form of this loss function, which is called *the reflected gamma loss function*, is

$$L_2(\delta,\tau) = k\left\{ 1 - (\frac{\delta}{\tau})^{\gamma^2} e^{-\gamma^2 (\frac{\delta}{\tau} - 1)}\right\} \tag{1.3}$$

Where $\gamma>0$ is a shape parameter and $k>0$ is the maximum loss parameter. The curve of this loss function is given in Figure 1. Note that we can write L_2 as a monotone function of the entropy loss function L_1 in the following way:

$$L_2(\delta,\tau) = k \left\{ 1 - e^{-\gamma^2(\frac{\delta}{\tau} - \ln\frac{\delta}{\tau} - 1)}\right\}$$
$$= k \left\{ 1 - e^{-\gamma^2 L_1(\delta,\tau)}\right\}$$

This can be approximated by $k\gamma^2 L_1(\delta,\tau)$, for small values

Keywords: Admissibility; Bayes estimation; Best scale-invariant estimator; Entropy loss; Minimax estimator; Sufficient statistic

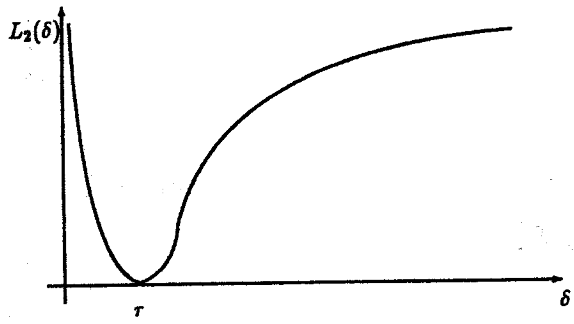

Figure 1. A reflected gamma loss function

of γ, which is a multiple of the entropy loss.

In this article, we study the problem of estimation of a scale parameter, using a relfected gamma loss. In Section 2, we introduce the best invariant estimator of the scale parameter τ under the Loss L_2. In Section 3, we consider a subclass of the exponential family and obtain the Bayes estimates of τ under the loss L_2. In Section 4, we discuss the admissibility of all linear functions of the sufficient statistic in the subclass of Section 3, for estimating the scale parameter τ, in the absence of a nuisance parameter.

Best Scale Invariant Estimator

Consider a random sample $X_1, ..., X_n$ from $\frac{1}{\tau}f(\frac{x}{\tau})$, where f is a known function and τ is an unknown scale parameter. It is desired to estimate τ under the loss function L_2. Let $\mathcal{Y}$ be a group of transformations in the form

$$\mathcal{Y} = \{g_c : g_c(x_1, ..., x_n) = (cx_1, ..., cx_n), c > 0\}$$

It is known (see Lehmann [10], pp 175-179) that the loss function L_2 and the decision problem are invariant under $\mathcal{Y}$ and the class of all scale-invariant estimators of τ is of the form,

$$\delta(\mathbf{X}) = \delta_0(\mathbf{X})/W(\mathbf{Z}),$$

where δ_0 is any scale-invariant estimator, $\mathbf{X} = (X_1, ..., X_n)$, and $\mathbf{Z} = (Z_1, ... Z_n)$ with $Z_i = \frac{X_i}{X_n}$; $i = 1, ..., n-1$, $Z_n = \frac{X_n}{|X_n|}$. Moreover the best scale-invariant (minimum risk equivariant (MRE)) estimator δ^* or τ is given by

$$\delta^*(\mathbf{X}) = \delta_0(\mathbf{X})/w^*(\mathbf{Z}),$$

where $w^*(\mathbf{Z})$ is a function of $\mathbf{Z}$ which maximizes

$$E_{\tau=1}\left[\left(\frac{\delta_0(\mathbf{X})}{w(z)}\right)^{\gamma^2} exp\ \{-\gamma^2\left(\frac{\delta_0(\mathbf{X})}{w(z)} - 1\right)\} \mid \mathbf{Z} = z\right]$$

In the presence of a location parameter as a nuisance parameter, the MRE estimator of τ is of the form

$$\delta^*(\mathbf{X}) = \delta_0(\mathbf{Y})/w^*(\mathbf{Z}),$$

where $\delta_0(\mathbf{Y})$ is any finite risk scale-invariant estimator of τ based on $\mathbf{Y} = (Y_1, ..., Y_{n-1})$, with $Y_i = X_i - X_n$, $i = 1, ..., n-1$, $\mathbf{Z} = (Z_1, ..., Z_{n-1})$, $Z_i = \frac{Y_i}{Y_{n-1}}$; $i = 1, ..., n-2$, and $Z_{n-1} = \frac{Y_{n-1}}{|Y_{n-1}|}$ and $w^*(\mathbf{Z})$ is any function of $\mathbf{Z}$ maximizing

$$E_{\tau=1}\left[\left(\frac{\delta_0(\mathbf{Y})}{w(z)}\right)^{\gamma^2} exp\ \{-\gamma^2\left(\frac{\delta_0(\mathbf{Y})}{w(z)} - 1\right)\} \mid \mathbf{Z} = z\right]$$

In many cases, when $\tau = 1$, we can find an equivariant estimator $\delta_0(\mathbf{X})$ or $\delta_0(\mathbf{Y})$ which has the gamma distribution with known parametes ν, η and is independent of $\mathbf{Z}$.

It follows that $\delta^* = \frac{\delta_0}{w^*}$ is the MRE estimator of τ where w^* is a number which maximizes

$$g(w) = \int_0^{+\infty} \left(\frac{x}{w}\right)^{\gamma^2} e^{-\gamma^2(\frac{x}{w}-1)} \left\{\frac{\eta^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\eta x}\right\} dx$$

$$= \frac{\eta^\nu e^{\gamma^2}}{\Gamma(\nu) w^{\gamma^2}} \frac{\Gamma(\nu+\gamma)^2}{(\eta + \frac{\gamma^2}{w})^{\nu+\gamma^2}}$$

$$= c(\eta, \nu, \gamma^2) \frac{w^\nu}{(\eta w + \gamma^2)^{\nu+\gamma^2}}$$

and c is a function of η, ν, γ^2. Now we can easily show that $w^* = \frac{\nu}{\eta}$ maximizes the function g. Hence, we have the following result:

Theorem 2.1. If $\delta_0(\mathbf{X})$ is a finite risk scale-invariant estimator of τ which has the gamma distribution with known parameters ν, η when τ=1. Then the MRE (minimum risk equivariant) estimator of τ under the reflected gamma loss function, is $\delta^*(\mathbf{X}) = \frac{\eta}{\nu}\delta_0(\mathbf{X})$.

Example 2.1. *(Exponential)*

Let $X_1, ..., X_n$ be a random sample from $E(0, \lambda)$ with density $\frac{1}{\lambda}e^{-\frac{x}{\lambda}}$; $x > 0$, and consider the estimation of λ under the loss (1.3). $\delta_0(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is an equivariant estimator which has Ga(n,1)-distribution when λ=1 and it follows from Basu's theorem that δ_0 is independent of $\mathbf{Z}$. Hence, the MRE estimator of λ under the loss L_2 is $\delta^*(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Example 2.1. *(continued)*

Suppose that $X_1, ..., X_n$ is a random sample of $E(\theta, \lambda)$

with density $\frac{1}{\lambda} e^{-(x-\theta)/\lambda}$, $x > \theta$, and consider the estimation of λ when θ is unknown. We know that $(X_{(1)}, \sum_{i=1}^{n}(X_i - X_{(1)})$ is a complete sufficient statistic for (θ, λ). It follows that $\delta_0(Y) = 2\sum_{i=1}^{n}(X_i - X_{(1)})$ has $Ga\,(n-1, \frac{1}{2})$-distribution, when $\lambda=1$, and from the Basu's theorem, $\delta_0(Y)$ is independent of Z and hence, $\delta^*(X) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - X_{(1)})$ is the MRE estimator of λ under the loss (1.3).

Example 2.2. (*Normal variance*)

Let $X_1, ..., X_n$ be a random sample of $N\,(0, \sigma^2)$ and consider the estimation of σ^2. $\delta_0(X) = \sum_{i=1}^{n} X_i^2$ is a finite risk scale-invariant estimator of σ^2 and is independent of Z, and when $\sigma^2 = 1$, $\delta_0(X)$ has $Ga\,(\frac{n}{2}, \frac{1}{2})$-distribution and hence, $\delta^*(X) = \frac{1}{n}\sum_{i=1}^{n}X_i^2$ is the MRE estimator of σ^2.

Example 2.2. (*Continued*)

Let $X_1, ..., X_n$ be a random sample from $N\,(\mu, \sigma^2)$, with a nuisance parameter μ. In estimating σ^2 under the loss (1.3), it follows that $\delta_0(Y) = \sum_{i=1}^{n}(X_i - \bar{X})^2$ is independent of Z and when $\sigma^2 = 1$, the distribution of $\delta_0(Y)$ is $Ga\,(\frac{n-1}{2}, \frac{1}{2})$.

Therefore, $\delta^*(X) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the MRE estimator of σ^2.

Example 2.3. (*Inverse Gaussian with zero drift*)

Let $X_1, ..., X_n$ be a random sample of $IG\,(\infty, \lambda)$ with density

$$f(x\,|\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2x}} \quad \text{if } x > 0,$$

and consider the estimation of λ. $\delta_0(X) = \sum_{i=1}^{n} X_i^{-1}$ has $Ga(\frac{n}{2}, \frac{1}{2})$-distribution and is independent of Z and hence, $\delta^*(X) = \frac{1}{n}\sum_{i=1}^{n}X_i^{-1}$ is the MRE estimator of λ.

Remark 2.1. It can be shown that the MRE estimators of the scale parameter τ in examples 2.1-2.3 dominate the MRE estimators of τ under the loss (1.1) and (1.2), relative to the reflected gamma loss function.

The Bayes Estimators

In this section, we consider the Bayesian estimation of the scale parameter τ in a subclass of one-parameter exponential families in which the complete sufficient statistic $\delta_0(X)$ has $Ga\,(\nu, \frac{\eta}{\tau})$-distribution, where $\nu > 0, \eta > 0$ are known.

Assume that the conjugate family of prior distributions for $\beta = \frac{1}{\tau}$ is the family of Gamma distribution $Ga\,(\alpha, \xi)$. Now, the posterior distribution of β is $Ga\,(\nu + \alpha, \xi + \eta\delta_0(x))$ and the Bayes estimate of τ is a function $\delta(x)$ which maximizes the function $E\,[\,(\delta\beta)^{\gamma^2} e^{-\gamma^2(\delta\beta - 1)} |\, X$

$$= \frac{(\delta e)^{\gamma^2}(\eta\delta_0(X) + \xi)^{\nu+\alpha}}{\Gamma(\nu + \alpha)} \int_0^{+\infty} \beta^{(\nu+\alpha+\gamma^2)-1} e^{-\beta(\xi + \eta\delta_0(x) + \delta\gamma^2)} d\beta$$

or the function,

$$g(\delta) = \frac{\delta^{\gamma^2}}{(\xi + \eta\delta_0(X) + \delta\gamma^2)^{\nu+\alpha+\gamma^2}}$$

δ is obtained from the relation $\frac{dg(\delta)}{d\delta} = 0$. Hence, the function g is maximized at

$$\delta_B(X) = \frac{\xi + \eta\delta_0(X)}{\nu + \alpha}$$
$$= \frac{\xi}{\nu + \alpha} + \frac{\nu}{\nu + \alpha}\delta^*(X),$$

where $\delta^*(X) = \frac{\eta}{\nu}\delta_0(X)$.

Remark 3.1. All estimators of the form $c\delta^*(X)+d$ with $0 < c < 1, d > 0$ are also B ayes estimators.

Example 3.1.

In examples 2.1, 2.2 and 2.3, all estimators of the form $c\delta^*(X)+d$, where $\delta^*(X)$ is the MRE estimator of τ, are Bayes estimators of the scale parameter τ, when $0 < c < 1, d > 0$.

Admissibility and Inadmissibility of $c\delta^*(X)+d$

In this section, we consider the question of admissibility of the linear estimators of the form $c\delta^*(X)+d$, relative to the reflected gamma loss function (1.3), with $\gamma = 1$. First, the class of inadmissible linear estimators is exhibited by Theorem 4.1 and then by Theorems 4.2 and 4.3, we obtain all linear admissible estimators.

Theorem 4.1. Under the assumptions of Section 3, the linear estimators $c\delta^*(X)+d$, under the loss function (1.3) with $\gamma = 1$, is inadmissible whenever one of the following conditions hold;

(i) $c < 0$ or $d < 0$
(ii) $c > 1, d \geq 0$
(iii) $0 \leq c\ 1, d = 0$

Proof.

First, note that the risk function of $c\delta^*(X)+d$ is

$$R(\tau, c\delta^*(X)+d) = k\left\{1 - E\left[\left(\frac{c\delta^*(X)+d}{\tau}\right) e^{-\left(\frac{c\delta^*(X)+d}{\tau}-1\right)}\Big|\tau\right]\right\}$$

$$= k\{1 - g(c,d,\beta)\}. \qquad (4.1)$$

where $\beta = \frac{1}{\tau}$ and

$$g(c,d,\beta) = e^{-d\beta+1}\left[\frac{c}{(\frac{c}{\nu}+1)^{\nu+1}} + \frac{d\beta}{(\frac{c}{\nu}+1)^{\nu}}\right]$$

$$= e^{-d\beta+1}\frac{(1+\frac{d\beta}{\nu})c + d\beta}{(\frac{c}{\nu}+1)^{\nu+1}} \qquad (4.2)$$

In case (i), $c\delta^*(X)+d$ takes on negative values with positive probability; therefore, $c\delta^*(X)+d$ is dominated by $\max(0; c\delta^*(X)+d)$.

To see (ii), note that the function g is maximized at $c=c^*$, for any given $d \geq 0$ and $\beta > 0$, where

$$c^* = \frac{1-d\beta}{1+\frac{d\beta}{\nu}} = 1 - \frac{\frac{\nu+1}{\nu}d\beta}{1+\frac{d\beta}{\nu}}$$

Since $c^* \leq 1$, for any $d \geq 0$, $\beta > 0$, we conclude that the estimator $c\delta^*(X)+d$ is dominated by $\delta^*(X)+d$.

Similarly, with condition (iii) we can show that the estimator $\delta^*(X)$ dominates $c\,\delta^*(X)$.

Before we present the admissibility theorems, we need the following results.

Lemma 4.1. *(Berger [3] P 545)* If $\Theta \subseteq R^m$ and $L(\theta, a)$ is a bounded function, which is continuous in θ for each $a \in A$. Suppose also that X has a density function which is continuous in θ for each $x \in X$. Then all decision rules have continuous risk functions.

Lemma 4.2. (*Berger [3] P 254*) Assume that $R(\theta, \delta)$ is continuous in θ for all decision rules δ and the prior Π gives positive probability to any open subset of Θ. Then a Bayes rule w.r.t. Π is admissible

Theorem 4.2. Under the assumptions of Theorem 4.1, $c\delta^*(X)+d$ is admissible for any choice of $0 \leq c < 1$, $d > 0$.

Proof.

By using Remark 3.1, any linear estimator $c\delta^*(X)+d$ with $0 < c < 1, d > 0$ is a Bayes e stimator and by Lemmas 4.1 and 4.2, we conclude that $c\delta^*(X)+d$ is admissible for any $0 < c < 1, d > 0$. We can also easily show that any estimator $\delta(X) = d$ is admissible, where $d > 0$.

The only case not covered yet is when $c=1$ and $d \geq 0$, which we consider in the following theorem.

Theorem 4.3. Under the assumptions of Theorem 4.1, for any $d \geq 0$, the estimator $\delta^*(X)+d$ is admissible with the loss function (1.3).

Proof.

We use the Blyth's technique [4] to prove this theorem. Suppose that $\delta^*(X)+d$ is inadmissible, then there exists an estimator δ_1 such that

$$R(\tau, \delta_1) \leq R(\tau, \delta^*(X)+d) \qquad \forall\tau > 0,$$

with strict inequality for some τ. Since $R(\tau, \delta_1)$ is a continuous function of τ (See Lemma 4.1), there exists $\varepsilon > 0$, $0 < \tau_0 < \tau_1$ such that

$$R(\tau, \delta_1) < R(\tau, \delta^*(X)+d) - \varepsilon \qquad \forall\tau \in (\tau_0, \tau_1)$$

Let Λ_α be an Inverse Gamma prior with parameters $\alpha > 0$, $\xi = 1$ for τ. Consider the following risks

r^{**}_α = Bayes risk of δ_1 w.r.t. Λ_α,
r^*_α = Bayes risk of $\delta^*(X)+d$ w.r.t. Λ_α,
r_α = Bayes risk of the Bayes solution w.r.t. Λ_α.

Then, with $\beta = \frac{1}{\tau}$, $\beta_0 = \frac{1}{\tau_1}$, $\beta_1 = \frac{1}{\tau_0}$, we can show that there exists an $\alpha_0 > 0$ and an $\varepsilon_0 > 0$ such that

$$\int_{\beta_0}^{\beta_1} \Lambda_\alpha(\beta)\, d\beta > \varepsilon_0$$

for all $\alpha < \alpha_0$ and therefore,

$$r^*_\alpha - r^{**}_\alpha \geq \varepsilon\int_{\beta_0}^{\beta_1} \Lambda_\alpha(\beta)\, d\beta$$

$$\geq \varepsilon\varepsilon_0 \qquad \forall\alpha < \alpha_0$$

Now, it can be easily shown that (by the relations (4.1) and (4.2))

$$r^*_\alpha = k\left\{1 - \frac{e}{(\frac{1}{\nu}+1)^{\nu+1}(d+1)^{\alpha+1}}\left[(d+1) + (\frac{1}{\nu}+1)d\alpha\right]\right\},$$

$$r_\alpha = k\left\{1 - \frac{e}{(\frac{1}{\nu+\alpha}+1)^{\nu+\alpha+1}}\right\},$$

and hence,

$r^*_\alpha - r_\alpha \to 0$ as $\alpha \to 0$

or

$$\frac{r^*_\alpha - r^{**}_\alpha}{r^*_\alpha - r_\alpha} \to +\infty \text{ as } \alpha \to 0$$

Thus, there exists a number $\alpha_0 > 0$ such that $r^{**}_{\alpha_0} < r_{\alpha_0}$ which contradicts the fact that r_{α_0} is the Bayes risk of the Bayes solution w.r.t. Λ_{α_0}.

Remark 4.1. By using the relations (4.1) and (4.2), we can find the risk function of $\delta^*(\mathbf{X})$ as

$$R(\tau, \delta^*(\mathbf{X})) = k\left\{1 - \frac{e}{(\frac{1}{v}+1)^{v+1}}\right\}$$

which is a constant number. Since $\delta^*(\mathbf{X})$ is admissible with a constant risk function, hence, $\delta^*(\mathbf{X})$ is minimax.

Acknowledgement

The authors would like to thank the referees for careful reading of the article and giving valuable comments. They are also grateful to Shiraz University Research Council for the support of this work.

References

1. Akaike, H. One entropy maximization principle, *In Application of Statistics (P. R. Krishnaiah, ed.)*, 27-41, North-Holand, Amesterdam, (1977).
2. Akaike, H. A new look at the Bayes procedure, *Biometrika*, **65**, 53-59, (1978).
3. Berger, J. O. *Statistical decision theory and bayesian analysis*, (2nd ed.), Springer-Verlag, New York, (1985).
4. Blyth, C. R. On minimax statistical decision procedures- and their admissibility., *Ann. Math. Statist.*, **22**, 22-42, (1951).
5. Dey, D. K., Ghosh, M. and Srinivasan, C. Simultaneous estimation of parameters under entropy loss, J. *Statist. Plann, Inference*, **15**, 347-363, (1987).
6. Haff, L. R. Empirical Bayes estimation of the multivariate normal covariance matrix, *Ann. Statist.*, **8**, 586-597, (1980).
7. Haff, L. R. Identities for the Wishart distribution with applications to regression., *Sankhya Ser.*, **B 44**, 245-258, (1982).
8. Ighodaro, A. O., Santner, T. and Brown, L. D. Admissibility and complete class results for the multinomial estimation problem with entropy and squared error loss., *N. Multivariate Anal.*, **12**, 469-479, (1982).
9. James, W. and Stein, C. Estimation with quadratic loss, *Proc. Fourth Berkele Symp. Math. Statist. Probab.*, **1**, 361-379, Univ. Californai Prss. (1961).
10. Lehmann, E. L. *Theory of point estimation*, John Wiley, New York, (1983).
11. Rukhin, A. L. and Anada, M. M. A. Risk behavior of variance estimators in mulktivariate normal distribution. *Statist. & Prob. Letters*, **13**, 159-166, (1992).
12. Spiring, F. A. The refected normal loss function, *The Canadian J. Statist.* **21**, 3, 321-330, (1993).
13. Yang, M. C. Ridge estimation of independent Poisson means under entropy loss, *Statistics & Decisions*, **10**, 1-23, (1992).