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AN INTRODUCTION TO THE THEORY OF DIFFERENTIABLE STRUCTURES ON INFINITE INTEGRAL DOMAINS

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Abstract

A special class of differentiable functions on an infinite integral domain which is not a field is introduced. Some facts about these functions are established and the special case of $\mathbb{Z}$ is studied in more detail.

Introduction

In this paper we introduce classes of functions over infinite integral domains which are not fields, called "differentiable" or "smooth" functions, which include polynomial functions. These concepts are based purely on the algebraic structure of the integral domain and do not involve the concept of "limit". This work was inspired by a reformulation of the definition of the derivative of a real-valued function of a real variable at a point, in the following manner:

Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at $t_\circ\in\mathbb{R}$. Consider the function

$$g(t)=\begin{cases} f'(t_\circ) & t=t_\circ \\ \dfrac{f(t)-f(t_\circ)}{t-t_\circ} & t\neq t_\circ \end{cases}$$

Observe that $g:\mathbb{R}\to\mathbb{R}$ is continuous at $t_\circ$, and we can state the following well-known result.

Theorem. A necessary and sufficient condition for $f:\mathbb{R}\to\mathbb{R}$ to be differentiable at $t_\circ\in\mathbb{R}$ is that there exists a function $g:\mathbb{R}\to\mathbb{R}$ continuous at $t_\circ$ such that

$$f(t)=f(t_\circ)+(t-t_\circ)g(t) \qquad t\in\mathbb{R}$$

The only property of continuous functions which enables us to give a definition of the derivative of a real function at a point, in a unique way, is the following:

For no value of $\alpha\in\mathbb{R}$, different from zero, the function

$\varphi_\alpha^{t_\circ}:\mathbb{R}\to\mathbb{R}$ given by

$$\varphi_\alpha^{t_\circ}(t)=\begin{cases} \alpha\neq\circ & t=t_\circ \\ \circ & t\neq t_\circ \end{cases}$$

is continuous at $t_\circ$. In other words the $\mathbb{R}$ −algebra of functions continuous at $t_\circ$ does not contain any $\varphi_\alpha^{t_\circ}$. If we substitute this algebra by $\mathscr{A}_{t_\circ}$, an algebra of functions from $\mathbb{R}$ into $\mathbb{R}$ subject to the following restiction:

$$\forall\alpha\in\mathbb{R}\ \ \alpha\neq\circ \implies \varphi_\alpha^{t_\circ}\notin\mathscr{A}_{t_\circ}$$

then the algebra

$$\underline{\mathscr{A}}_{t_\circ}=\{f:\mathbb{R}\to\mathbb{R}\,|\,\exists g\in\mathscr{A}_{t_\circ}\ni\forall t\in\mathbb{R}, f(t)=f(t_\circ)+(t-t_\circ)g(t)\}$$

is called ***the algebra of differentiable functions at $t_\circ$ with respect to $\mathscr{A}_{t_\circ}$***. If $f\in\underline{\mathscr{A}}_{t_\circ}$ can be written as

$$f(t)=f(t_\circ)+(t-t_\circ)g(t),$$

for some $g\in\mathscr{A}_{t_\circ}$, then $g(t_\circ)$, which is uniquely determined by f, is called ***the derivative of f at $t_\circ$ with respect to $\mathscr{A}_{t_\circ}$***. Using only algebraic operations of IR, a suitable $\mathscr{A}_{t_\circ}$ will not be uniquely determined. But if we substitute IR by an infinite integral domain R which is not a field, we can determine $\mathscr{A}_{t_\circ}$ uniquely by using only algebraic operations of R.

The definition of differentiability has been extended to R −modules (here R is an infinite integral domain). This extention provides us with a large number of interesting problems which are non-trivial even when $R=\mathbb{Z}$.

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Our aim, here, is only to introduce a simple type of functions on an infinite integral domain which is not a field, called "differentiable" and to prove some very simple basic theorems. Other matters are treated in [1] and [2].

§ 2. Differentiable Functions

Throughout the rest of this paper R is an infinite integral domain which is not a field. The set of all functions from cofinite subsets of R into R will be denoted by $F_\circ$, and the domain of $y \in F_\circ$ will be denoted by $\mathcal{D}(y)$. For $\lambda \in R$ and y, z in $F_\circ$, yz and $y+\lambda z$ are two elements of $F_\circ$ given by

$$\mathcal{D}(yz) = \mathcal{D}(y+\lambda z) = \mathcal{D}(y) \cap \mathcal{D}(z)$$

$$(yz)(\mu) = y(\mu)\, z(\mu)$$

$$(y+\lambda z)(\mu) = y(\mu) + \lambda z(\mu)$$

It is clear that with the above definitions of addition and multiplication $F_\circ$ is a commutative algebra over R with the constant function $e: R \to R$, where $e(\lambda)=1$ for all $\lambda \in R$, as the unit element. Let x be the identity function on R, and let $F, F_{(\lambda)}$ and Σ_λ be the following subalgebras of $F_\circ$:

$$F = \{ y \in F_\circ \mid \mathcal{D}(y) = R \},\ F_{(\lambda)} = \{ y \in F_\circ \mid \lambda \in \mathcal{D}(y) \},$$

$$(*)\ \Sigma_\lambda = \{ y \in F_{(\lambda)} \mid \exists y_\lambda \in F_{(\lambda)}, \exists \mu \in R - \{\circ\} \ni \mu\, [y - y(\lambda) e] = (x - \lambda e)\, y_\lambda \}$$

Fundamental Lemma 2.1. The function $y \in F_\circ$ defined by

$$y(\mu) \begin{cases} \eta \neq \circ & \mu = \lambda \\ \circ & \mu \neq \lambda \end{cases}$$

is not an element of $\Sigma_\lambda^\bullet$.

Proof. Let $y \in \Sigma_\lambda$. By $(*)$ there exists $y_\lambda \in F_{(\lambda)}$, $\theta \in R$ such that

$(**)$ $\quad \theta \neq \cdot$ and $\theta(y - \eta e) = (x - \lambda e) y_\lambda$.

Since for each $\mu \neq \lambda$, $(\mu - \lambda) y(\mu) = -\theta \eta$ and $\mathcal{D}(y)$ is cofinite, there exists $\delta \neq$ in R such that $\delta \theta^2 \eta^2 + \lambda \in \mathcal{D}(y_\lambda)$ and $(\delta \theta^2 \eta^2)\, y_\lambda(\delta \theta^2 \eta^2 + \lambda) = -\theta \eta$.

The last equality implies that

$$\theta\eta\, [\theta\eta\delta\, y_\lambda(\delta \theta^2 \eta^2 + \lambda)] + 1 = \circ$$

From this relation and the fact that R is an integral domain we have

$$\theta\eta\, [-\delta\, y_\lambda(\delta \theta^2 \eta^2 + \lambda)] = 1 .$$

Thus $\theta\eta$ is invertible in R. Now the relation $(**)$ and the definition of y yields

$$(\mu - \lambda)[-y_\lambda(\mu).(\theta\eta)^{-1}] = \begin{cases} \circ & \mu = \lambda \\ 1 & \mu \neq \lambda . \end{cases}$$

This relation implies that R is a field, which is a contradiction.

Definition 2.2 For $\lambda \in R$, $y \in F_\circ$ is called differentiable at λ if there exists $y_1 \in \Sigma_\lambda$ such that

$$y = y(\lambda) e + (x - \lambda e) y_1 .$$

The set of all elements of $F_\circ$, differentiable at λ will be denoted by $\mathcal{A}_\lambda$.

For $H \subset R$, $\bigcap_{\lambda \in H} \mathcal{A}_\lambda$ is the algebra of functions from cofinite subsets of R into R, differentiable at all points of H.

Let S , $T \subset R$, $\lambda \in R$. We say that ***S absorbs T*** with respect to λ if for each $\mu \in T$, there exists μ' in R such that $\mu\mu' + \lambda \in S$.

Lemma 2.3. Assume that S absorbs T with respect to λ and T absorbs R with respect to $\circ \in R$ and let $S = A \cup B$. Then A or B absorbs T with respect to λ .

Proof. Suppose that neither A nor B absorbs T with respect to λ Then

$$(\dagger)\ \mid \exists \mu \in T \mid \forall \mu' \in R,\ \mu\mu' + \lambda \notin A) \wedge (\exists \varepsilon \in T \mid \forall \varepsilon' \in R,\ \varepsilon\varepsilon' + \lambda \notin B) .$$

Since T absorbs R, there exists $\alpha \in R$ such that $\alpha(\mu\varepsilon) \in T$ and since S absorbs T with respect to λ , there exists $\beta \in R$ such that $\beta\,\alpha(\mu\varepsilon) + \lambda \in S$. But in this case $\beta\,\alpha\mu(\varepsilon) + \lambda \in B$ or $\beta\,\alpha\varepsilon(\mu) + \lambda \in A_1$ which is in contradiction with the relation $(\dagger)$.

Lemma 2.4. Let A absorbs B with respect to λ and B absorbs C with respect to $\circ \in R$ Then A absorbs C with respect to λ .

The proof of the lemma is clear.

Lemma 2.5. Let $y \in \bigcap_{\alpha \in \mathcal{D}(y)} \mathcal{A}_\alpha$. Assume that $S \subset \mathcal{D}(y)$ absorbs $\mathcal{D}(y) - \{\circ\}$ with respect to all $\lambda \in \mathcal{D}(y)$. If $y|_S$ is zero then y is zero.

Proof. If $y \neq \circ$, there exists $\lambda \in \mathcal{D}(y)$, such that $y(\lambda) = \eta \neq \circ$. Then by definition there exists $y_1 \in \Sigma_\lambda$ such that

$$y = \eta e + (x - \lambda e) y_1$$

Since $\mathcal{D}(y)$ is cofinite it absorbs $R - \{\circ\}$ with respect to $\circ \in R$. Thus by lemma (2.4) there exists an $\alpha \in R$ such that $\alpha\eta^2 + \lambda \in S$. So

$$\eta + \alpha\eta^2 y_1(\alpha\eta^2 + \lambda) = \circ .$$

This means that η is invertible. Let $z = y/\eta$. Then $z|s$ is zero, and

$$z = e + (x - \lambda e) y_1/\eta$$

Let ε be a non-zero element of R. By lemma (2.4) there exists $\beta \in R$ such that $\beta\varepsilon + \lambda \in S$. Whence

$$e + \varepsilon(\eta^{-1}.\beta\, y_1(\beta\varepsilon + \lambda)) = \circ$$

This means that R is a field, which is a contradiction.

Definition 2.6. With the above notations ***a derivation of*** $\mathcal{R}_\lambda$ ***based at*** λ is a linear functional α of $\mathcal{R}_\lambda$ satisfying the following condition

$$\alpha(yz) = \alpha(y)z(\lambda) + y(\lambda)\alpha(z).$$

Lemma 2.7. The mapping

$$\eta_\lambda : \mathcal{R}_\lambda \longrightarrow R$$

given by $\eta_\lambda(y) = y_1(\lambda)$ where $y = y(\lambda)e + (x - \lambda e)y_1$ is a derivation of $\mathcal{R}_\lambda$ based at λ.

The proof of the above lemma is clear.

Assume that $\mathcal{R}_\lambda^{\#}$ denotes the set of all derivations of $\mathcal{R}_\lambda$ based at λ. Clearly $\mathcal{R}_\lambda^{\#}$ is an R−module.

Lemma 2.8. The mapping $\mu \rightarrow \mu.\eta_\lambda$ from R into $\mathcal{R}_\lambda^{\#}$ is an isomorphism of R−modules.

Proof. Since $\eta_\lambda(x) = 1$, the mapping is clearly an injective homomorphism of R−modules. On the other hand, let $D \in \mathcal{R}_\lambda^{\#}$ and $D(x) = \mu$. Then for $y = y(\lambda)e + (x - \lambda e)y_1$ we have $D(y) = \mu y_1(\lambda) = \mu\eta_\lambda(y)$. Therefore $D = D(x)\eta_\lambda$, and the mapping is surjective.

Corollary 2.9. Let D_1, D_2 be two derivations of $\mathcal{R}_\lambda$ based at λ.
Then

$$D_1 = D_2 \Leftrightarrow D_1(x) = D_2(x)$$

For $y \in \mathcal{R}_\lambda$, $\frac{dy}{dx}(\lambda)$ is defined by

$$\frac{dy}{dx}(\lambda) = y^{(1)}(\lambda) = \eta_\lambda(y).$$

This is called the ***first order derivative of*** y ***at*** λ. (Note that $\frac{dy}{dx}(\lambda)$ is uniquely determined by y and λ.)

Theorem 2.10. Let $y \in \mathcal{R}_\lambda$ and $z \in \mathcal{R}_{y(\lambda)}$ be such that $z \circ y \in F_\circ$. Then $z \circ y \in \mathcal{R}_\lambda$ and $\frac{d}{dx}(z \circ y)(\lambda) = z^{(1)}(y(\lambda)) \cdot y^{(1)}(\lambda)$.

Proof. Let $z = z(y(\lambda))e + (x - \lambda e)z_1$. Then

$$z(y(\mu)) = z(y(\lambda)) + (y(\mu) - y(\lambda))\, z_1(y(\mu))$$

$$= z_\circ y(\lambda) + (\mu - \lambda)(y_1(\mu).\, z_1(y(\mu))).$$

i.e. $z_\circ y = z_\circ y(\lambda)e + (X - \lambda e)(y_1 . z_1^\circ y)$. It is clear that $y_1 . z_1^\circ y \in \Sigma_\lambda$ and $(y_1 z_1^\circ y)(\lambda) = y_1(\lambda) \cdot z_1(y(\lambda)) = z^{(1)}(y(\lambda)) . y^{(1)}(\lambda)$

Theorem 2.11. Let $y \in \mathcal{R}_\lambda$ be invertible in $F_\circ$. Then $z = 1/y \in \mathcal{R}_\lambda$ and $\frac{dz}{dx}(\lambda) = -y^{(1)}(\lambda)[y(\lambda)]^{-2}$

Proof. Direct computation shows that

$$1/y - (1/y(\lambda))e = -(x - \lambda e)y_1[y(\lambda)y]^{-1},$$

where $y = y(\lambda)e + (x - \lambda e)y_1$.

Remark 2.12. Let $y \in F_\circ$ be differentiable on its domain of definition. Define $y^{(1)} \in F_\circ$ as follows:

$$\mathcal{D}(y)^{(1)} = \mathcal{D}(y),\ y^{(1)} : \lambda \longmapsto \frac{dy}{dx}(\lambda)$$

Let $y^{(1)}$ be differentiable at $\lambda \in R$. Then we say that ***y is two times differentiable at*** λ and $\frac{dy^{(1)}}{dx}\lambda = y^{(2)}(\lambda)$ is called the ***second derivative of y at*** λ. By recurrence we can define, for all $n \in \mathbb{N}$ the n-th order derivative of y, if it exists. The O-th order derivative of y is y itself. y is ***infinitely differentiable*** if for all $n \in \mathbb{N}$, y has n-th order derivative.

Definition 2.13. Let $\mathcal{R}_\wedge$ be the maximal subalgebra of the R−algebra $F_\circ$ satisfying the following condition

$$\forall y \in \mathcal{R}_\wedge, \forall \lambda \in \mathcal{D}(y), \exists z \in \mathcal{R}_\wedge \text{ such that } y = y(\lambda)e + (x - \lambda e)z.$$

Clearly $\mathcal{R}_\wedge$ contains the R−algebra of polynomial functions and by (2.1) it is uniquely determined. Each $y \in \mathcal{R}_\wedge$ is called a ***Smooth function from*** $\mathcal{D}$ ***(y) into R,*** or briefly an ***R−smooth function.*** From this definition it is clear that every R−smooth function is infinitely differentiable in its domain of definition.

Example 2.14. Let R be the subring of Q defined by

$$R = \left\{\frac{a}{1+2b} \,\middle|\, a, b \in \mathbb{Z}\right\}.$$

It is clear that R is an integral domain and it is not a field. Let $\varphi : R \rightarrow R$ be given by

$$\varphi(\lambda) = (1 + 2\lambda)^{-1}$$

Assume that $\mathcal{R}$ is the subalgebra of $F_\circ$ generated by e, x, φ. Direct computation shows that $\mathcal{R}$ is a subalgebra of $\mathcal{R}_\wedge$. Therefore φ is a smooth function.

Remark 2.15. We have proved that there exist uncountable integral domains which admit non-polynomial smooth functions. Recently, N. Borujerdian has also proved that the algebra of smooth functions on each countable integral domain which is not a field is uncountable. But the existence of non-polynomial smooth functions on general infinite integral domains has not yet been proven.

§ 3. The Special Case $\mathbb{Z}$

Proposition 3.1. Let y be a non-zero globally differentiable (differentiable at each point of its domain) function from $\mathfrak{D}(y) \subset \mathbb{Z}$ into $\mathbb{Z}$. Then

$$\#\{\mu \in \mathfrak{D}(y) \mid y(\mu) = 0\} < \infty$$

Proof. Let $y(\lambda) > 0$, and $y = y(\lambda)e + (x - \lambda e)y_1$. Then for $\eta \in \mathbb{Z}$ we have $y(\eta) = y(\lambda) + (\eta - \lambda)y_1(\eta)$. From this equality it is clear that if $\#\{\mu \in \mathfrak{D}(y) \mid y(\mu) = 0\} = \infty$, then $y(\lambda)$ is divisible by infinitely many distict integers $(\eta - \lambda)$, which is absurd.

Corollary 3.2. Every globally differentiable function y from $\mathfrak{D}(y)$ into $\mathbb{Z}$ that is bounded, is constant.

Corollary 3.3. The constant functions $\varphi\pm : \mathbb{Z} \to \pm 1$, are the only globally differentiable functions on $\mathbb{Z}$ which have globally differentiable inverses.

Corollary 3.4. Let y be globally differentiable. Then $\mathbb{Z} - y(\mathbb{Z})$ is an infinite set or is empty.

Proposition 3.5. Let y be a $\mathbb{Z}$-smooth function. If there exists $\lambda \in \mathfrak{D}(y)$ such that for all $n \in \mathbb{N}$, $y^{(n)}(\lambda) = 0$, then $y = 0$

Proof. Let $(x - \lambda e)^n v_n$ be the $(n-1)$-th order expansion of y around $\lambda \in \mathbb{Z}$. i.e. $y = (x - \lambda e)^n y_n$. Let $\lambda \neq \mu \in [\mathfrak{D}(y) - \{\lambda \pm 1\}] - \lambda$. Then $y(\mu + \lambda) = \mu^k . \eta$, where $k \in \mathbb{N}$, and if $y(\mu + \lambda) \neq 0$, η is not divisible by μ. Since for $n > k$ we also have $y(\mu + \lambda) = \mu^n y_n(\mu + \lambda)$, we have $y(\mu + \lambda) = 0$. On the other hand $\mathfrak{D}(y) - \{\lambda \pm 1\}$ is an infinite set and $y : \mathfrak{D}(y) \to \mathbb{Z}$ is zero on $\mathfrak{D}(y) - \{\lambda \pm 1\}$. Therefore by (3.1) y is zero.

Corollary 3.6. Let y be a $\mathbb{Z}$-smooth function. Then

$$y = \text{Constant} \Leftrightarrow y^{(1)} = 0$$

As mentioned earlier N. Borujerdian has proved that there exists an uncountable set of smooth functions on $\mathbb{Z}$. Among these is the following:

$$\varphi : \mathbb{Z} \to \mathbb{Z}$$

$$\varphi(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n (\lambda^2 - 1)^n (\lambda^2 - 4)^n \times \cdots \times (\lambda^2 - n^2)^n \qquad \alpha_i \in \mathbb{Z}.$$

References

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