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FREE SEMIGROUPS AND IDEMPOTENTS IN T^∞

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Abstract

The known theory for an oid T shows how to find a subset T^∞ of βT, which is a compact right topological semigroup [1]. The success of the methods in [2] for obtaining properties of T^∞ has prompted us to see how successful they would be in another context. Thus we find (Theorem 4.8) that T^∞ contains copies of free semigroups on 2^c generators, is an immediate consequence of the stronger result and that it contains a cancellative subsemigroup (Theorem 4.7). Also obtained is a new proof of the known result in [6] that T^∞ contains 2^c idempotents.

1. Introduction

The most striking conclusions of this paper are easy to state. Throughout this paper, we will let T be a standard oid (see Section 2 for precise definition). Moreover, T is countable [2]. We will always assume that T is discrete, so that the Stone-Cech compactification βT of T is the set of ultrafilters on T, the points of T being identified with the principal ultrafilters.

It is well known that βT has a compact right topological semigroup "at infinity" T^∞, i.e. for fixed $v\in T^\infty$, the map $\mu\rightarrow\mu v$ is continuous. (See [1] for details). It is also known that T^∞ is non-commutative. Indeed its centre is empty. Further, T^∞ contains 2^c disjoint left ideals of the form $T^\infty v$, where $v\in H_k$, $k\in N$ (see Section 2 for definition), and the minimal right ideals are not closed. (We refer the reader to [2] for these facts). We shall prove that it contains copies of free semigroups on 2^c generators and we obtain a simpler proof than has so far been given that the number of different idempotents in T^∞ is 2^c. Finally, H_∞ is nowhere dense in T^∞ (Proposition 4.10).

Keywords: Cardinal function ($c(x)$ = card (supp$x$)); Compact right topological semigroup "at infinity" T^∞; Oids; Special suboids; supp$x$ (support of the element x)

2. Definitions and Preliminaries

Let $x=(x/n)_{n\in N}$ be any sequence consisting of 1, s and ∞, s. Write $1.1=1, 1.\infty=\infty.1=\infty$. We define supp $(x(n))_{n\in N}=\{n\in N: x(n)=\infty\}$. Write $T=\{(x(n))_{n\in N}$: supp $(x(n))_{n\in N}$ is finite and non-empty$\}$. A commutative standard oid is the set T together with the product xy defined in T, if and only if, (supp$x$) $\cap$ (supp$y$) $=\phi$ to be $(x(n)\ y(n))$. Thus the product $x(n)\ y(n)$ is required to be defined only if either $x(n)$ or $y(n)$ is 1. Of course, the product in T is associative where defined, and supp (xy) = (supp$x$) $\cup$ (supp$y$) whenever xy is defined in T. (Oids are discussed in [6]).

Any commutative standard oid T can be considered as $\bigoplus_{n=1} \{1,\infty\}\backslash\{(1,1,\ldots,1,\ldots)\}$. (In [6] oids could have any index set). Write $u_n=(1,1,\ldots,\infty,1,\ldots)$ (with ∞ in the nth place).

Put $U=\{u_n: n\in N\}$. Thus U is a countable subset of T. Moreover, every element $x\in T$ can be written as the finite product $x=u_{i_1}u_{i_2}\ldots u_{i_k}$ where $i_1<i_2<\ldots<i_k$, supp$x$ $=\{i_1,i_2,\ldots,i_k\}$. For $x,y,\in T$, supp$x$ < supp$y$ means that $r<s$ if $r\in$ supp x, $s\in$ supp$y$, and supp $x_\alpha\rightarrow\infty$ for some net (x_α)

in T will mean that for arbitrary $k \in N$, eventually $\min(\mathrm{supp}\, x_\alpha) > k$. The compact space βT is the Stone-Cech compactification of the discrete space T. Then βT produces a compact right topological semigroup "at infinity" T^∞ defined by

$$T^\infty = \{\mu \in \beta T : \mu = \lim_\alpha x_\alpha,\ \mathrm{supp}\, x_\alpha \to \infty\},$$

with the multiplication $\mu\nu = \lim_\alpha \lim_\beta x_\alpha y_\beta$ if $\mu = \lim_\alpha x_\alpha$, $\nu = \lim_\beta y_\beta$. (See [1] for details). For a function f from a discrete space T into a compact space X, the unique continuous extension of f to βT is denoted by f^β. The cardinal function is the map $c: T \to N$ given by $c(x) =$ card (supp$x$) (that is, the number of elements for the support of x). Then if (supp$x$) $\cap$ (supp$y$) $= \phi$, so that xy is defined, $c(xy) = c(x) + c(y)$. It follows easily that c extends to a homomorphism c^β from T^∞ into the one-point compactification $N \cup \{\infty\}$. Now write $H_k = T^\infty \cap (c^\beta)^{-1}(k)$, $k \in N$, and $H_\infty = T^\infty \cap (c^\beta)^{-1}(\infty)$. Then $T^\infty = H_1 \cup H_2 \cup \ldots \cup H_\infty$. So each $\mu \in H_k$ is the limit of a net (x_α) with $c(x_\alpha) = k$ for each α and card $(H_k) = 2^c$, $k \in N$ (see [2], Remark 5.8). Further, H_k is open, $H_k H_m \subseteq H_{k+m}$ for all $k, m \in N$, so that $H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots$ is a subsemigroup of T^∞.

3. Idempotents

The present section is devoted to the existence of idempotents in the compact right topological semigroup T^∞. Of course T^∞ contains a minimal idempotent ([3], Theorem 3.11). In the search for idempotents of T^∞ a major role is played by substructures called special suboids. Let us first give the following definition of special suboids.

Definition 3.1. Let T be a commutative standard oid and let $(k_n)_{n=1}^{\infty}$ be a strictly increasing subsequence of N. The special suboid of an oid T corresponding to $(k_n)_{n=1}^{\infty}$ is denoted by $S(k_n)$ and defined by

$$S(k_n) = \{(x(k))_{k \in N} \in T : x(k) = 1 \text{ for } k \neq k_n \text{ for all } n \in N\}.$$

Note that for an infinite subset $A \subseteq N$, the special suboid of the standard oid T corresponding to the strictly increasing sequence of A is denoted by $S(A)$.

The following is the main result of this section. For this result, we use non-principal ultrafilters on N (see [4] for details). If ν is a non-principal ultrafilter on N and $A \in \nu$, then A is an infinite subset of N, and so generates a special suboid $S(A)$ of an oid T. Moreover, the number of non-principal ultrafilters on N is 2^c [8].

Theorem 3.2. T^∞ contains at least 2^c idempotents.

Proof. Let ν be a non-principal ultrafilter on N, and let $A \in \nu$. Let $S(A)$ be the special suboid of T corresponding to A. Then $S^\infty(A)$ is a compact right continuous subsemigroup of T^∞. Take $B \in \nu$, then $A \cap B \in \nu$, so that $S^\infty(A \cap B)) \subseteq S^\infty(A) \cap S^\infty(B)$. By the finite intersection property $\bigcap_{A \in \nu} S^\infty(A)$ is non-empty and hence is a compact right continuous subsemigroup of T^∞. So it contains a minimal idempotent, e_ν say. Now suppose that ν_1 and ν_2 are two different non-principal ultrafilters on N, and let $A \in \nu_1$ with $N \backslash A \in \nu_2$. Then $S^\infty(A) \cap S^\infty(N \backslash A) = \phi$ ([1], Proposition 7.1), and hence $[\bigcap_{A \in \nu_1} S^\infty(A)] \cap [\bigcap_{A \in \nu_2} S^\infty(A)] = \phi$. Thus $e_{\nu_1} \neq e_{\nu_2}$. This proves our assertion.

Remark 3.3. It should be noted that for an idempotent e in T^∞, then $c^\beta(e) = \infty$. We denote the set of all idempotents in T^∞ by $E(T^\infty)$. So we obtain that $E(T^\infty) \subseteq \{\nu \in T^\infty : c^\beta(\nu) = \infty\}$.

4. Free Semigroups

Our aim in this section is concerned with the free semigroups on 2^c generators in the compact right topological semigroup "at infinity" T^∞. Let us first establish some definitions and the results which will be required in this section.

Definition 4.1. Let $x \in T$, $x = u_{i_1} u_{i_2} \ldots u_{i_r}$ where $i_1 < i_2 < \ldots < i_r$, and let $k \in N$. We define $\sigma_k : T \to T$ by

$$\sigma_k(u_{i_1} u_{i_2} \ldots u_{i_r}) = \begin{cases} u_{i_1} u_{i_2} \ldots u_{i_r} & \text{if } k \geq r \\ u_{i_1} u_{i_2} \ldots u_{i_{r-k}} & \text{if } k < r. \end{cases}$$

Then σ_k extends to a unique continuous function σ_k^β from βT into itself.

Theorem 4.2. Let $\mu \in \beta T$, $\nu \in T^\infty$ with $c^\beta(\nu) = k$, $k \in N$. Then $\sigma_k^\beta(\mu\nu) = \mu$.

Proof. Let $x_\alpha \to \mu$, $y_\beta \to \nu$, for some nets (x_α), (y_β) in T with supp $y_\beta \to \infty$. Then eventually $c(y_\beta) = k$, since $\{k\}$ is open in $N \cup \{\infty\}$. In view of the definition of σ_k, for large β, $\sigma_k(x_\alpha y_\beta) = x_\alpha$. Since $\sigma_k^\beta(\mu\nu) = \lim_\alpha \lim_\beta \sigma_k(x_\alpha y_\beta)$, it follows that $\sigma_k^\beta(\mu\nu) = \mu$, as claimed.

Proposition 4.3. Let $\mu_1, \mu_2 \in T^\infty$, $\nu \in H_k$, $k \in N$. Then $\mu_1 \nu = \mu_2 \nu$ implies that $\mu_1 = \mu_2$.

Proof. By Theorem 4.2, $\mu_1 = \sigma_k^\beta(\mu_1 \nu) = \sigma_k^\beta(\mu_2 \nu) = \mu_2$ and the result follows.

Definition 4.4. Let $x \in T$, $x = u_{i_1} u_{i_2} \ldots u_{i_r}$ where $i_1 < i_2 < \ldots < i_r$, and let $k \in N$. We define $\varphi_k : T \to T$ by

$$\varphi_k(u_{i_1} u_{i_2} \ldots u_{i_r}) = \begin{cases} u_{i_1} u_{i_2} \ldots u_{i_r} & \text{if } k \geq r \\ u_{i_{k+1}} u_{i_{k+2}} \ldots u_{i_r} & \text{if } k < r. \end{cases}$$

Then φ_k extends to a unique continuous mapping φ_k^β from βT into itself.

Theorem 4.5. Let $\mu \in \beta T$, with $c^\beta(\mu) = k$, $k \in N$, and let $\nu \in T^\infty$. Then $\varphi_k^\beta(\mu\nu) = \nu$.

Proof. Analogous to that of Theorem 4.2.

Proposition 4.6. Let $\eta_1, \eta_2 \in T^\infty$, $\nu \in H_k$, $k \in N$. Then $\nu\eta_1 = \nu\eta_2$ implies that $\eta_1 = \eta_2$.

Proof. Analogous to that of Proposition 4.3.
The next result is an immediate consequence of Propositions 4.3 and 4.6.

Theorem 4.7. The semigroup $H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots$ is cancellative.

We now come to the principal result of this section.

Theorem 4.8. For each $k \in N$, H_k generates a free semigroup in T^∞ on 2^c generators.

Proof. Let $\mu_1 \mu_2 \ldots \mu_p = \nu_1 \nu_2 \ldots \nu_q$ where $\mu_1, \mu_2, \ldots, \mu_p, \nu_1, \nu_2, \ldots, \nu_q \in H_k$, $k \in N$. Since $c^\beta(\mu_1 \mu_2 \ldots \mu_p) = kp$, $c^\beta(\nu_1 \nu_2 \ldots \nu_q) = kq$, it follows that $p = q$. By applying φ_k^β (similarly for σ_k^β) to the both sides, we obtain that $\mu_2 \ldots \mu_p = \varphi_k^\beta(\mu_1 \mu_2 \ldots \mu_p) = \varphi_k^\beta(\nu_1 \nu_2 \ldots \nu_q) = \nu_2 \ldots \nu_q$. An application of Theorem 4.7 completes the proof.

Remark 4.9. Let $x \in T$ with $x = u_{i_1} u_{i_2} \ldots u_{i_r}$, $i_1 < i_2 < \ldots < i_r$, and let $k \in N$. Define $\lambda_k : T \to T$ by

$$\lambda_k(u_{i_1} u_{i_2} \ldots u_{i_r}) = \begin{cases} u_{i_1} u_{i_2} \ldots u_{i_r} & \text{if } r \leq k \\ u_{i_1} u_{i_2} \ldots u_{i_k} & \text{if } k < r. \end{cases}$$

Then λ_k extends to a unique continuous function λ_k^β: $\beta T \to \beta T$, so that $\lambda_k^\beta(\mu\nu) = \mu$, whenever $\mu \in \beta T$ with $c^\beta(\mu) = k$, $\nu \in T^\infty$. Thus we obtain an alternative proof of 4.8 by using λ_k, $k \in N$.

Proposition 4.10. H_∞ is nowhere dense in T^∞.

Proof. For each n, write $x_n = u_n u_{n+1} \ldots u_n^2$. Put $X = \{x_n : n \in N\}$. Let 1_X be the indicator function of X (that is, the function whose value is 1 on X and O on $T \backslash X$). Then $(1_X^\beta)^{-1}(1) \cap T^\infty$ is a non-empty open set in T^∞. Now X is countable and discrete, so that $(\mathrm{cl}_{\beta T} X)$ is homeomorphic to βN and $(\mathrm{cl}_{\beta T} X) \backslash X$ is homeomorphic to N^* ($= \beta N \backslash N$). Thus $\mu \in (\mathrm{cl}_{\beta T} X) \backslash X$, if and only if, $\mu = \lim_j x_{n_j}$ for some subnet (x_{n_j}) of (x_n) with $n_j \to \infty$. Further, $(\mathrm{cl}_{\beta T} X) \backslash X = (1_X^\beta)^{-1}(1) \cap T^\infty$. So if $\mu \in (\mathrm{cl}_{\beta T} X) \backslash X$, then $c^\beta(\mu) = \infty$, $\mu \in T^\infty$. Hence $(1_X^\beta)^{-1}(1) \cap T^\infty \subseteq H_\infty$, and the result now follows.

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