ON THE PERIODIC SOLUTIONS OF A CLASS OF nTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS*

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Abstract

The nth order differential equation $x^{(n)} + \sum_{i=1}^{n} c_{i-1}(t)x^{(i-1)} + f(t,x) = e(t)$, $n > 3$ is considered. Using the Leray-Schauder principle, it is shown that under certain conditions on the functions involved, this equation possesses a periodic solution.

We consider the nth order differential equation

$$x^{(n)} + \sum_{i=2}^{n} c_{i-1}(t)x^{(i-1)} + f(t,x) = e(t),\ n > 3 \tag{1}$$

where $c_i(t)$, $i = 1, 2, \ldots, n-1$, $e(t)$ are continuous for $t \in [0, w]$ and $f(t,x)$ is continuous on $[0,w] \times IR$. Furthermore we assume all solutions of initial value problem for (1) can be extended to $[0,w]$.

Theorem 1

In addition to the above hypotheses assume

i) $|f(t,x)| \leq \gamma|x| + \beta$, $t \in [0, w]$, $|x| < \infty$

ii) $\sum_{i=2}^{n} \gamma_{i-1} (\frac{w}{\pi})^{n-i+1} + \gamma(\frac{w}{\pi})^n < 1$,

where γ and β are positive constants and $\gamma_j = \max |c_j(t)|$, $j = 1, 2, \ldots, n$, $t \in [0, w]$.

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Then Equation (1) has a solution satisfying

$$x^{(i)}(0) + x^{(i)}(w) = 0,\quad i = 1, 2, \ldots, n-1 \tag{2}$$

Proof

First we look at the following differential equation

$$x^{(n)} + \sum_{i=2}^{n} c_{i-1}(t)x^{(i-1)} = \mu[e(t) - f(t,x)] \tag{3}$$

where $\mu \in [0, 1]$ and find an estimate for the magnitude of its solutions satisfying boundary conditions (2).

We shall make use of Wirtinger's inequality written in the following form. Assume $x(t) \in c^{n-1}[0, w]$ and $x(t+w) + x(t) = 0$, then

$$\|x^{(i-1)}\|_{1/2} \leq (\frac{w}{\pi})^{n-i+1} \|x^{(n)}\|_{1/2} \tag{4}$$

where $\|x\|_{1/2} = [\int_0^w |x(t)|^2\, dt]^{1/2}$.

Let $x(t)$ be any solution of (3), then

$$|x^{(n)}(t)| \leq \sum_{i=2}^{n} \gamma_{i-1} |x^{i-1}(t)| + \mu\, [|e(t)| + \gamma|x(t)| + \beta]$$

Applying Minkowski's inequality, we obtain

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$$\|x^{(n)}\|_{1/2} \leq \sum_{i=2} \gamma_{i-1} \|x^{(i-1)}\|_{1/2} + \mu\, [\|e\|_{1/2} + \gamma \|x\|_{1/2} + \beta\sqrt{w}\,]$$

By Wirtinger's inequality

$$\|x^{(n)}\|_{1/2} \leq \sum_{i=2}^{n} \gamma_{i-1} \left(\frac{w}{\pi}\right)^{n-i+1} \|x^{(n)}\|_{1/2}\; \mu[\|e\|_{1/2} + \gamma \left(\frac{w}{\pi}\right)^{n} \|x^{(n)}\|_{1/2} + \beta\sqrt{w}\,]$$

Hence,

$$\|x^{(n)}\|_{1/2}\, \{1 - \sum_{i=2}^{n} \gamma_{i-1} \left(\frac{w}{\pi}\right)^{n-i+1} - \mu\gamma \left(\frac{w}{\pi}\right)^{n}\} \leq \mu \{\|e\|_{1/2} + \beta\sqrt{w}\}$$

Considering assumption (ii) we can write

$$\|x^{(n)}\|_{1/2} \leq \mu \frac{\|e\|_{1/2} + \beta\sqrt{w}}{1 - \sum_{i=2}^{n} \gamma_{i-1} \left(\frac{w}{\pi}\right)^{n-i+1} - \mu\gamma \left(\frac{w}{\pi}\right)^{n}}$$

Since $0 \leq \mu \leq 1$, we get

$$\|x^{(n)}\|_{1/2} \leq \mu\Delta_0$$

$$\Delta_0 = \frac{\|e\|_{1/2} + \beta\sqrt{w}}{1 - \sum_{i=2}^{n} \gamma_{i-1} \left(\frac{w}{\pi}\right)^{n-i+1} - \gamma \left(\frac{w}{\pi}\right)^{n}} \qquad (5)$$

Next, write

$$x^{(i-1)}(t) = x^{(i-1)}(0) + \int_0^t x(\tau)d\tau\,;\; i = 1, 2, \ldots, n$$

Hence,

$$x^{(i-1)}(w) = x^{(i-1)}(0) + \int_0^w x(\tau)d\tau$$

Using boundary conditions (2) we get

$$x^{(i-1)}(w) = -x^{(i-1)}(0) = \frac{1}{2}\int_0^w x(\tau)d\tau;\; i = 1, 2, \ldots, n$$

As a result

$$|x^{(i-1)}(t)| \leq \frac{1}{2}\int_0^w |x^{(i)}(\tau)|d\tau,$$

And by Holder's inequality

$$|x^{(i-1)}(t)| \leq \frac{1}{2}\sqrt{w}\, \|x^{(i)}\|_{1/2}$$

Using Wirtinger's inequality

$$|x^{(i-1)}(t)| \leq \frac{1}{2}\sqrt{w} \left(\frac{w}{\pi}\right)^{n-i} \|x^{n}\|_{1/2}, \quad i = 1, 2, \ldots, n$$

Finally, using inequality (5) we obtain the following estimates on the magnitudes of the solutions of (3)

$$|x^{(i-1)}(t)| \leq \frac{1}{2}\sqrt{w} \left(\frac{w}{\pi}\right)^{n-i+1} \mu\, \Delta_0 \qquad (6)$$

For $\mu = 0$ we obtain

$$x^{(i-1)}(t) = 0,\; t \in [0, w],\; i = 1, 2, \ldots, n$$

Hence, we have shown that homogeneous equation

$$x^{(n)} + \sum_{i=2}^{n} c_{i-1}(t) x^{(i-1)} = 0$$

has no nontrivial solution satisfying boundary conditions (2). This implies the existence of Green's function $G(t,s)$. Using Green's function we can write the solution of Equation (3) with boundary condition (2) in the form

$$x(t) = \mu \int_0^w G(t,s)\, [e(s) - f(s, x(s))]\, ds \qquad (7)$$

Next we consider the space $c^n[0,w]$ and define the norm

$$\|x\|_{c^n} = \max |x^{(i-1)}(t)|;\; i = 1, 2, \ldots, n,\; t \in [0, w]$$

Now let

$$B_\rho = \{x(t) \in c^n[0, w]: \|x\|_{c^n} \leq \rho\}$$

where

$$\rho = \max_i \{\frac{1}{2}\sqrt{w} \left(\frac{w}{\pi}\right)^{n-i+1}\},\; i = 1, 2, \ldots, n$$

and define the operator L on B_ρ by

$$(Lx)(t) = \mu \int_0^w G(t,s)\, [e(s) - f(s, x(s))]ds \qquad (8)$$

It follows from (6) that Equation (1) has no solution on the sphere $S_R = \{x : \|x\|_{c^n} = R\}$, $R > \rho$. Hence, by the Leray-Schauder principle and complete continuity of the operator, we conclude that (7) has at least one solution in the open ball $\{x: \|x\|_{c^n} < R\}$ and therefore it has a solution in B_ρ.

Thus, we have shown that Equation (3) has a solution for $\mu=1$ satisfying boundary conditions (2).

Corollary 1

If in addition to the hypotheses of Theorem 1, we assume (iii) $c_i(t)$, $i=1,2,\ldots,n$ are periodic with period w.
(iv) $e(t)$ is $2w$ periodic, that is, $e(t,2w)=e(t)$ and in addition $e(t+w)+e(t)=0$.
(v) $f(t,x)$ is w-periodic in t and in addition $f(t,-x)=-f(t,x)$.
Then Equation (1) has a $2w$-periodic solution with zero mean value.

Proof

A $2w$-periodic extension of solution of Equation (1) can be defined as

$$z(t)=\begin{cases} x(t) & 0\leq t\leq w \\ -x(t+w) & -w\leq t\leq 0 \end{cases}$$

First we note that boundary conditions (2) imply the continuity of $z(t)$ and its derivatives up to and including $(n-1)st$ derivative. From assumptions (iii), (iv) and (v), one can easily conclude that $z(t)$ is a solution of (1) satisfying periodic boundary conditions

$$z^{(i)}(-w)=z^{(i)}(w) \quad i=0,1,2,\ldots,n-1 \tag{9}$$

To show $z(t)$ has zero mean value we look at

$$\int_0^{2w} z(t)\,dt=\int_0^{w} z(t)\,dt+\int_w^{2w} z(t)\,dt$$
$$=\int_0^{w} z(t)\,dt+\int_0^{w} z(t+w)\,dt=0$$

Similar results can be obtained for the case of differential equation

$$x^{(n)}+\sum_{i=2}^{n}\phi_{i-1}(x^{(i-2)})x^{(i-1)}+f(t,x)=e(t) \tag{10}$$

where ϕ_i, $i=1,2,\ldots,n$ are continuous and they are even with respect to their arguments.

Theorem 2

In addition to the above hypotheses and the assumptions of Theorem 1 and Corollary 1, regarding the functions f and e, we further assume

$$\sum_{i=2}^{n}M_{i-1}\left(\frac{w}{\pi}\right)^{n-i+1}+\gamma\left(\frac{w}{\pi}\right)^{n}<1,$$

where $m_i=\max|\phi_i(x)|$, $i=1,2,\ldots,n-1$. Then Equation (10) admits a $2w$-periodic solution with zero mean value.

Proof

The proof follows along the same lines as in the proof of Theorem 1. Here, again, we can show differential equation (9) has a solution satisfying boundary conditions (2). Now we construct a $2w$-periodic extension of solution $x(t)$ of Equation (10) satisfying periodic boundary conditions (9). The rest of the proof follows as in Corollary 1.

Next we consider differential equation

$$x^{(n)}+\sum_{i=2}^{n}c_{i-1}(t)x^{(i-1)}+f(t,x,x',\ldots,x^{(n-1)})=e(t) \tag{11}$$

where $c_i(t)$ and $e(t)$ satisfy the hypotheses (iii) and (iv) of Corollary 1 and f is w-periodic in t. Furthermore, we assume

$$\sum_{i=2}^{n}\gamma_{i-1}\left(\frac{w}{\pi}\right)^{n-i+1}<1 \tag{12}$$

where $\gamma_i=\max|c_i(t)|$, $i=1,2,\ldots,n-1$, $t\in[0,w]$. Then the following theorem can easily be shown.

Theorem 3

In addition to the above hypotheses assume

v) $|f(t,x_i)|\leq F$ for all x_i, $i=1,2,\ldots,n-1$
vi) $f(t,x_i)=-f(t,-x_i)$, $i=1,2,\ldots,n-1$

where $f(t,x_i)$ stands for $f(t,x_1,x_2,\ldots,x_{n-1})$, then Equation (11) has a $2w$-periodic solution with zero mean value.

Corollary 2

The results of Theorem 3 remain valid if instead of assumption v) we assume v') $|f(t,x_i)|\leq\delta_0+\delta\sum_{i=1}^{n}|x_i|$ provided

$$\sum_{i=2}^{n}(\delta+\gamma_{i-1})\left(\frac{w}{\pi}\right)^{n-i+1}+\delta\left(\frac{w}{\pi}\right)^{n}<1$$

Example

Consider a one-degree-freedom system of quarter car model shown in Figure 1. The nonlinear suspension spring has the stiffness k and is proportional to the cube of the displacement x.

There is a nonlinear viscous damper with coefficient c and a nonlinear damper c_1 with velocity squared damping behavior. The actuator A is assumed to have a force proportional to the derivative of acceleration. The vertical

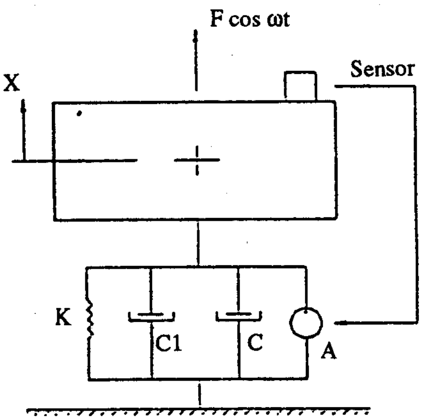

Figure 1

motion of the system is considered under an induced periodic external excitation $F_0 \cos w_0 t$.

The following differential equation is obtained for the displacement x

$$Ax''' + mx'' + cx' + c_1|x'|x' + kx^3 = F_0 \cos w_0 t$$

or

$$x''' + \frac{m}{A}x'' + \frac{c}{B}x' + f(x,x') = \frac{F_0}{A} \cos w_0 t \qquad (13)$$

We apply Theorem 3 for the case $n = 3$. From inequality (12) we obtain

$\gamma_1 (w/\pi)^2 + \gamma_2 (w/\pi) < 1$, where $\gamma_1 = m/A$, $\gamma_2 = c/A$ and $w = \pi/w_0$. We obtain

$$w_0 > 2m\,[(c^2 + 4mA)^{1/2} - c]^{-1}$$

We also note $f(x,x') = (c_1|x'|x' + kx^3)/B$ satisfies condition (vi) of Theorem 3.

Next let $|x^{(i-1)}(t)| \leq \delta$, $t \in [0, w]$, i = 1,2,3. Then for every δ there exists an $F(\delta)$ such that

$$F(\delta) = \frac{1}{A}(c_1\delta^2 + k\delta^3)$$

The bound for $|x^{(i-1)}(t)|$ takes the form

$$|x^{(i-1)}(t)| \leq \frac{1}{2}\left(\frac{\pi}{w_0}\right)^{1/2} w_0^{i-4}\, \mu\Delta_0$$

where

$$\Delta_0 = \frac{F_0/A + F(\delta)(\pi/w_0)^{1/2}}{1 - (c/A)\,w_0^{-1} - (m/A)\,w_0^{-2}}$$

Hence if w_0 is large enough, then for some $\delta > 0$ we will have

$$\frac{1}{2}(\pi/w_0)^{1/2} w_0^{i-4} \mu\Delta_0 < \delta \;,\; i = 1, 2, 3$$

Now applying Theorem 1, the existence of a $2w$ periodic solution of Equation (13) can be proved.

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