A QUARTIC POTENTIAL FOR THE NUCLEONIC QUARKS

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Abstract

We assume that each valence quark in a nucleon is in a phenomenological modified harmonic oscillator potential of the form: $\frac{1}{2}(1+\gamma_0)$ $(ar^2+br+cr^3+dr^4)$, where a, b, c and d are constants and γ_0 is one of the Dirac matrices. Then by making use of a suitable ansatz, the Dirac equation has a very simple solution which is exact. We then have calculated the static properties of the nucleon in the ground state with and without center of mass correction. The results are encouraging. PACS index 12.35 kW and 13.40 fn.

1. Introduction

An important step towards a better understanding of quark dynamics was made with the introduction of the MIT bag model. Here the relativistic motion of the quarks in the aforementioned potential is described more satisfactorily. The static properties of the nucleons with zero orbital angular momentum come out quite well in our model, except for the nucleon charge-radius which is somewhat large, and this is due to the tail of the wave function outside the nucleon [1]. When making the quark system less relativistic, the nucleon charge radius decreases and reaches the actual value. However, the corresponding properties of higher excited states can not be easily calculated within this framework. One easily sees that a potential of the form $(1+\gamma_0)M(r)$ has the desired properties. Physically this is an equal admixture of a scalar potential and the time component of a vector potential. The MIT square-well potential is all scalar while a one-gluon exchange would contribute to the time component of a vector potential. In our model, we have an equal admixture of both.

Keywords: Dirac; Harmonic; Nucleonic; Quarks; Phenomenology; Potential; Proton; Spinor; Static Properties

In section (2), we have used this potential to calculate the relativistic wave function for valence quarks. By using this wave function, we can calculate the recoil-corrected wave function.

In section (3), we have calculated some of the static properties and the effective radius of the nucleon, in the zero orbital angular momentum state.

2. Relativistic Wave Function for Three Valence Quarks in a Nucleon

The Dirac equation for a single valence quark in a central potential U(r) is:

$$[\gamma_o\varepsilon + i\,\bar{\gamma}.\bar{\nabla} - (m + U(r)]\psi(\bar{r}) = 0 \tag{1}$$

where we take U(r) to have the following form:

$$U(r) = \frac{1}{2}(1 + a\gamma_0)\,M(r) \tag{2}$$

The parameter a can take any value. We take it to be equal to 1. This case is important because it leads to an exact SU (2) symmetry and hence to spin-orbit doublet degeneracy [2, 3, 4], as it was studied by Bell and Ruegg [3]. For a = 0, the potential is scalar only [5, 6, 7]. In general, M(r) can be taken to have any form [3, 4], but it should be small in the baryon center (asymptotic freedom), and should be infinite beyond some distance from the center of the nucleon (quark confinement)[8, 9]. Here, we work with the following phenomenological potential M(r):

$$M(r) = ar^2 + br + cr^3 + dr^4 \tag{3}$$

Now, we write the solutions of equation (1) in the form

$$\psi_{jj_3}(r) = \begin{pmatrix} \phi \\ X \end{pmatrix} = N \begin{pmatrix} g_k(r)\, y^j_{j1,3} \\ if_k(r)\, y^j_{j1,3} \end{pmatrix} \tag{4}$$

where

$$y^j_{j1,3} = \left(l \tfrac{1}{2} j_3 - \tfrac{1}{2} \tfrac{1}{2} \mid jj_3\right) y_1^{j_3 - \frac{1}{2}}(\hat{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(l \tfrac{1}{2} j_3 - \tfrac{1}{2} \tfrac{1}{2} \mid jj_3\right) y_1^{j_3 + \frac{1}{2}}(\hat{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Here (1) indicates the familiar Clebsch-Gordan coefficients. Substitution of (4) into (1) yields:

$$\begin{cases} \dfrac{d\, g_k(r)}{dr} + \dfrac{k+1}{r} g_k(r) - m f_k(r) = \varepsilon f_k(r) \\ \dfrac{d\, f_k(r)}{dr} + \dfrac{-k+1}{r} f_k(r) - (m + M(r)) g_k(r) = -\varepsilon g_k(r) \end{cases} \tag{5}$$

We eliminate $f_k(r)$ between these two equations. The resulting equation is

$$g''_k(r) + \frac{2}{r} g'_k(r) + \left[\frac{-k(k+1)}{r^2} + \varepsilon^2 - m^2 - (\varepsilon + m) M(r)\right] g_k(r) = o \tag{6}$$

If we let $g_k(r) = \frac{1}{r}\phi_k(r)$, then we get:

$$\phi''_k(r) + \left[\frac{-k(k+1)}{r^2} + \varepsilon^2 - m^2 - (\varepsilon + m) M(r)\right] \phi_k(r) = o \tag{7}$$

Let

$$\lambda = \varepsilon^2 - m^2 \tag{8}$$

$$M_1(r) = (\varepsilon + m) M(r) \tag{9}$$

$$a_1 = (\varepsilon + m)a\,;\; b_1 = (\varepsilon + m)b\,;\; c_1 = (\varepsilon + m)c,\; d_1 = (\varepsilon + m)d \tag{10}$$

Then Equation (7) reduces to

$$\varphi''_k(r) + [\lambda - M_1(r) - \frac{k(k+1)}{r^2}]\, \varphi_k(r) = \tag{11}$$

where

$$M_1(r) = a_1 r^2 + b_1 r + c_1 r^3 + d_1 r^4 \tag{12}$$

Here a_1, c_1 and d_1 are postive and b_1 is negative.

A solution of (11) for $M_1(r)$ similar to ours has recently been obtained by Znojil [10], using the method of continued fractions. On the other hand, a solution of the Schrödinger-type equation (7) has been obtained for the potential (12) in a different context [11, 12, 13 and 14]. To solve (7), we make use of an ansatz similar to that of Ref. [11], i. e. we let

$$\varphi_k(r) = \exp(h(r)) \tag{13}$$

with

$$h(r) = -\frac{1}{2}\alpha r^2 - \frac{1}{3}\beta r^3 + \delta \ln r \tag{14}$$

Inserting (13) into (11), one gets for the ground state, i. e. k = -1

$$\phi_{-1}(r) = r\exp\left(-\frac{\sqrt{a_1}}{2} r^2 + \frac{b_1}{12} r^3\right) \tag{15}$$

with

$$\lambda = \varepsilon^2 - m^2 = 3\sqrt{a_1} \tag{16}$$

and we have the following constraint between the parameters a_1, b_1, c_1 and d_1:

$$c_1 = -b_1 \frac{\sqrt{a_1}}{2} \tag{17}$$

$$b_1 = -4\sqrt{d_1} \tag{18}$$

Hence the upper component of the Dirac spinor confined by our potential (12) is

$$g_{-1}(r) = \exp - \left(\sqrt{a_1}\, r^2 - \frac{b_1}{12} r^3\right) \tag{19}$$

From (5) we can find the lower component $f_{-1}(r)$ of the Dirac spinor. Then the normalized spin $\frac{1}{2}$, positive parity solution of a quark in the potential (12), in the ground state is*

$$\psi_{\frac{1}{2}\frac{1}{2}}(r) = \frac{N}{\sqrt{4\pi}}\begin{bmatrix} 1 \\ -i\frac{\vec{\sigma}.\hat{r}}{\varepsilon+m}(-\sqrt{a_1}r+\frac{b_1}{4}r^2)\end{bmatrix}\exp\left(\frac{-\sqrt{a_1}}{2}r^2+\frac{b_1}{12}r^3\right) \quad (20)$$

Assuming the mass of the nucleon (≈938 MeV) to be the sum of the three valence-quark energies, i.e. neglecting the center of mass motion (as in Bogolioubov model), one gets

$$\varepsilon = 313 \text{ MeV or } \varepsilon = 1.584 \text{ fm}^{-1}$$

By using $g_{-1}(r)$ and $f_{-1}(r)$, one can calculate the recoil corrected function $\tilde{g}_{-1}(r)$ and $f_{-1}(r)$ [see Ref. 5].

3. The Static Properties of the Nucleon

3-1. Fitting with the Ratio $\frac{g_A}{g_V}$

We have Equation (16) and the wave function (20) with two unknown parameters a_1 and b_1. If we use the experimental result $\frac{g_A}{g_V} = 1.26$ as a constraint, we can calculate a_1 and b_1.

3-2. Proton Magnetic Moment and Proton Charge-Radius

By using the standard definition of magnetic moment, one can find the general expression for the magnetic moment of a quark in its ground state, which is

$$\mu_q = -\frac{2}{3}Ne^2\int_0^\infty r^3 g(r) f^{**}(r)\, dr \quad (21)$$

One can show that the proton magnetic moment μ_p is equal to

$$\mu_p = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d = \mu_q$$

where μ_u and μ_d are the magnetic moments of u and d quarks respectively.

The proton charge radius is:

$$<r^2_{em}>_p = \int_0^\infty r^2 \psi_{\frac{1}{2}\frac{1}{2}}(r)\, \psi_{\frac{1}{2}\frac{1}{2}}(r)\, d^3x \quad (22)$$

where $\psi_{\frac{1}{2}\frac{1}{2}}(r)$ is the quark wave function, given by (20).

In Table 1, we have calculated μ_p and $<r^2_{em}>_p^{\frac{1}{2}}$ for various quark masses. The rather large values for the charge-radius in Table 1 is due to the tail of the wave function outside the bag. We shall ignore the tail of the wave function outside the nucleon. Then the charge radius will be close to the experimental value. On the other hand, the net effect of recoil corrections works always in such a way that it effectively reduces charge radius. This result is consistent with the findings of Refs. [16] and [17].

In the presence of the center of mass correction, the above static properties can be found as in Ref. [5]. Using the method of Ref. [5] for the correction of center mass motion, we get:

$$1.21 \le \left(\frac{g_A}{g_V}\right)_b \le 1.26 \quad (23)$$

where the index b indicates inclusion of the center of mass correction. Similarly, one can show that

$$2.408 \le (\mu_p)_b \le 3.080 \quad (24)$$

and

$$1.112 \text{ fm} \le <r^2_{em}>_b^{\frac{1}{2}} \le 1.260 \text{ fm} \quad (25)$$

for various quark masses. These bounds on $\frac{g_A}{g_V}$, μ_p and $<r^2_{em}>_p^{\frac{1}{2}}$ are found by a numerical calculation.

3-3. Proton Effective Radius and Corrected Charge Radii

We can define an effective radius R for the nucleon with the aid of the wave function (20). We have taken R to be the value of r for which the upper component of the spinor, in the absence of the recoil correction, is a fraction $e^{-1/2}$ of its value for r = 0.

This definition gives the following results for various

* The wave function $\psi_{\frac{1}{2}\frac{1}{2}}(\hat{r})$ does not satisfy Lorentz invariance in the sense that the small component associated with the center of mass motion is not treated properly. This is justified as long as the nucleon as a whole moves nonrelativistically, i.e. if $\frac{|p|}{\sqrt{p^2+M^2}+M} < 1$, where M is the nucleon mass.

** The corresponding Equation (14) in Ref. [1] is incorrect.

values of quark masses. For 120 MeV $\leq m_q \leq$ 312 MeV, we get

$$1.092 \text{ fm} \leq R \leq 1.21 \text{ fm}$$

If we ignore the tail of the wave function (20) outside the nucleon and normalize this wave function inside the nucleon, then the charge radius of the proton will become very close to the experimental value (Table 4).

This shows that the results are reasonable. The term proportional to $\exp(+\frac{b}{12}r^3)$ in the wave function has improved the results relative to the simple harmonic case [Ref. 1]. If we let b = 0 then from constraints (17), (18) c = d = 0, we get the potential and wave function, and the results of Ref [1].

Conclusion

In this work we have taken the nucleonic quarks to be under the influence of the phenomenological modified harmonic oscillator potential.

$$U(r) = \frac{1}{2}(1+\gamma_0)(ar^2 + br + cr^3 + dr^4)$$

Table 1. The static properties and the confining potential in a nucleon for (a) without and (b) with center of mass correction, and the effective-radius of a nucleon for various quark masses

m_q	$\frac{g_A}{g_V}$	$\langle r^2_{em}\rangle_p^{\frac{1}{2}}$	μ_p	R	$M_1(r) = a_1r^2+b_1r+c_1r^3+d_1r^4$
120 MeV	1.26[a] 1.21[b]	1.50[a] fm 1.26[b] fm	3.271[a] 2.788[b]	1.179[a] fm	$M_1(r) = 0.51\ r^2 - 0.03\ r + 0.01\ r^3 + 0.00006\ r^4$
153 MeV	1.26[a] 1.223[b]	1.41[a] fm 1.241[b] fm	3.240[a] 2.62[b]	1.167[a] fm	$M_1(r) = 0.41\ r^2 - 0.5\ r + 0.162\ r^3 + 0.016\ r^4$
250 MeV	1.26[a] 1.235[b]	1.22[a] fm 1.135[b] fm	2.753[a] 2.476[b]	1.131[a] fm	$M_1(r) = 0.09\ r^2 - 2.55\ r + 0.382\ r^3 + 0.406\ r^4$
300 MeV	1.26[a] 1.25[b]	1.13[a] fm 1.113[b] fm	2.497[a] 2.450[b]	1.098[a] fm	$M_1(r) = 0.004\ r^2 - 4.2\ r + 0.138\ r^3 + 1.102\ r^4$
313 MeV	1.26	1.112 fm	2.414	1.092[a] fm	$M_1(r) = -4.6\ r + 1.323\ r^4$

Table 2. The static properties and the bag-radius of a nucleon for harmonic quark bag model [1]

m_q (MeV)	$\frac{g_A}{g_V}$	$\langle r^2_{em}\rangle_p^{\frac{1}{2}}$	μ_p	R_{bag} fm	Potential $M_1 = a_1 r^2$
0	0.93	1.48 fm	4.00 n.m	1.153	$M_1(r) = 0.7\ r^2$
120	1.27	1.53 fm	3.546 n. m	1.402	$M_1(r) = 0.51\ r^2$
153	1.28	1.60 fm	3.44 n.m	1.562	$M_1(1) = 0.41\ r^2$
201.5	1.45	1.807 fm	3.293 n.m	2.045	$M_1(r) = 0.239\ r^2$
250	1.55	2.275 fm	3.156 n.m	3.33	$M_1(r) = 0.09\ r^2$
300	1.65	4.77 fm	2.074 n.m	15.15	$M_1(r) = 0.004\ r^2$
310	1.66	10.35 fm	0.569 n.m	71.43	$M_1(1) = 0.0002\ r^2$
313	1.67	∞ fm	0	∞	$M_1(r) = 0$

Table 3. Comparison between the results of our model Table 1 and the model [1] Table 2 for (120 MeV $\leq m_q \leq$ 313 MeV) and the experimental values of proton

Our model	Harmonic quark bag model [1]	Experiments
Fitted with $\frac{g_A}{g_V} = 1.26$	$1.292 \leq \frac{g_A}{g_V} \leq 1.67$	1.254 ± 0.004
2.414 n.m $\leq \mu_p \leq$ 3.271 n.m	$0 \leq \mu_p \leq$ 3.48 n.m	2.792 n.m
0.805 fm $\leq \langle r^2_{em}\rangle^{\frac{1}{2}} \leq$ 0.87 fm	1.57 fm $\leq \langle r^2_{em}\rangle^{\frac{1}{2}} \leq \infty$	0.88 ± 0.03 fm
1.092 fm $\leq R \leq$ 1.179 fm	1.206 fm $\leq R_{eff} < \infty$	0.88 ± 0.03 fm

This table shows that our model has certainly improved the results of the harmonic quark bag model [1]

Table 4. Varian of R and $\langle r^2_{em}\rangle^{\frac{1}{2}}$ with m_q, ignoring the tail of the value function

m_q	120 MeV	153 MeV	250 MeV	300 MeV	312 MeV
R	1.21 fm	1.167 fm	1.131 fm	1.098 fm	1.092 fm
$\langle r^2_{em}\rangle^{\frac{1}{2}}_p$	0.87 fm	0.862 fm	0.85 fm	0.81 fm	0.805 fm

We have chosen the parameters a, b, c, d in a suitable manner for the quark masses in the range 120 MeV $\leq m_q \leq$ 313 MeV, and we have tabulated them in Table 1. For this potential, we have found the quark wave function and then the static properties of the nucleon without (a) or with (b) center of mass correction.

Our results, with the exception of charge-radius, are in good agreement with the experimental values and if we ignore the tail of the wave function outside the nucleon, the charge radius will become close to the experimental result.

If we compare the static properties (in cases a and b) in Table 1, we get the following results. The values for $\frac{g_A}{g_V}$ and $\langle r^2_{em}\rangle^{\frac{1}{2}}_p$ and μ_p, in the absence of center of mass correction, are increased by about (0, 5%), (0, 15%), and (0, 22%) respectively, depending on the quark masses.
This is similar to the results of Refs. [16] and [17].

Now, if we compare the results of Tables 1 and 2, we see that our model gives better results especially when nucleonic quarks become less relativistic. In this case, the results of the harmonic quark bag model [Ref. 1] are unacceptable, while the results of our model are close to the experimental values.

References

1. Ravndal, F. *Physics Letter*, **113 B**, (1), 57-60, (1982).
2. Smith, G. B. and Tassie, J. J. *Ann. Phys.*, NY **65**, 352, (1971).
3. Bell, J. S. and Ruegg, H. *Nucl Phys.*, **98B**, 151, (1975); Bell, J.S. and Ruegg, H. *Ibid.* **104B**, 546, (1976).
4. Ferreira, P.L., Helayel, J. A. and Zagury, N. *Nuovo Cm.*, **55A**, 215, (1980).
5. Tegen, R., Brockmann, R. and Weise, W. *Z. Phys.*, **A307**, 339, (1982); Tegen, R., Schedl, M. and Weise, W. *Physics Letters*, **125**, 9, (1983).
6. Ferreira P. L. and Zagury, N. *Lett. Nuovo Cm.*, **20**, 150-511, (1977).
7. Tegen, R. *Phys. Rev. Lett.*, **62**, 1724, (1989); Zagury, N. *Lett. Nuovo Cm.*, **20**, 157-511, (1977).
8. Nambu, Y. and Jona-lasino, G. *Phys. Rev.*, **122**, 345, (1961).
9. Brockmann, R., Weise, W. and Werner, E. Preprint (1982).
10. Znojil, M. *J. Math. Phys.*, **31**, 108 ,(1990); Znojil, J. *Ibid.*, **30**, 23 (1982); Znojil, M. *J. Phys.*, **A15, 2111**, (1982).
11. Kaushal, R. S. *Ann. Phys.*, (NY), **206**, 102, (1991).
12. Papp, E. *Phys. Lett.*, **A157**, 192, (1991).
13. Kaushal, R. S. **A142**, 57, (1989).
14. Oezelik, S. and Simsek, M. *Phys. Lett.*, **A152**, 145, (1991).
15. Scadron, M.D. *Advanced quantum theory and its applications*, p. 351. Springer-Verlag, (1991).
16. Myhrer, F. *Phys. Lett.*, **110B**, 353, (1982).
17. Detar, C. *Phys. Rev.*, **D24**, 752,762, (1981).