QUANTUM TUNNELING IN MEDIUMS WITH LINEAR AND NONLINEAR DISSIPATION

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Abstract

We have applied the method of integration of the Heisenberg equation of motion proposed by Bender and Dunne, and M. Kamella and M. Razavy to the potential $V(q) = \frac{1}{2}v^2q^2 - \frac{1}{3}\mu^3q^3$ with linear and nonlinear dissipation. We concentrate our calculations on the evolution of basis set of Weyl Ordered Operators and calculate the mean position, velocity, the commutation relation [q, p], and the energy of particle. According to our results, the particle which is confined in the well at t= 0, has some oscillations before tunneling. If dissipation is proportional to velocity it inhibits tunneling but when it is quadratic in velocity it facilitates tunneling. Thus, we can continue our calculation easily for every power of velocity.

I. Introduction

Quantum tunneling is an important physical phenomenon that occurs in a variety of different physical systems, so it is natural that many people show an interest in studying it. Moreover, since there is no true absolute isolated system in the universe, quantum tunneling in the presence of dissipative forces has also been extensively studied [1-3]. Phenomenologically the frictional forces are often assumed to have a simple dependence on the velocity of the particle. Although we have included the nonlinear dissipation quadratic in velocity, in our calculation because of the importance of ohmic systems, the former is discussed in more detail.

M. Kamela and M. Razavy [6] have shown that for the problem of quantum tunneling with an anharmonic potential, one can use the time evolution of Weyl Ordered Operators introduced by C. M. Bender and G. Dunne [4-5]. They then integrated the Heisenberg equation of motion and found the solution of operator differential equation. Numerical results obtained show that the particle which is initially located at some point Q_0, first goes toward the minimum of the potential, similar to classical dynamics, and then, after spending a short time in the well, escapes from it. We followed the same general procedure but our numerical results are in some sense more interesting, i.e. the particle has some oscillations in the well before tunneling occurs. We found a very interesting result for the trajectory trajectory of the particle in phase space. The constancy of energy and equal time commutation relation, ETCR, in the absence of dissipation (which is the same for both linear and nonlinear dissipation) were used to check the validity of our results. Although linear dissipation inhibits tunneling, it seems that when the dissipation term is quadratic in velocity, it facilitates tunneling. This surpising phenomenon in some sense corresponds to the results of Nieto and coworkers [7].

It may be noted that in this formulation there is no need to construct a Lagrangian or a Hamiltonian for the system. In addition to this advantage, the Heisenberg equation of motion is much more effective to use in solving problems than the Schrödinger equation in quantum field theory. Although we have applied the method of integration of the Heisenberg equation of motion to a simple example, we hope that by generalizing it we can solve some previously unsolved problems in field theory.

The calculational procedure is explained in Sec. II. In Sec. III we apply the method to an anharmonic potential. Finally, in Sec. IV we briefly discuss our results.

Keywords: Dissipation; Quantum friction; Quantum

II. Position and Momentum and the Time Evolution of the Basis Set

Our purpose is to solve the operator differential equations of motion of a particle with unit mass:

$$\frac{dp}{dt} = -\lambda p + f(q); \quad \frac{dq}{dt} = p \tag{1}$$

subject to the initial conditions $p(O) = p_o$, $q(O) = q_o$, such that in the absence of damping, they preserve the equal time commutation relation $[q(t), p(t)] = i$, where in (1) λ is the damping coefficient and $f(q)$ is any polynomial in q, so that the only possible singularities in the solution of (1) are at fixed poles.

C. M. Bender and G. Dunne introduced the following Weyl Ordered Operators [4-5]:

$$T_{m,n} = \frac{1}{2^n} \sum_{k=0}^{n} \frac{m!}{k!(m-k)!} q^k p^m q^{n-k} \tag{2}$$

which by using ETCR relation has the equivalent form:

$$T_{m,n} = \frac{1}{2^m} \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} p^j q^n p^{m-j} \tag{3}$$

They also showed that these basis sets $T_{m,n}$ form a closed algebra under multiplication and satisfy some interesting commutation and anticommutation relations. For example, the following was proved (for $m, n, r, s \in Z^+$):

$$T_{m,n} T_{r,s} = \sum_{j=0}^{\infty} \frac{(i/2)^j}{j!} \sum_{k=0}^{j} (-1)^{j-k} \frac{j!}{k!(j-k)!} \times \frac{\Gamma(n+1)\Gamma(m+1)\Gamma(r+1)\Gamma(s+1)}{\Gamma(n-k+1)\Gamma(m+k-j+1)\Gamma(r-k+1)\Gamma(s+k-j+1)} T_{m+r-j,\, n+s-j} \tag{4}$$

From the definition (2) or (3) we obviously have: $q(t) = T_{0,1}(t)$, and $p(t) = T_{1,0}(t)$.

To solve (1) we start with the identity:

$$q(t) = e^{iHt} q(0) e^{-iHt} \tag{5}$$

where H is the Hamiltonian. This relation can be written as:

$$q(\Delta t) - q(0) + \frac{(\Delta t)}{1!}\left(\frac{dq}{dt}\right)_0 + \frac{(\Delta t)^2}{2!}\left(\frac{d^2q}{dt^2}\right)_0 + \ldots \tag{6}$$

We now write q and p in terms of $T_{m,n}$'s and denote the time $j\Delta t$ by t_j, with j an integer, we finally have:

$$q(t_{j+1}) = T_{0,1}(t_j) + (\Delta t) T_{1,0}(t_j) + \frac{(\Delta t)^2}{2!}[-\lambda T_{1,0}(t_j) + f(T_{0,1}(t_j))] + \frac{(\Delta t)^3}{3!}\{\lambda^2 T_{1,0}(t_j) - \lambda f(T_{0,1}(t_j)) + \frac{1}{2}[T_{1,0}(t_j) f'(T_{0,1}(t_j)) + f'(T_{0,1}(t_j)) T_{1,0}(t_j)]\} + \ldots \tag{7}$$

$$p(t_{j+1}) = T_{1,0}(t_j) + (\Delta t)[f(T_{0,1}(t_j)) - \lambda T_{1,0}(T_j)] + \frac{(\Delta t)^2}{2!}[2\lambda T_{1,0}(t_j) - \lambda f(T_{0,1}(t_j))] + \frac{1}{2}[T_{1,0}(t_j) f'(T_{0,1}(t_j)) + f'(T_{0,1}(t_j)) T_{1,0}(t_j)] + \ldots \tag{8}$$

The calculational procedure is as follows: if $\{T_{m,n}(t_j)\}$'s are known at t_j, then $q(t_{j+1})$ and $p(t_{j+1})$ can be calculated from (7) and (8). Using these we can obtain $T_{m,n}(t_{j+1})$ from (2) or (3):

$$T_{m,n}[q(t_{j+1}), p(t_{j+1})] = T_{m,n}(t_{j+1}) \tag{9}$$

Using $T_{m,n}(t_{j+1})$ we can calculate $p(t_{j+2})$ and $q(t_{j+2})$ at $(2+j)\Delta t$, ... Therefore, if we have $\{T_{m,n}(0)\}$'s we can find position and velocity at any time by iteration. Note that the core of the above procedure is finding the $C_{m,n}$'s which are the time dependent real coefficients:

$$q(t) = \Sigma_{m,n} C_{m,n}(t) T_{m,n}(0) \tag{10}$$

$$p(t) = \Sigma_{m,n} \dot{C}_{m,n}(t) T_{m,n}(0) \tag{11}$$

For dissipative force quadratic in velocity the operator differential equations are:

$$\frac{dp}{dt} = -\lambda p^2 + f(q); \quad \frac{dq}{dt} = p \tag{12}$$

In this case, we obtain the following Taylor series which are more complicated than (7) and (8):

$$q(t_{j+1}) = T_{0,1}(t_j) + (\Delta t) T_{1,0}(t_j) + \frac{(\Delta t)^2}{2!}[-\lambda T_{2,0}(t_j) + f(T_{0,1}(t_j))] + \frac{(\Delta t)^3}{3!}\{2\lambda^2 T_{3,0}(t_j) - \lambda[T_{1,0}(t_j) f(T_{0,1}(t_j)) + f(T_{0,1}(t_j)) T_{1,0}(t_j)] + \frac{1}{2}[T_{1,0}(t_j) f'(T_{0,1}(t_j)) + f'(T_{0,1}(t_j)) T_{1,0}(t_j)]\} + \ldots \tag{13}$$

$$p(t_{j+1}) = T_{1,0}(t_j) + (\Delta t)[-\lambda T_{2,0}(t_j) + f(T_{0,1}(t_j))] + \frac{(\Delta t)^2}{2!}\{2\lambda^2 T_{3,0}(t_j) - \lambda[T_{1,0}(t_j) f(T_{0,1}(t_j)) + f(T_{0,1}(t_j)) T_{1,0}(t_j)] + \frac{1}{2}[T_{1,0}(t_j) f'(T_{0,1})(t_j)) + f'(T_{0,1}(t_j)) T_{1,0}(t_j)]\} + \ldots \tag{14}$$

The above procedure and the general form of the final relations (10) and (11) do not change for a nonlinear one, $-\lambda p^2$. However, it is clear that the coefficients $C_{m,n}$ and $\dot{C}_{m,n}$ are different in the three cases: no dissipation, linear and quadratic velocity and dependent dissipative forces.

III. Tunneling through an Anharmonic Potential

As a simple example to check the validity of the above procedure we apply it to the anharmonic potential:

$$V(q) = \frac{1}{2}\nu^2 q^2 - \frac{1}{3}\mu^3 q^3 \quad (15)$$

where μ and ν are constants. By introducing the dimensionless quantities:

$$\theta = \nu t,\ Q(\theta) = \left(\frac{\mu^3}{\nu^2}\right) q(t),\ P(\theta) = \left(\frac{\mu^3}{\nu^3}\right) p(t) \quad (16)$$

we have:

$$\frac{dP}{d\theta} = -\lambda' P - Q + Q^2,\ \frac{dQ}{d\theta} = P \quad (17)$$

$$[Q(\theta), P(\theta)] = i\gamma \quad (18)$$

$$\tau_{m,n}(P, Q, \theta) = \left(\frac{\mu^3}{\nu^2}\right)^n \left(\frac{\mu^3}{\nu^3}\right)^m T_{m,n} \quad (19)$$

where $\lambda' = \frac{\lambda}{\nu}$ and $\gamma = \frac{\mu^6}{\nu^5}$.

We choose the normalized Gaussian wave packet:

$$\psi(Q - Q_0) = \left(\frac{1}{\pi\gamma}\right)^{1/4} \exp[-(Q - Q_0)^2/2\gamma] \quad (20)$$

to represent the particle. In a straightforward manner we obtain:

$$<0|\tau_{m,n}|0>_{Q0} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} Q_0^{n-k}(m-1)!!\,(k-1)!!\,\left(\frac{\gamma}{2}\right)^{(m+k)/2} \times \delta_{m,\,even}\,\delta_{k,\,even} \quad (21)$$

The energy of the particle in the absence of dissipation at $\theta = o$ is:

$$<0|H|0>_{Q0} = \frac{1}{2}[\gamma(1 - Q_0) + Q_0^2] - \frac{1}{3}Q_0^3 \quad (22)$$

Note that although the maximum of the potential is at $Q_0=1$ with a height of $V_0= 1/6$, the position of the barrier is at $Q_0=0.9$, with a height of 0.167.

With the new dimensionless variables, (9) and (10) become:

$$<0|Q(0)|0>_{Q0} = \sum_{m,n} d_{m,n} <0|\tau_{m,n}|0> \quad (23)$$

$$<0|P(0)|0>_{Q0}0 = \sum_{m,n} d_{m,n} <0|\tau_{m,n}|0> \quad (24)$$

where:

$$d_{m,n} = \left(\frac{\nu^2}{\mu^3}\right)^{(n-1)} \left(\frac{\nu}{\mu}\right)^{3m} C_{m,n}$$

IV. Discussion of the Numerical Results

In usual quantum mechanics, if the initial state of a particle is an eigenstate of an observable which commutes with the Hamiltonian, its expectation value does not depend on the time (stationary states). But if the expectation value is taken with respect to a superposition of energy eigenstates

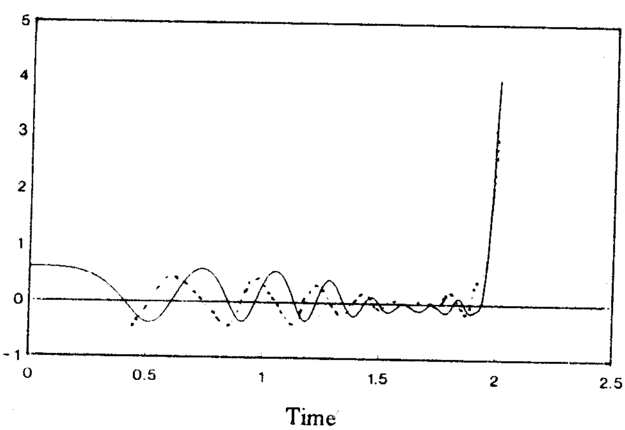

Figure 1. The expectation value of the position and momentum operators as a function of time $\theta = \nu t$. ($Q_0 = 0.6$, $P_0 = 0$, $\gamma = 0.1$ and $\lambda' = 0$). The dashed line refers to momentum and the solid line to position.

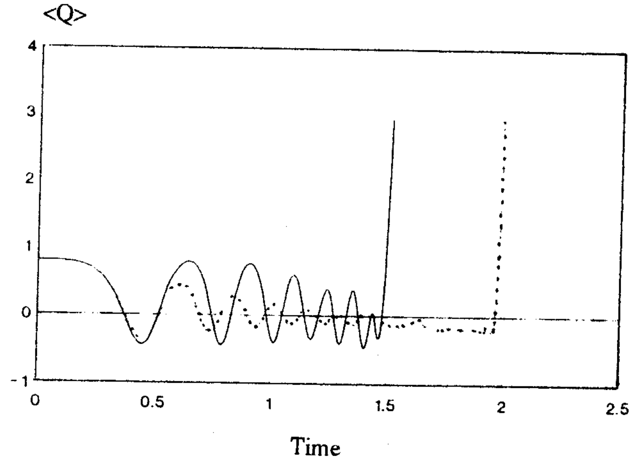

Figure 2. The expectation value of the position operator as a function of time $\theta = \nu t$ in the absence of dissipation ($\lambda' = 0$, solid line) and in the presence of it. ($\lambda' = 0.1$, dashed line), both for $Q_0 = 0.8$ and $\gamma = 0.1$

(nonstationary states), which is considered in this paper, it will involve oscillatory terms whose angular frequency can be found in every textbook on quantum mechanics. The earlier works [6] do not show the oscillations. Our calculations include up to terms of order $(\Delta\theta)^9$ in the Taylor series expansion (7), (8). As we see from Figure 1, P and Q have some oscillations before tunneling. We also see from Figures 1, 2 and 3 that this feature doesn't change qualitatively in the absence or presence of dissipation (linear or nonlinear).

According to our results, tunneling occurs for every value of λ'. The only difference is that when $\lambda'=0$ (undamped motion), the particle escapes faster from the well - as we may expect, because in this case the probability of staying in the well increases. This agrees with the earlier works of M. Razavy *et al.* [3, 6] and A. O. Calderia *et al.* [2]. The dependence of time tunneling of λ' is more complicated, and there's no simple relation between time of tunneling and damping constant λ'. It is also not possible to define a critical damping, (Fig. 4).

The greater the value of Q_0 in the Fortran Program Code (thus, the greater the energy), the sooner the particle escapes.

For $Q_0 \leq 0.85$, $Q_0 = 0.9$ (just at the maximum height of the barrier) or even $Q_0 = 0.95$, no qualitative change is observed, which is meaningless in classical mechanics. Thus, this shows the quantum nature of our results.

Classically, the orbit of a harmonic oscillator with the

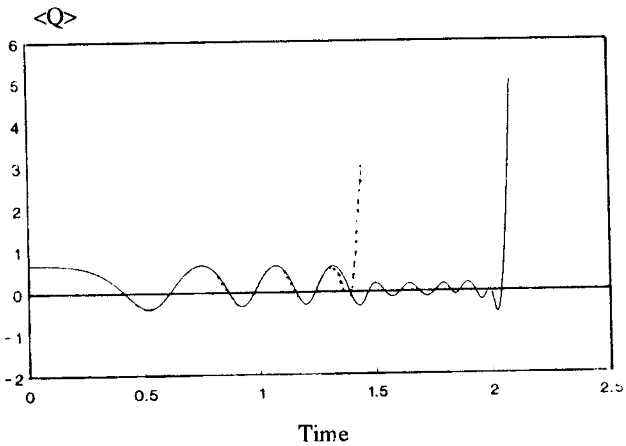

Figure 3. The expectation value of the position operators as a function of time $\theta = vt$. Solid line for $\lambda'=0$ (no dissipation) and dashed line for $\lambda'=0.1$ (nonlinear dissipation, quadratic in velocity), both for $Q_0 = 0.65$ and $\gamma = 0.1$.

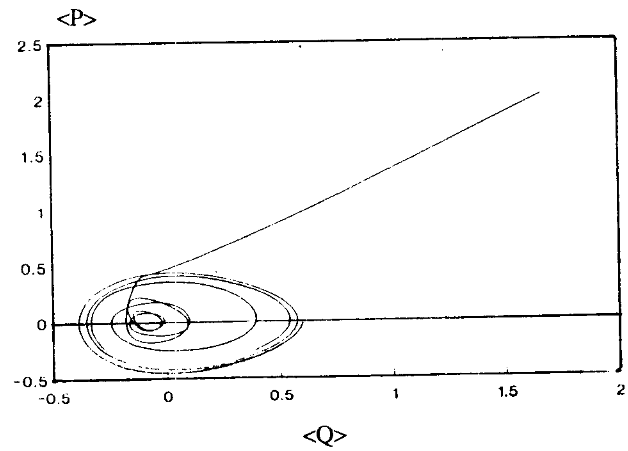

Figure 5. The phase diagram of the particle's motion in the absence of dissipation ($Q_0 = 0.6$, $P_0 = 0$, $\gamma = 0.1$ and $\lambda' = 0$).

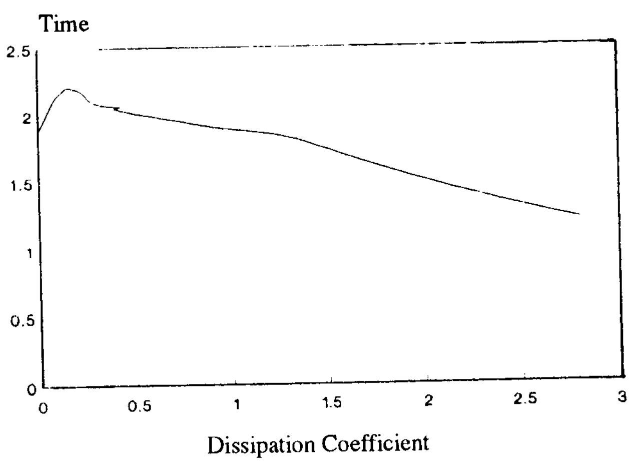

Figure 4. The variation of time of tunneling as a function of damping coefficient λ', for linear dissipation.

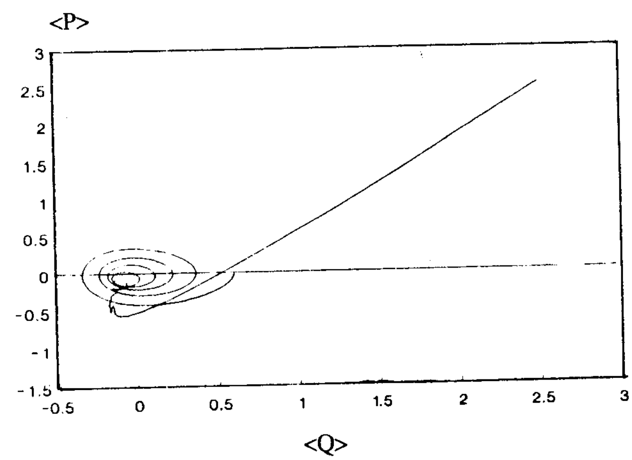

Figure 6. The phase diagram of the particle's motion in the presence of linear dissipation ($Q_0 = 0.6$, $P_0 = 0$, $\gamma = 0.1$ and $\lambda' = 0.1$).

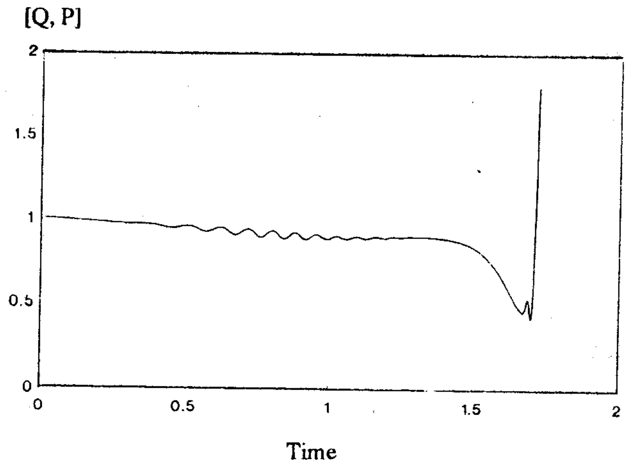

Figure 7. The expectation value of equal time commutation relation as a function of time $\theta = \nu t$. in the presence of damping force, (Q.=0.7, P.= 0, γ = 0.1 and λ'= 0.1).

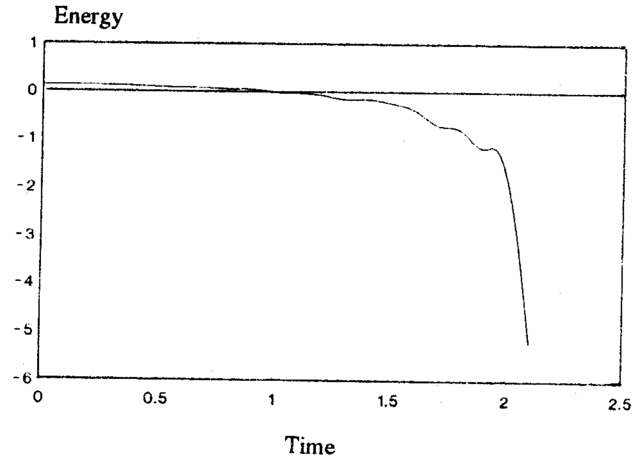

Figure 8. The dissipation of energy of the particle in the presence of dissipation as a function of time $\theta = \nu t$. (Q.=0.65, P.=0, γ = 0.1 and λ'= 0.1).

Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$ in the phase space is a circle with a radius of $r^2 = p_0^2 + q_0^2$. And, in the presence of damping, this radius will decrease with time and finally tend to zero, i.e. the particle comes to rest. But in quantum mechanically, the uncertain relation requires that the energy never becomes zero, whether dissipation is present or not. Our results, as shown in Figures 5 and 6, indicate that the trajectory of the particle in the phase space diagram illustrates this consistency. The long diagonal lines in Figures 5 and 6 obviously indicate escaping from the potential well.

Our numerical results show that when dissipation is quadratic in velocity, it facilitates tunneling, which is surprising in classical mechanics. This picture is similar to the results of Neito *et al.* [7] which show the existence of resonances in quantum tunneling in an asymmetric double well. One possible explanation for this result is that there may be an effective potential corresponding to the term - $\lambda' p^2$, so that it changes the width and height of the barrier of the main potential in such a way that it facilitates tunneling.

Figure 7 shows the variation of ETCR with time with λ'=0.1, and Figure 8 shows the dissipation of energy in the presence of a damping force.

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