Vol. 5 No. 1, 2
Winter & Spring 1994

BATCH TRANSFERRING IN PRODUCTION-INVENTORY SYSTEMS

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Abstract

In most production-inventory systems, the products are transferred from the production centre to the inventory (warehouse) in subbatchs rather than one at a time. In this paper, mathematical models of such systems for both deterministic and stochastic cases of demand are developed and analysed. Comparison of these new models with the EBQ model shows a considerable loss when applying the rules of EBQ model to the batch transferring systems.

Introduction

One assumption usually made in the construction of Economic Batch Quantity (EBQ) models for production-inventory systems [5, 6] is that the items are transferred from the production centre to the inventory (warehouse) one at a time and as soon as they are produced. Also in line with this assumption, the actual transferring cost is ignored. This assumption is realistic only for systems in which the production centre and the main inventory are located in the same building and there is no cost involved in moving the items. Frequently, the produced items are transferred in group or subbatchs such as truck-loads, boxes, etc. In some systems, the production centre and the warehouse are so far from each other that it is very costly or impossible to move the items in units and as soon as they are produced. Instead, the products are accumulated up to a specified level then transferred to the warehouse as a subbatch of production batch quantity.

Regarding the above considerations, we developed mathematical models for the mentioned batch transferring production-inventory systems. In section 2, the case of deterministic demand is considered and analysed for minimisation of the cost function. Section 3 includes the case of random demand. In both cases, we assume that there is no interruption to production during the production cycle and the products are all good enough to meet demands.

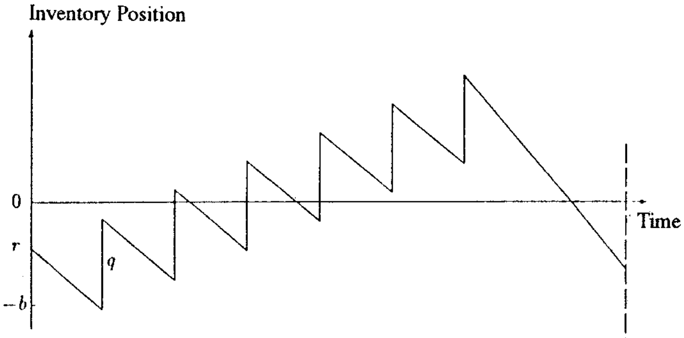
Keywords: Mathematical modelling; Production and Inventory control; Economic batch quantity; Mixed-integer linear programming

Throughout this paper, we denote the unit holding cost per unit time at the production centre and the inventory respectively by c' and c, the unit shortage cost per unit time and back order cost per unit by $\hat{\pi}$ and π respectively and set-up cost by A.

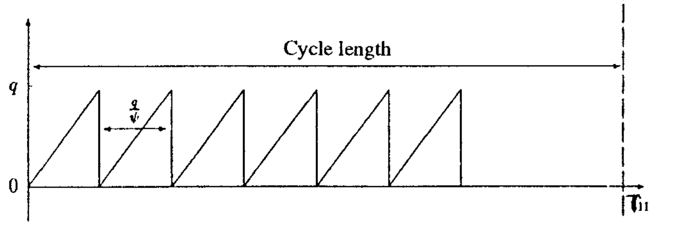
Deterministic Demand

Suppose the size of a subbatch is q units and during each production cycle a batch of size $Q=nq$, where n is a positive integer, is produced. The products are held at the production centre until one subbatch is completed. Then the subbatch is transferred into the inventory with the cost of a monetary units. As soon as the n subbatches (or Q units) are produced, the production process is stopped until the time when the inventory position reaches the re-order level (end of cycle). Suppose that the production rate is ψ and products are demanded at the constant rate of λ with a uniform pattern. Let r be the re-order level and b the maximum back-orders allowed. Figure 1 illustrates the inventory position and stock level at the production centre.

Let the time taken to produce the j^{th} subbatch of a cycle be called production period j, $1 \leq j \leq n$. The lengths of all these periods are the same and equal $\frac{q}{\psi}$. The time elapsed between the arrival of the n^{th} group into the inventory and the crossing of the re-order level is called the $n+1^{st}$ period. Let the inventory position at the beginning of the j^{th} period be denoted by I_j for $j = 1,2,..., n+1$.



(a) Inventory position during one production cycle (when n=6 and r=<0)



(b) Stock level at the production centre during one production cycle (n=6)

Figure 1. Inventory position

It is easy to show that

$$I_j = q - b + (j-2)\frac{\delta q}{\psi} \qquad j = 1,2,..., n+1 \quad (1)$$

where $\delta = \psi - \lambda$ and number of back orders incurred in the j^{th} period respectively. Then the total amount of holding, shortage and back orders, during one production cycle, will be

$$C_1 = \sum_{j=1}^{n+1} C_j^{(1)} \quad (2)$$

$$C_2 = \sum_{j=1}^{n+1} C_j^{(2)} \quad (3)$$

$$C_3 = \sum_{j=1}^{n+1} C_j^{(3)} \quad (4)$$

Since exactly nq units of the items have been demanded during the cycle, the cycle length is $\frac{nq}{\lambda}$. Consequently, we get the total system cost per unit time as

$$K(q,b,n) = \frac{\lambda a}{q} + \frac{\lambda A}{nq} + c.\frac{\lambda C_1}{nq} + \hat{\pi}.\frac{\lambda C_2}{nq} + \pi.\frac{\lambda C_3}{nq} + \frac{\lambda c' q}{2\psi} \quad (5)$$

To determine C_1, C_2, C_3, Figure 1 indicates that when $I_j \leq 0$ there will not be any stock on hand in the j^{th} period and hence there will be only shortage costs incurred. $I_j \leq 0$ implies $j \leq 1 + \frac{\psi b}{\delta q} - \frac{\lambda}{\delta}$. Let $m = \left[1 + \frac{\psi b}{\delta q} - \frac{\lambda}{\delta}\right]$

where $[x]$ is the largest integer less than or equal to x. Thus, for periods $j = 1,2,...,m$ all demands are backlogged. Now if $1 \leq n < m$ the system will be in shortage mode throughout every cycle. Hence, hereafter we take $n \geq$max. $\{1, m\}$. A quick calculation shows that

$$\left.\begin{aligned} C_1^{(1)} + C_{n+1}^{(1)} &= \frac{I_{n+1}^2}{2\lambda} \\ C_1^{(2)} + C_{n+1}^{(2)} &= \frac{b^2}{2\lambda} \\ \text{and} \quad C_1^{(3)} + C_{n+1}^{(3)} &= b \end{aligned}\right\} \quad (6)$$

Now consider the inventory position in the other periods. If there exists an integer $j \leq n$ such that $I_j > 0$ and $I_j - \frac{\lambda q}{\psi} < 0$, then there will be both carrying and shortage costs incurred in the j^{th} period. These inequalities imply

$$1 + \frac{\psi b}{\delta q} - \frac{\lambda}{\delta} < j < 1 + \frac{\psi b}{\delta q} \quad (7)$$

Since, when $1 + \frac{\psi b}{\delta q}$ is integer and $j = 1 + \frac{\psi b}{\delta q}$ there will be zero shortage costs in the period j, for the simplicity of computations we extend (7) to

$$1 + \frac{\psi b}{\delta q} - \frac{\lambda}{\delta} < j \leq 1 + \frac{\psi b}{\delta q} \quad (8)$$

If $I_j - \frac{\lambda q}{\psi} > 0$ there will only be holding costs in the j^{th} period. If we let $k = \left[1 + \frac{\psi b}{\delta q}\right]$ then we get

$$C_j^{(1)} = \begin{cases} 0 & 1 < j \leq m \\ \dfrac{I_j^2}{2\lambda} & m < j \leq k \\ \dfrac{(2I_j - \frac{\lambda q}{\psi})q}{2\psi} & k < j \leq n \end{cases} \quad (9)$$

$$C_j^{(2)} = \begin{cases} \dfrac{(\frac{\lambda q}{\psi} - 2I_j)q}{2\psi} & 1 < j \le m \\ \dfrac{(I_j - \frac{\lambda q}{\psi})^2}{2\lambda} & m < j \le k \\ 0 & k < j \le n \end{cases} \quad (10)$$

and

$$C_j^{(3)} = \begin{cases} \dfrac{\lambda q}{\psi} & 1 < j \le m \\ \dfrac{\lambda q}{\psi} - I_j & m < j \le k \\ 0 & k < j \le n \end{cases} \quad (11)$$

Consequently, using (2)-(4), we get

$$C_1 = \frac{I_{n+1}^2}{2\lambda} + \sum_{j=max.\{2,m+1\}}^{k} \frac{I_j^2}{2\lambda} + \sum_{j=k+1}^{n} \frac{(2I_j - \frac{\lambda q}{\psi})q}{2\psi} \quad (12)$$

$$C_2 = \frac{b^2}{2\lambda} + \sum_{j=2}^{m} \frac{(\frac{\lambda q}{\psi} - 2I_j)q}{2\psi} + \sum_{j=max.\{2,m+1\}}^{k} \frac{(I_j - \frac{\lambda q}{\psi})^2}{2\lambda} \quad (13)$$

$$C_3 = b + \sum_{j=2}^{m} \frac{\lambda q}{\psi} + \sum_{j=max.\{2,m+1\}}^{k} (\frac{\lambda q}{\psi} - I_j) \quad (14)$$

which completes the cost expression of the model.

For the systems in which back orders are not permitted, the cost function takes a very simple form. In such systems, the maximum number of back orders allowed, b, is zero and as a result $m = 0$ and $k = 1$. Using (12)-(14) we have

$$K(q, n) = \frac{\lambda a}{q} + \frac{\lambda A}{nq} + \frac{cq}{2\psi}(n\delta + \lambda) + \frac{c'\lambda q}{2\psi} \quad (15)$$

Model Optimisation

To obtain the optimal values of the decision variables of the models, the annual cost function K is minimised with respect to the variables under control. To do this, for the no backlog model we first show that for a given value of q, $K(q, n)$ denoted by $K_q(n)$ is a unimodal function of n.

Let n_1, n_2 and n_3 be any triple of positive integers such that $n_1 \le n_2 \le n_3$. The connecting line between $(n_1, k_q(n_1))$ and $(n_3, k_q(n_3))$ is

$$Y_x = K_q(n_1) + \frac{x - n_1}{n_3 - n_1}\left[K_q(n_3) - K_q(n_1)\right] \quad (16)$$

and

$$Y_{n_2} - K_q(n_2) = \frac{\lambda A}{q} \cdot \frac{(n_3 - n_1)(n_3 - n_2)(n_2 - n_1)}{n_1 n_2 n_3} \quad (17)$$

which is non-negative and confirms the unimodality of $K_q(n)$.

Now if the system is such that the number of items per subbatch, q, is pre-specified and hence unchangeable, then the only variable to decide on is n. The optimal value of n, n^* say, in this case is equal to the smallest positive integer n satisfying

$$K_q(n+1) - K_q(n) \ge 0 \quad (18)$$

or

$$n(n+1) \ge \frac{2\lambda\psi A}{c\delta q^2} \quad (19)$$

For the other systems, where q is to be decided on as well, by solving $\frac{\partial K_{(q,n)}}{\partial q} = 0$ we get

$$q = \{\frac{2\lambda\psi(na + A)}{n[c(n\delta + \lambda) + c'\lambda]}\}^{\frac{1}{2}} \quad (20)$$

Then q^* and n^*, the optimal values of q and n, are determined by using (19) and (20) iteratively until two consecutive values of q obtained are equal. Of course, to start this process an initial value for n is needed. It can be any natural number, but a close choice to n^* will decrease the number of iterations needed for convergence. For many cases, whatever initial value for n was chosen the number of iterations did not exceed 5.

Minimisation of $K(q, b, n)$, the annual cost function of the model with back orders, is not so straightforward. According to the relation that holds between n, m and k, some terms of the expressions in (12)-(14) may vanish and the explicit form of K changes. That is, for different values of q, b and n we have different explicit forms of the cost expression. Once these forms are known, then numerical methods can be exploited to obtain the best values for q, b and n in each case.

It is clear from the definition of m and k that $m \le k$. With regard to the assumption of $m \le n$, made earlier, the possible relationships between m, k and n are as follows:

(a) Case of $m \le 1 \le k < n$. These conditions are equivalent to:

$$b < q,$$
$$\psi b < (n-1)\delta q,$$
$$2 \le n.$$

The terms of the cost function in this case become

$$C_1 = \frac{\delta q^2}{2\lambda\psi} n^2 + q(\frac{q}{2\psi} - \frac{b}{\lambda})n + \frac{\delta^2 q^2}{6\lambda\psi^2} k^3$$

$$-\frac{\delta q}{2\lambda\psi}\left(\frac{\delta q}{2\psi}+b\right)k^2+\frac{1}{\lambda}\left(\frac{\delta^2 q^2}{6\psi^2}+\frac{\delta bq}{\psi}+b^2\right)k, \quad (21)$$

$$C_2=\frac{\delta^2 q^2}{6\lambda\psi^2}k^3-\frac{\delta q}{2\lambda\psi}\left(\frac{\delta q}{2\psi}+b\right)k^2+\frac{1}{2\lambda}\left(\frac{\delta^2 q^2}{6\psi^2}+\frac{\delta bq}{\psi}+b^2\right)k \quad (22)$$

and

$$C_3=-\frac{\delta q}{\psi}k^2+\left(\frac{\delta q}{2\psi}+b\right)k \quad (23)$$

(b) Case of $1 < m \leq k < n$. In this case we have

$$C_1=\frac{\delta q^2}{2\lambda\psi}n^2+q\left(\frac{q}{2\psi}-\frac{b}{\lambda}\right)n+\frac{\delta^2 q^2}{6\lambda\psi^2}(k^3-m^3)$$
$$-\frac{\delta q}{2\lambda\psi}\left(\frac{\delta q}{2\psi}+b\right)k^2+\frac{\delta q}{2\lambda\psi}\left(\frac{\psi-3\lambda}{2\psi}q+b\right)m^2$$
$$+\frac{1}{2\lambda}\left(\frac{\delta^2 q^2}{6\psi^2}+\frac{\delta bq}{\psi}+b^2\right)k-\frac{1}{2\lambda}\left(\frac{\psi^2+13\lambda^2-8\lambda\psi}{6\psi^2}q^2 \quad (24)\right.$$
$$\left.+\frac{\psi-3\lambda}{\psi}bq+b^2\right)m+\frac{1}{2\lambda}\left(b-\frac{\lambda q}{\psi}\right)^2,$$

$$C_2=\frac{\delta^2 q^2}{6\lambda\psi^2}(k^3-m^3)-\frac{\delta q}{2\lambda\psi}\left(\frac{\delta q}{2\psi}+b\right)k^2+\frac{\delta q}{2\lambda\psi}\left(\frac{\psi-3\lambda}{2\psi}q+b\right)m^2$$
$$+\frac{1}{2\lambda}\left(\frac{\delta^2 q^2}{6\psi^2}+\frac{\delta bq}{\psi}+b^2\right)k-\frac{1}{2\lambda}\left(\frac{\psi^2+13\lambda^2-8\lambda\psi}{6\psi^2}q^2 \quad (25)\right.$$
$$\left.+\frac{\psi-3\lambda}{\psi}bq+b^2\right)m+\frac{1}{2\lambda}\left(b-\frac{\lambda q}{\psi}\right)^2,$$

and

$$C_3=-\frac{\delta q}{2\psi}k^2+\left(\frac{\delta q}{2\psi}+b\right)k+\frac{\delta q}{2\psi}m^2-\left(\frac{\psi-3\lambda}{2\psi}q+b\right)m$$
$$+\left(b-\frac{\lambda q}{\psi}\right) \quad (26)$$

The constraints on m, k and n are, in this case, equivalent to

$$q \leq b$$
$$\psi b < (n-1)\,\delta q$$
$$3 \leq n$$

(c) Case of $m \leq 1 \leq n \leq k$ or equivalently

$$b < q$$
$$(n-1)\delta q \leq \psi b$$
$$1 \leq n$$

Here the components of the annual cost function are:

$$C_1=\frac{\delta^2 q^2}{6\lambda\psi^2}n^3+\frac{\delta q}{2\lambda\psi}\left(\frac{\lambda+\psi}{2\psi}q-b\right)n^2+\frac{1}{2\lambda}\left(\frac{\delta^2+6\lambda\psi}{6\psi^2}q^2-\frac{\lambda+\psi}{\psi}bq\right.$$
$$\left.+b^2\right)n, \quad (27)$$

$$C_2=\frac{\delta^2 q^2}{6\lambda\psi^2}n^3-\frac{\delta q}{2\lambda\psi}\left(\frac{\delta q}{2\psi}+b\right)n^2+\frac{1}{2\lambda}\left(\frac{\delta^2 q^2}{6\psi^2}+\frac{\delta bq}{\psi}+b^2\right)n, \quad (28)$$

and

$$C_3=\frac{-\delta q}{2\psi}n^2+\left(\frac{\delta q}{2\psi}+b\right)n \quad (29)$$

(d) Case of $1 < m \leq n \leq k$. In this final case we get

$$C_1=\frac{\delta^2 q^2}{6\lambda\psi^2}(n^3-m^3)+\frac{\delta q}{2\lambda\psi}\left(\frac{\lambda+\psi}{2\psi}q-b\right)n^2+\frac{1}{2\lambda}\left(\frac{\delta^2+6\lambda\psi}{6\psi^2}q^2\right.$$
$$\left.-\frac{\lambda+\psi}{\psi}bq+b^2\right)n+\frac{\delta q}{2\lambda\psi}\left(\frac{\psi-3\lambda}{2\psi}q+b\right)m^2 \quad (30)$$
$$-\frac{1}{2\lambda}\left(\frac{\psi^2+13\lambda^2-8\lambda\psi}{6\psi^2}q^2+\frac{\psi-3\lambda}{\psi}bq+b^2\right)m+\frac{1}{2\lambda}\left(b-\frac{\lambda q}{\psi}\right)^2$$

$$C_2=\frac{\delta^2 q^2}{6\lambda\psi^2}(n^3-m^3)-\frac{\delta q}{2\lambda\psi}\left(\frac{q}{2\psi}+b\right)n^2+\frac{1}{2\lambda}\left(\frac{\delta^2 q^2}{6\psi^2}+\frac{\delta bq}{\psi}+b^2\right)n$$
$$+\frac{\delta q}{2\lambda\psi}\left(\frac{\psi-3\lambda}{2\psi}q+b\right)m^2-\frac{1}{2\lambda}\left(\frac{\psi^2+13\lambda^2-8\lambda\psi}{6\psi^2}q^2+\frac{\psi-3\lambda}{\psi}bq\right.$$
$$\left.+b^2\right)m+\frac{1}{2\lambda}\left(b-\frac{\lambda q}{\psi}\right)^2 \quad (31)$$

$$C_3=\frac{-\delta q}{2\psi}n^2+\left(\frac{\delta q}{2\psi}+b\right)n+\frac{\delta q}{2\psi}m^2-\left(\frac{\psi-3\lambda}{2\psi}q+b\right)m+b-\frac{\lambda q}{\psi}$$
$$(32)$$

The constraints on q, b and n in this case are

$$q \leq b$$
$$\psi(b-q) < (n-1)\,\delta q$$
$$(n-1)\,\delta q \leq \psi b$$
$$2 \leq n$$

Now for a given set of a system's specifications (data) we have four constrained minimisation problems; we seek to minimise $k(q, b, n)$ with different terms and constraints for the cases (a)-(d). For each case, when the related objective function is minimised, we get the best value of q, b and n satisfying the relevant set of constraints. The optimal values for the decision variables are those which yield the lowest annual cost value amongst the four cases.

It is worth stating the minimisation numerical method applied to the problem. As was mentioned, minimisation is a problem in the field of non-linear mixed-integer programming. In 1967, an approach based on the interior penalty concept was introduced by Gellatly and Marcal for general non-linear problems [2]. Gisvold and Moe [3] later applied this method to solve some design problems in the field of non-linear mixed-integer programming. Rao [8] has explained the method and suggested the range of constants and initial values used in the method. The method adds two penalty terms to the objective function, one for violating the constraints and one for non-integrality of the integer variables. Then it minimises the new function in which variables are unconstrained.

The data in Table 1 were used to illustrate the numerical results of the model applying the above minimisation method. The results in Table 2, where $k_0 = k(q_0, b_0, n_0)$, show that any one of the four cases, (a)-(d), raised upon the systems parameters, may happen to be optimum. In fact, if shortage costs are relatively high, then either case (a) or (c) becomes optimum. This happens because in these two cases the number of production periods in which all demands are back ordered (m) is none or at most one (data set no. I and II). That is, for high shortage costs the optimal q and b are such that the inventory builds up quickly after each production cycle starts. For low costs of shortage either case (b) or (d), in which the number of periods without any stock on hand is greater than one, becomes optimum, which means more shortages are allowed to occur during a cycle.

Table 1. List of four different sets of data

Set.	Data							
No.	λ	ψ	C	C'	π	$\hat{\pi}$	A	a
I	40	45	6.0	5.2	5.0	4.5	25	7.0
II	125	200	0.3	0.2	0.15	0.27	10	6.0
III	90	110	7.0	7.0	0.5	0.3	60	4.0
IV	16	20	10.0	10.0	1.0	0.5	40	0.5

Now consider a system in which the batch transferring procedure is in action but the optimal values of the decision variables are determined according to the EBQ model's formulae. That is, the batch quantity which is to be produced during a production cycle, Q, and the maximum number of shortages permitted, b, are determined as

$$Q^* = [(\frac{2\lambda\psi A}{c\delta} - \frac{(\lambda\psi)}{c(c+\hat{\pi})})(\frac{c+\hat{\pi}}{A})]^{\frac{1}{2}}$$

and

$$b^* = \frac{\delta(cQ^* - \lambda\pi)}{\psi(c+\hat{\pi})}$$

but the products are transferred to the inventory in subbatch of size q such that $nq = Q^*$. Of course the cost of such a system depends on how n and q are chosen. In fact, although the product of n and q is fixed to Q^*, but the system's cost and hence the loss made due to the deviation from optimality varies a lot. In some systems the variation or the loss may even exceed 400 per cent. Table 3 shows these variations and loss percentages for the systems I and II of Table 1. The %inc. in the table is equal to $100 \times \frac{[k(q,b^*,n) - k_0]}{k_0}$ where k_0 is the optimal annual cost of the system shown in Table 2.

Table 2. Optimal and sub-optimal values for q, b and n and the related annual cost value in different cases and systems

Data Set	Case	q_0	b_0	n_0	k_0
I	a*	11.080	4.924	7	71.124*
	b	8.408	8.408	21	125.295
	c	14.687	4.897	4	102.978
	d	13.758	15.285	11	163.378
II	a	58.709	33.331	5	34.1105
	b	81.481	81.927	5	32.3065
	c*	123.013	92.217	3	29.4041*
	d	85.715	96.399	4	32.7177
III	a	18.530	13.476	6	138.709
	b	27.903	131.093	27	85.268
	c	57.179	57.179	4	92.496
	d*	33.368	115.045	18	81.006*
IV	a	4.377	3.502	6	52.1005
	b*	3.766	22.960	32	29.094*
	c	17.140	17.140	3	38.5671
	d	6.361	20.781	16	29.4166

-* optimal case/optimal cost value

The interesting point is that even if the number of subbatch per cycle, n, is set to its optimal value when Q^* is used, there will still be a considerable loss, e.g. 9.7% in system I and 7.6% in system II.

Random Demand

For this case we assume that the production and transferring processes of the items have the same properties as the deterministic case of demand. But here the number of items demanded during any defined period of time is random. Let the probability that x units will be demanded in a period of length t be $f(x; t)dx$ in the continuous case and $p(x; t)$ in discrete cases and the expected rate of demand be λ. Also assume that demands in different periods are independent and identically distributed and that the re-order level, r, is non-negative.

Table 3. The effect of applying the rules of EBQ model to the group transferring systems

n	System I $Q^* = nq = 74.286$, $b^* = 2.6$			System II $Q^* = nq = 206.332$, $b^* = 28.38$		
	q	$k(q, b^*, n)$	% inc.	q	$k(q, b^*, n)$	% inc.
1	74.28	391.65	450.6	206.33	56.25	31.3
2	37.14	134.63	89.2	103.166	30.62	4.1
3	24.76	105.93	48.9	68.77	31.66	7.6
4	18.57	88.22	23.8	51.58	33.72	14.6
5	14.875	84.73	19.1	41.26	36.58	24.4
6	12.38	83.70	17.6	34.38	40.18	36.6
7	10.61	78.08	9.7	29.47	43.81	49.0
8	9.28	80.91	13.7	25.79	47.21	60.5
9	8.25	83.97	17.9	22.92	50.94	73.2
10	7.42	86.94	22.2	20.63	54.71	86.0
12	6.19	89.87	26.3	17.19	61.99	110.8
14	5.30	92.99	30.7	14.73	69.28	135.6
16	4.64	96.39	35.5	12.89	76.63	160.7
18	4.12	107.27	50.9	11.46	83.99	185.6
20	3.71	111.62	56.9	10.31	91.27	210.4
25	3.09	127.45	79.2	8.25	109.50	272.4

Here again we define the j^{th} production period as the time interval taken to produce the j^{th} subbatch of items. The length of each period is $\frac{q}{\psi}$.

Clearly

$$I_1 = r,$$
$$I_j = r + (j-1)q - X_{\frac{(j-1)q}{\psi}}, \quad j > 1 \qquad (1)$$

where $X_{\frac{(j-1)q}{\psi}}$ is the demand during $j-1$ production periods of total length $\frac{(j-1)q}{\psi}$. Since the lengths of all production periods are equal and demands are i.i.d., the distribution of $X_{\frac{(j-1)q}{\psi}}$ will be $f^{(j-1)}(x;\frac{q}{\psi})$, $(j-1)$-fold convolution of $f(x;\frac{q}{\psi})$. Therefore, if we let $c_j(i)$ be the average cost of holding stock and incurred shortages in the j^{th} ($j > 1$) production period, given that the starting inventory position of the period is $i_j = r + (j-1)q - x$, then the contribution to the average cost per cycle from period j is

$$\int_0^\infty f^{(j-1)}(x;\frac{q}{\psi})\, c_j(i)\, dx \qquad 1 \le j \le n+1 \qquad (34)$$

where $F^{(0)}(0;\frac{q}{\psi}) = 1$ and $f^{(0)}(x;\frac{q}{\psi}) = 0, x > 0$.

First the average unit years of storage is evaluated. At any time point $t (0 < t < \frac{q}{\psi})$ during the j^{th} period inventory on hand is equal to

$$\int_0^\infty (i_j - x) f(x;t)\, dx + \int_{i_j}^\infty (x - i_j) f(x,t)\, dx = i_j - \lambda t$$
$$+ \int_{i_j}^\infty (x - i_j) f(x,t)\, dx \qquad (35)$$

Consequently, the expected unit years of storage incurred in the j^{th} period ($j > 1$) becomes

$$C_j^{(1)} = \int_0^{\frac{q}{\psi}} \{i_j - \lambda t + \int_{i_j}^\infty (x - i_j) f(x;t) dx\} dt \qquad (36)$$

The average unit years of shortage in the j^{th} period is

$$C_j^{(2)} = \int_0^{\frac{q}{\psi}} \int_{i_j}^\infty (x - i_j) f(x;t) dx dt \qquad (37)$$

And finally the expected number of back-logged units at the end of the j^{th} period is

$$C_j^{(3)} = \int_{i_j}^\infty (x - i_j) f(x;\frac{q}{\psi}) dx \qquad (38)$$

Having found the expected number of unit years of storage and the average number of back orders for the j^{th} period, $C_j(i)$ can be written as

$$C_j(i) = c \cdot C_j^{(1)} + \hat{\pi} C_j^{(2)} + \pi \cdot C_j^{(3)}$$
$$= c \cdot (i_j - \frac{\lambda q}{2\psi})\frac{q}{\psi} + (c + \hat{\pi}) \int_0^{\frac{q}{\psi}} \int_{i_j}^\infty (x - i_j) f(x;t) dx dt \qquad (39)$$
$$+ \pi \int_{i_j}^\infty (x - i_j) f(x;\frac{q}{\psi}) dx, \quad 1 < j \le n$$

For the first production period, since we have assumed that $\lambda \le \psi$, the starting inventory position on average is r. Thus, if we let $i_1 = r$ then $C_1(i)$ will be the contribution cost from the first period.

For the $n + 1^{st}$ period, things are different. Since it has been assumed that r, the re-order level, is non-negative, there will not be any shortage cost in this period. There are holding costs only if $I_{n+1} > r$ and in this case the average number of unit years of storage is $\frac{(I_{n+2}^2 - r^2)}{2\lambda}$. Hence, the contribution to the expected cycle cost of carrying stocks and shortages from the last period in a cycle is

$$C_{n+1}=\frac{c}{2\lambda}\int_0^{nq}(i_{n+1}^2-r^2)f^{(n)}(x;\frac{q}{\psi})dx \qquad (40)$$

Now in order to express the average annual cost of the system, it remains to determine the expected cycle length. The total length of the first n production periods is exactly $\frac{nq}{\psi}$. The length of the $n+1^{st}$ period is a random variable dependent on I_{n+1} and the demand in that period. Since the demand is uniform over time, it can be evaluated as

$$\frac{E[I_{n+1}-r]}{\lambda}=\frac{nq}{\lambda}-\frac{1}{\lambda}E[X_{\frac{nq}{\psi}}]=\frac{nq}{\lambda}-\frac{nq}{\psi} \qquad (41)$$

Thus, the expected cycle length becomes $\frac{nq}{\lambda}$. Therefore, the average annual cost is

$$K(q,n,r)=\frac{\lambda a}{q}+\frac{\lambda A}{nq}+\frac{\lambda}{nq}[\sum_{j=1}^{n}\int_0^{\infty}f^{(j-1)}(x;\frac{q}{\psi})C_j(i)dx$$

$$+C_{n+1}]+\frac{c'\lambda q}{2\psi} \qquad (42)$$

For discrete cases of demand with the same argument we have

$$K(q,n,r)=\frac{\lambda a}{q}+\frac{\lambda A}{nq}+\frac{\lambda}{nq}[\sum_{j=1}^{n}\sum_{x=0}^{\infty}p^{(j-1)}(x,\frac{q}{\psi})C_j(i)$$

$$+C_{n+1}]+\frac{c'\lambda q}{2\psi} \qquad (43)$$

Here the cost function is differentiable, in both cases, with respect to q and r, but it is so complicated that it is impossible to evaluate q or r from $\frac{\partial K}{\partial q}=0$ or $\frac{\partial K}{\partial r}=0$. Therefore, to find the optimal values of the decision variables, a numerical method such as the one used for the deterministic cost has to be exploited.

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