THE ANALOGUE OF WEIGHTED GROUP ALGEBRA FOR SEMITOPOLOGICAL SEMIGROUPS

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Abstract

In [1,2,3], A. C. Baker and J.W. Baker studied the subspace $M_a(S)$ of the convolution measure algebra $M_b(S)$ of a locally compact semigroup. H. Dzinotyiweyi in [5,7] considers an analogous measure space on a large class of C-distinguished topological semigroups containing all completely regular topological semigroups. In this paper, we extend the definitions to study the weighted semigroup algebra $M_a(S, \omega)$, where ω is a weight function on a C-distinguished semitopological semigroup S. We will show that this subspace is a convolution measure algebra. As a corollary, this answers in the affirmative a question raised by J.W. Baker and H. Dzinotyiweyi in [6].

Definitions and Preliminary Results

Throughout the paper, S [resp. X] will denote a Hausdorff semitopological semigroup [resp. topological space]. Let k_x denote the strongest topology on X which agrees with the original topology on the compact subset of X. The topological space X is said to be a k-space if $k_x = T_x$. By T_x' we mean the weakest topology on X such that whenever a bounded real valued function which is continuous with respect to the topology k_x, then it is continuous with respect to T_x. All notations and terminology in the subject of measure theory are as in [4] and [7]. We denote by $K(X)$ [resp. $B(X)$] the set of all compact [resp. Borel] subsets of X. Also by $C_b(X)$ [resp. $C_b(X, k_x)$], we mean the set of all real-valued bounded continuous functions on (X, T_x) [resp. (X, k_x)]. We note $C_b(X) \subseteq C_b(X, k_x)$ and denote $\|f\|_\infty := \sup\{|f(x)|: x \in X\}$, for $f \in C_b(X)$. If $C_b(X)$ separate points of X, we say X is C-distinguished. Clearly, the family of C-distinguished spaces contain all completely regular spaces.

Let $M_b(X)$ [resp. $M_b(X, k_x)$] be the set of all bounded Radon measures on (X, T_x) [resp. (X, k_x)]. If $\mu = \mu^+ - \mu^-$ be the Hahn decomposition of $\mu \in M_b(X)$, then we write $\widetilde{\mu} = (\mu^+)_t - (\mu^-)_t \in M_b(X, k_x)$, where $(\mu^+)_t$ is the unique extension Radon measure of μ^+ on (X, k_x) which agrees on compacta, (see [4, p. 18]). We recall that $K(X, T_x) = K(X, k_x)$ and $B(X, T_x) \subseteq B(X, k_x)$, so $v_t(E) := \sup\{v(K): K \text{ is a compact subset of } E\}$, for $v \in M_b(X)$ and $E \in B(X, k_x)$.

In the following we give an alternate proof to a Glicksberg's result for general case, (see [9], [11]), noting that Glicksberg's proof can be modified, by using this method, to get this extended version.

For easy reference, we mention the following consequence of [4, p. 20-21].

(1.1) Lemma. Let X be a Hausdorff space and $f: X \to [0, +\infty]$ be an arbitrary function, Then,

(i) Let $A_{i,n} = \{x \in X: f(x) > \frac{i}{2^n}\}$ and $f_n = \sum_{i=1}^{\infty} \frac{1}{2^n} \chi_{A_{i,n}}$. Then

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$0 \leq f_n \leq f$ and f_n increases to f. In particular if f is lower semicontinuous function, then there exists a sequence $s_{n,k} = \sum_{i=1}^{k} \frac{1}{2^n} \chi_{A_{i,n}}$ of simple lower semicontinuous functions such that $\lim_n \lim_k s_{n,k} = f$.

(ii) If a net (f_α) of lower semicontinuous functions $X \to [0, +\infty]$ is increasing with $\sup_\alpha (f_\alpha) = f$ and $\mu \in M_b^+(X)$. Then,

$$\int_X f d\mu = \sup_\alpha \{ \int_X f_\alpha d\mu \} = \lim_\alpha \int_X f_\alpha d\mu.$$

(iii) Let $f: X \to [0, +\infty]$ be a Borel -measurable function and $K(X)$ be directed by inclusion. If $\mu \in M_b^+(X)$, then

$$\int_X f d\mu = \sup \{ \int_C f d\mu : C \in K(X) \}.$$

(1.2) Theorem (Glicksberg's Extended Version). Let (X, T_x) and (Y, T_Y) be Hausdorff topological spaces and F: $X \times Y \to \mathbb{R}$ be a bounded separately continuous function. If $\mu \in M_b(X)$ and $\nu \in M_b(Y)$, then

(i) The map $x \to \int_Y F(x,y)\, d\nu(y)$ [resp. $y \to \int_X F(x,y), d\mu(x)$] is k_x [resp. k_Y] continuous.

(ii) $\int_X \int_Y F(x,y) d\nu(y) d\mu(x) = \int_Y \int_X F(x,y) d\mu(x) d\nu(y)$.

Proof. Without loss of generality, we can assume that F, μ and ν are positive.

(i) Let $\bar{x}$ denote the point mass at $x \in X$. Then by (1.1),

$$\int_Y F(x,y)\, d\nu(y) = \nu({}_xF), \text{ where } {}_xF(y) := F(x,y)$$
$$= \sup \{ \int_D {}_xF(y) d\nu(y) : D \in K(Y) \}.$$

But the map $x \to \int_D {}_xF(y)\, d\nu(y)$ is continuous on each compact subset C of X, by the Glicksberg theorem, (see [9]). Hence the map $x \to \int_D F(x, y)\, d\nu(y)$ is k_x-continuous on X, for each $D \in K(Y)$. Since the family of functions

$\{x \to \int_D F(x,y)\, d\nu(y) : D \in K(Y)\}$ is directed upward to $x \to \int_Y F(x,y) d\nu(y)$, so the map $x \to \int_Y F(x,y)\, d\nu(y)$ is K_x-lower semicontinuous, by (1.1) (ii).

Similarly, the map $x \to \int_Y (\|F\|_\infty - F)(x,y)\, d\nu(y) = \|F\|_\infty \nu(Y) - \int_Y F(x,y)\, d\nu(y)$ is k_x - lower semicontinuous. Therefore, the map $x \to \int_Y F(x,y)\, d\nu(y)$ is K_x-continuous.

By the same argument, the map $y \to \int_X F(x,y)\, d\mu(x)$ is k_Y-continuous.

(ii) Since the family of K_x-continuous functions $\{x \to \int_D F(x,y) d\nu(y) : D \in K(Y)\}$ is directed upward to k_x-continuous map $x \to \int_Y F(x,y)\, d\nu(y)$, by (i), so $\{\int_C (\int_D F(x, y)\, d\nu(y))\, d\tilde{\mu}(x) : C \in K(X), D \in K(Y)\}$ is directed upward to the integral $\int_X (\int_Y F(x,y)\, d\nu(y))\, d\tilde{\mu}(x)$, by (1.1) (iii). But the measures $\tilde{\mu}$ and μ are concentrated on a σ-compact set and $\tilde{\mu}$ agree with μ on compacta. Hence

$$\int_X \int_Y F(x,y) d\nu(y) d\mu(x) =$$
$$\sup \{ \int_C \int_D F(x,y)\, d\nu(y)\, d\mu(x) : C \in K(X), D \in K(Y) \} =$$
$$\sup \{ \int_D \int_C F(x,y)\, d\mu(x)\, d\nu(y) : C \in K(X), D \in K(Y) \}$$
$$= \int_Y \int_X F(x,y)\, d\mu(x) d\nu(y).$$

(1.3) Corollary [11]. Let X, Y be Hausdorff completely regular topological spaces and $F: (X, T_x) \times (Y, T_Y) \to \mathbb{R}$ be a bounded separately continuous function. If $\mu \in M_b(X, T_x)$, $\nu \in M_b(Y, T_Y)$, then

(i) The map $x \to \int_Y F(x,y)\, d\nu(y)$ [resp. $y \to \int_X F(x,y)\, d\mu(x)$] is T_x [resp. T_Y] continuous.

(ii) $\int_X \int_Y F(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X F(x,y)\, d\mu(x) d\nu(y)$.

Weighted Convolution Measure Algebras $M_b(S, \omega)$

In [9], I. Glicksberg showed that $M_b(S)$ with the usual convolution is a Banach algebra, when S is compact. Later, C.J. Wong [18] studied the space $M_b(S)$, where S is a locally compact semitopological semigroup. Also, H. Kharaghani [12] considered $M_b(S)$ on Čech-complete spaces included in locally compact and complete metric semitopological semigroups S. It is to be noted that H. Dzinotyiweyi [5] showed that $M_b(S)$ is a convolution measure algebra on a large class of C-distinguished spaces containing all completely regular topological semigroups S. Finally, A. Janssen [11] proved $M_b(S)$ need not be a Banach algebra with usual convolution under the assumption that S is a completely regular semitopological semigroup. In this section, we will introduce a convolution "*" for which $(M_b(S), *)$ be a (non associative) Banach algebra.

Let $\omega : S \to (0, +\infty)$ be a Borel measurable weight function, that is $\omega(st) \leq \omega(s)\, \omega(t)$, where $s, t \in S$ for which $1/\omega$ is bounded on compacta. Various authors have considered the space of weighted measure algebra $M(\omega)$ consisting of all complex measures μ such that $|\mu| \omega \in M_b(S)$, (see for example [8], [14]). The space $M(\omega)$ need not be complete and the norm-algebra $M(\omega)$ is different from $l_1(S,\omega) := \{f : S \to \mathbb{R} \mid \sum_{s\in S} |f(s)|\,\omega(s) < \infty\}$, where S has discrete topology. For these reasons we have chosen a different definition for the weighted convolution measure algebra $M_b(S,\omega)$.

Let $C_b(S,\omega) = \{f : S \to \mathbb{R} \mid \frac{f}{\omega} \in C_b(S)\}$. Then $C_b(S,\omega)$ with the usual addition and the following multiplication,

$$f.g(x) = \frac{f(x)\,g(x)}{\omega(x)}, \text{ for } x \in S \text{ and } f, g \in C_b(S,\omega)$$

with the norm, $\|f\|_\omega := \sup\{|\frac{f}{\omega}(x)| : x \in S\}$, is a Banach algebra. Also the map $f \to \frac{f}{\omega}$ from $(C_b(S,\omega),.)$ onto $C_b(S)$ with pointwise multiplication is an isometric isomorphism. In [5], H. A. M. Dzinotyiweyi showed that $M_b(S) = C_\beta(S)^*$ as Banach algebra, where $C_\beta(S)$ is $C_b(S)$ with the strict-topology. In the following, we define $M_b(S,\omega)$ such that the identity $M_b(S,\omega) = C_\beta(S,\omega)^*$ holds.

Let $M_b^+(S,\omega)$ be the set of all Radon measures μ on S, that is inner-regular and finite on compacta, such that $\mu\omega \in M_b^+(S)$ where

$$\mu\omega(E) = \int_E \omega d\mu, \text{ for } E \in B(S).$$

If $\varphi : M_b^+(S,\omega) \times M_b^+(S,\omega) \to C_b(S,\omega)^*$ be defined by $(\mu,\nu) \mapsto I_\mu - I_\nu$, where

$$I_\mu - I_\nu(f) = \int_S f d\mu - \int_S f d\nu, \quad \text{for } f \in C_b(S,\omega).$$

In general, φ need not be injection. Following [15], let "$\simeq$" be an equivalence relation on $M_b^+(S,\omega) \times M_b^+(S,\omega)$ defined by,

$(\mu,\nu) \simeq (\mu',\nu')$ if and only if $\mu + \nu' = \mu' + \nu$

and $[\mu,\nu]$ be the equivalence class of (μ,ν), then we define,

$$M_b(S,\omega) = \{[\mu,\nu] : \mu,\nu \in M_b^+(S,\omega)\}.$$

Let also $C_\beta(S,\omega)$ denote $C_b(S,\omega)$ with the ω-strict topology, in the obvious way. One can show that $M_b(S,\omega)$ with the norm $\|[\mu,\nu]\|_\omega := \|\mu\omega - \nu\omega\|$ and regard $M_b(S,\omega)$ as a norm space over $\mathbb{R}$ is a Banach space isometric isomorphism to $C_\beta(S,\omega)^*$.

Let us turn our attention to make $M_b(S,\omega)$ into a convolution measure algebra. Since $(C_b(S,\omega),.)$ is a Banach algebra, thus one can define a multiplication on $C_\beta(S,\omega)^*$ and so $M_b(S,\omega)$ such that it be a Banach algebra. In the following, we define a convolution "*" on $M_b(S,\omega)$, where S is C-distinguished semitopological semigroup, such that $\mu * \nu(K) = \int_S\int_S \chi_K(xy)d\mu(x)\,d\nu(y)$, for each compact set $K \subseteq S$ and $\mu,\nu \in M_b^+(S,\omega)$.

Since $\frac{1}{\omega}$ is bounded on compacta and $\mu = (\mu\,\omega)\frac{1}{\omega}$, so each measure $\mu \in M_b^+(S,\omega)$ is σ-finite. Let μ, $\nu \in M_b^+(S,\omega)$ and,

$$\lambda(C) := \int_S\int_S \chi_C(xy)d\mu(x)\,d\nu(y), \text{ for } C \in K(S).$$

Then the family of k-continuous maps $\{y \to \int_S f(xy)d\mu(x) : f \in C_b(S), f \geq \chi_C\}$ is directed downward to the map $y \to \int_S \chi_C(xy)d\mu(x)$, by [7, p. 174]. Hence, the map $y \to \int_S \chi_C(xy)d\mu(x)$ is k-upper semicontinuous function on S. Thus the family $\{\int_S\int_S f(xy)d\mu(x)\,d\nu(y) : f \in C_b(S) \text{ and } f \geq \chi_C\}$ is directed downward to $\lambda(C)$, see (1.1). In other words, $\lambda(C) = \inf\{I(f) : f \in C_b(S) \text{ and } f \geq \chi_C\}$, where $I(f) = \int_S\int_S f(xy)\,d\mu(x)\,d\nu(y)$, for $f \in C_b(S)$. But I is a positive linear functional on $C_b(S)$. Thus by the same argument as is used in [4, p. 36], one can show that λ is a Radon-content, that is

$\lambda(C_2) - \lambda(C_1) = \sup\{\lambda(C) : C$ is a compact subset of $C_2 \backslash C_1\}$, where C_1 and C_2 are compact subsets of S such that $C_1 \subseteq C_2$. It is to be noted that,

$$\lambda(C) \leq \int_S\int_S \frac{1}{\omega}\chi_C(xy)d\mu\omega(x)\,d\nu\omega(y) \leq \|\frac{1}{\omega}\|_C \cdot \|\mu\|_\omega \|\nu\|_\omega$$

is finite, for each $C \in K(S)$.

Therefore, the Radon-part λ_t of λ is a Radon measure, by [4, p. 18]. We define,

$\mu * \nu(E) := \lambda_t(E) = \sup\{\lambda(C) : C$ is a compact subset of $E\}$ and $\mu.\nu(E) := \int_S\int_S \chi_E(xy)d\mu(x)\,d\nu(y)$, for $E \in B(S)$.

(2.1) Definition. Let $[\mu,\nu]$, $\{\mu',\nu'\}$ $M_b(S,\omega)$ and $\lambda \in \mathbb{R}$. Then

(i) $[\mu,\nu] + [\mu',\nu'] = [\mu + \mu', \nu + \nu']$

(ii) $[\mu,\nu] * [\mu',\nu'] = [\mu * \mu' + \nu * \nu', \mu * \nu' + \mu' * \nu]$

(iii) $\lambda.[\mu,\nu] = \begin{cases} [\lambda\mu, \lambda\nu] & \text{if } \lambda \geq 0 \\ [-\lambda\nu, -\lambda\mu] & \text{otherwise.} \end{cases}$

It is easy to show that $M_b(S,\omega)$ is a vector space.

Let $\{G_\alpha\}$ be a family of open sets directed upward to G. Then the family of k-lower semicontinuous maps $\{y \to \int_s \chi_{G_\alpha}(xy)d\mu(x)\}$ is directed upward to the k-lower semicontinuous map $x \to \int_s \chi_G(xy)d\mu(x)$, by (1.1). Hence, by (1.1) (ii), the family $\{\mu.\nu(G_\alpha)\}$ is directed upward to $\mu.\nu(G)$, that is $\mu.\nu$ is a τ-smooth measure. In general $\mu.\nu$ is not Radon measure, by [11], so $\mu.\nu \neq \mu * \nu$. But $\mu*\nu$ is the maximal Radon measure on S coincide $\mu.\nu$ on compacta. If S is Čech complete space, that is S is G_δ-set in the Stone Čech compactification of S, then every τ-smooth measure is Radon-measure, see [13]. In particular, if S is either a locally compact or complete metric space, then the inner-convolution "*" is equal to the usual-convolution ".". In the following we give an alternative proof for the equality of "*" and ".", in this case, without using the Stone Cech compactification.

(2.2) Theorem. Let S be either a locally compact or complete metric semitopological semigroup. Then $(M_b(S,\omega),.)$ is a convolution measure algebra.

Proof. (i) Suppose S is a locally compact space. Then for each $x \in S$ there exists a relatively compact neighborhood V_x, say. Let $\mathcal{G}$ be the family of all finite union of these V_x, where $x \in S$. If $G = \bigcap_{k=1}^{n} V_{x_k} \in \mathcal{G}$, then $\overline{G} = \bigcup_{k=1}^{n} \overline{V_{x_k}}$ is compact. Let $\mu, \nu \in M_b^+(S)$. Then,

$$\begin{aligned}\mu.\nu(S) &= \sup\{\mu.\nu(G): G \in \mathcal{G}\}\\ &\leq \sup\{\mu.\nu(\overline{G}): G \in \mathcal{G}\}\\ &\leq \sup\{\mu.\nu(C): C \in K(S)\} \leq \mu.\nu(S).\end{aligned}$$

Thus $\mu.\nu(S) = \mu*\nu(S)$. Let $\mathcal{E} = \{E \in B(S): \mu.\nu(E) = \mu*\nu(E)\}$. One can show that ε is a σ-algebra containing closed sets, so $\mu*\nu = \mu.\nu$. Since each measure in $M_b^+(S,\omega)$ is σ-finite, so it is easy to show that $\mu*\nu = \mu.\nu$, for all $\mu, \nu \in M_b^+(S,\omega)$.

(ii) Suppose S is a complete metric space. Then for each $n \in \mathbb{N}$, define $\mathcal{G}_n$ be the family of all finite unions of open balls $B(x, \frac{1}{n})$, where $x \in S$. Then it is clear, $\mathcal{G}_n \nearrow S$, so $\mu.\nu(S) = \sup\{\mu.\nu(G): G \in \mathcal{G}_n\}$. Hence for each $\varepsilon > 0$ and $n \in \mathbb{N}$, there exist $G_n \in \mathcal{G}_n$ such that $\mu.\nu(S \setminus G_n) < \varepsilon/2^n$.

Put $G_0 = \bigcap_{n=1}^{\infty} G_n$. Then,

$$\mu.\nu(S\setminus\overline{G}_0) \leq \mu.\nu(S\setminus G_0) \leq \sum_{n=1}^{\infty} \mu.\nu(S\setminus G_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Also G_0 is totally-bounded, so $\overline{G_0}$ is compact. Hence,

$$\mu.\nu(S) = \sup\{\mu.\nu(C): C \in K(S)\} = \mu*\nu(S).$$

Therefore "*" coincides with ".". The rest of the proof is routine.

We now state the main theorem of this section.

(2.3) Theorem. Let S be a C-distinguished semitopological semigroup such that either
(i) K^2 is compact, for each compact K in S, or
(ii) $x^{-1}K$ and Kx^{-1} are compact, for each compact $K \subseteq S$.
Then $(M_b(S,\omega),*)$ is a convolution measure algebra.

Proof. (i) By a similar argument as is used in [7, p. 6-7] is immediate.
(ii) Let $\mu, \nu, \eta \in M_b^+(S)$ and $K \subseteq S$ be compact. Then,

$$\begin{aligned}(\mu*\nu)*\eta(K) &= \int_s\int_s \chi_k(az)d\mu*\nu(a)\,d\eta(z)\\ &= \int_s \mu.\nu(Kz^{-1})\,d\eta(z)\\ &= \int_s\int_s\int_s \chi_k(xyz)d\mu(x)d\nu(y)d\eta(z)\\ &= \int_s\int_s \chi_k(xb)d\mu(x)d\nu.\eta(b)\\ &= \int_s \nu*\eta(x^{-1}K)\,d\mu(x)\\ &= \mu*(\nu*\eta)(K).\end{aligned}$$

By inner-regularity, $(\mu*\nu)*\eta = \mu*(\nu*\eta)$. Thus "*" is associative.

Let $k = \sum_{i=1}^{n} a_i\chi_{K_i}$, where $a_i \in \mathbb{R}^+$, $n \in \mathbb{N}$ and K_i be compact subset of S, for $1 \leq i \leq n$. Let also $\mu, \nu \in M_b^+(S,\omega)$. Then,

$(\mu*\nu)(\omega\chi_c) = \sup\{\int_s k\,d\mu*\nu : k \leq \omega\chi_c\}$, see [4, p. 36-37]

$$\begin{aligned}&= \sup\{\mu.\nu(k): k \leq \omega\chi_c\}\\ &\leq \mu.\nu(\omega\chi_c) = \int_s\int_s \omega\chi_c(xy)\,d\mu(x)d\nu(y)\\ &\leq \|\mu\|_\omega\|\nu\|_\omega, \text{ for each compact } C \text{ in } S.\end{aligned}$$

Therefore $\|\mu*\nu\|_\omega \leq \|\mu\|_\omega\|\nu\|_\omega$. The rest of the proof is easy.

(2.4) Corollary. Let S be a C-distinguished topological semigroup, or semitopological group. Then $(M_b(S,\omega),*)$ is a Banach algebra.

The following example shows that the measure algebra $M(\omega)$ is not complete.

(2.5) Example. Let $S = (\mathbb{N}, +)$ with the discrete topology and $\omega(x) = e^{-x}$, for $x \in S$. Then $M(\omega) = l_1(\mathbb{N}) \cap l_1(\mathbb{N}, \omega)$ is not complete.

Proof. Let $\lambda_n = e^n / n^2$, for $n \in \mathbb{N}$, and $f_k(n) = \begin{cases} \lambda_n & \text{if } n \le k \\ 0 & \text{otherwise.} \end{cases}$ Then (f_k) is Cauchy in $(M(\omega), \|.\|\omega)$, which is not convergent [for, if $f_k \to f$, then $f = (\lambda_n)$ and

$\sum_{n=1}^{\infty} |f(n)| = \infty$, so $f \notin l_1(\mathbb{N})$].

The following example shows that "*" need not be associative.

(2.6) Example. Let S, μ', ν', be as in [11, p. 77] and $\eta' = \bar{e}$, where $e := (e_\alpha)$ and $e_\alpha : [0,1] \to (0,1]$ be defined by $e_\alpha(x) = 1$, for $x \in [0,1]$. Then,

(i) Let $C = \{e\}$. Then $(\mu' * \nu') * \eta'(C) = \mu'' * \nu'(S) \neq \mu'.\nu'(S) = \mu' * (\nu' * \eta')(C)$.

(ii) Let $I(f) = \int_S \int_S f(xy)\, d\mu'(x)\, d\nu'(y)$, for $f \in C_b(S)$. Then I is not strictly continuous, (c.f. [7], p. 6-7).

Weighted Convolution Measure Algebra $M_a(S,\omega)$

A.C. Baker and J.W. Baker in [1,2,3] introduced and studied the convolution measure algebra $M_a(S)$, absolutely continuous measures, analogous to the group algebra $L^1(G)$, for a locally compact topological semigroup S. Later, several authors studied this algebra, for example [5] and [16]. In particular, in [6] Dzinotyiweyi asked whether $M_a(S)$ can be made into a convolution measure algebra, whenever S is a semitopological semigroup.

In this section, we give an affirmative answer to this question and show that this space has the advantage that if $\mu \in M_b(S)$ and $\nu \in M_a^l(S)$, then $\mu * \nu = \mu.\nu$. For a suitable definition of $M_a^l(S, \omega)$ analogous to $M_a^l(S)$, see [7]. Also $M_a^l(S, \omega)$ is a solid and left ideal of $M_b(S,\omega)$.

Let $\eta = [\mu, \nu] \in M_b(S, \omega)$. Then $\eta\,\omega := \mu\omega - \nu\omega \in M_b(S)$, so by the Hahn decomposition theorem, there exist unique ξ^+, ξ^- in $M_b(S)$ such that $\eta\,\omega = \xi^+ - \xi^-$ and $\xi^+ \perp \xi^-$. Put $\eta^+ = (\xi^+)\frac{1}{\omega}$ and $\eta^- = (\xi^-)\frac{1}{\omega}$. Then $\eta = [\eta^+, \eta^-]$ such that $\eta^+ \perp \eta^-$. Let $|\eta| := \eta^+ + \eta^-$ and $A \subseteq M_b(S,\omega)$. If A satisfies the following conditions, then A is called (weighted) convolution measure algebra.

(i) A is a norm-closed subalgebra of $M_b(S,\omega)$.

(ii) A is solid, that is if $\eta \in M_b(S,\omega)$ and $\xi \in$ A such that $|\eta| \ll |\xi|$ implies $\eta \in$ A.

We define, $M_a^l(S, \omega) = \{\eta \in M_b(S,\omega) : |\eta|\omega \in M_a^l(S)\}$, where $M_a^l(S) = \{\mu \in M_b(S)$: The map $x \to |\mu|(x^{-1}C)$ is continuous for each $C \in K(S)\}$. Similarly, one can define $M_a^r(S,\omega)$ and $M_a(S, \omega) := M_a^l(S,\omega) \cap M_a^r(S,\omega)$.

Throughout this section, S is a C-distinguished semitopological semigroup endowed with the k_s-topology (or k-topology).

(3.1) Lemma. Let $h : S \to [0, +\infty]$ be a Borel-measurable function and $\mu \in M_b^+(S)$, $\nu \in M_a^l(S)^+$. Then,

(i) the map $x \to \int_S h(xy)\, d\nu(y)$ is k-lower semicontinuous (k-L.S.C.).

(ii) $\mu * \nu(h) = \mu.\nu(h) = \int_S \int_S h(xy)\, d\mu(x)\, d\nu(y) = \int_S \int_S h(xy)\, d\nu(y)\, d\mu(x)$.

Proof. (i) Let $E \subseteq S$ be a Borel set, $x \in S$. Then by (1.1) (iii),

$$\bar{x} * \nu(E) = \sup\{\nu(x^{-1}K) : K \text{ is a compact subset of } E\}.$$

But the map $x \to \nu(x^{-1}K)$ is k- continuous, so the map $x \to \nu(x^{-1}E)$ is k-L.S.C. Similarly, let $E^c = S \setminus E$. Then the map $x \to \nu(x^{-1}E^c) = \nu(S) - \nu(x^{-1}E)$ is k-L.S.C. Hence the map $x \to \nu(x^{-1}E)$ is k-continuous.

Let $(s_{n,m})$ be a sequence of positive, Borel measurable simple functions increasing to h, pointwise. (see (1.1)). Then $\int_S s_{n,m}(xy)\, d\nu(y)$ increasingly converge to $\int_S h(xy)\, d\nu(y)$. But the map $x \to \int_S s_{n,m}(xy)\, d\nu(y)$ is k-L.S.C. Hence (i) follows.

(ii) Let $E \subseteq S$ be a Borel set and $K(S)$ be directed by inclusion. Then the family of k-continuous functions $\{x \to \nu(x^{-1}C) : C \in K(S)\}$ is directed upward to the map $x \to \nu(x^{-1}E)$, by (1.1) (iii). Hence by (1.1) (ii),

$$\mu * \nu(K) = \int_S \nu(x^{-1}K)\, d\mu(x) \nearrow \int_S \int_S \chi_E(xy)\, d\nu(y)\, d\mu(x).$$

Therefore, $\mu * \nu(E) = \sup\{\mu * \nu(K) : K$ is a compact subset of $E\} = \mu.\nu(E)$.

By a standard argument and applying (1.1) (ii) is immediate.

We now state the main theorem of this paper. As a corollary this answers the open question raised in [6].

(3.2) Theorem. $M_a^l(S,\omega)$ is a Banach algebra, left ideal and solid in $M_b(S,\omega)$.

Proof. (i) First of all we show that $M_a^l(S, \omega)$ is solid. Let $\nu \in M_a^l(S,\omega)$ and $\mu \in M_b(S, \omega)$ such that $|\mu| \ll |\nu|$. Then it is easy to show that $|\mu|\omega \ll |\nu|\omega$. Since the map,

$$y \rightarrow \int_s \chi_K(xy)\, d|\mu|\omega(x) = |\mu|\omega * \bar{y}\ (K)$$
$$= \inf\{\int_s f(xy) d|\mu|\omega(x): f \in C_b(S) \text{ and } f \geq \chi_K\}$$

is k-upper semicontinuous, by (1.2) and [7, p.174]. Thus by a similar argument as is used in [7, p. 10], one can show that $|\mu|\omega \in M_a^l(S)$. Thus $\mu \in M_a^l(S,\omega)$.

(ii) Now we show that $M_a^l(S,\omega)$ is a left ideal of $M_b(S,\omega)$. Let $\mu \in M_b^+(S,\omega)$, $\nu \in M_a^l(S,\omega)^+$ and $K \subseteq S$ be compact. Then,

$$\mu\omega * \nu\omega(x^{-1}K) = \int_s \nu\omega((xa)^{-1}K)\, d\mu\omega(a), \text{ by (3.1)},$$

and the map $(x,a) \rightarrow \nu\omega((xa)^{-1}K)$ is k-separately continuous bounded function.

Thus by (1.2), the map $x \rightarrow \mu\omega * \nu\omega(x^{-1}K)$ is k-continuous. But $(\mu * \nu)\omega = (\mu.\nu)\omega \leq (\mu\omega).(\nu\omega) = (\mu\omega) * (\nu\omega)$ and $M_a^l(S)$ is solid, by (i), so $(\mu * \nu)\omega \in M_a^l(S)$.

That is, $\mu * \nu \in M_a^l(S,\omega)$. In general, let $\xi \in M_b(S,\omega)$ and $\eta \in M_a^l(S,\omega)$. Then $|\xi * \eta| \leq |\xi| * |\eta| \in M_a^l(S,\omega)^+$, so $\xi * \eta \in M_a^l(S,\omega)$, by (i).

(iii) $M_a^l(S,\omega)$ is a closed subalgebra of $M_b(S,\omega)$. For, let $\xi, \eta \in M_a^l(S,\omega)$ and $\lambda \in \mathbb{R}$. Then, $|\xi + \lambda\eta| \ll |\xi| + |\lambda||\eta| \in M_a^l(S,\omega)$. Thus $\xi + \lambda\eta \in M_a^l(S,\omega)$, by (i). Let $\eta_n = [\mu_n, \nu_n] \rightarrow \eta = [\mu,\nu]$ in $M_a^l(S,\omega)$, $f_n(x) := (\mu_n\omega - \nu_n\omega)(x^{-1}K)$ and $f(x) := (\mu\omega - \nu\omega)(x^{-1}K)$, for $x \in S$ and compact set K. Then $\|f_n - f\|_\infty \leq \|\eta_n - \eta\|_\omega$, so f is k-continuous. That is, the map $x \rightarrow |\eta|\omega(x^{-1}K)$ is k-continuous. Thus $\eta \in M_a^l(S,\omega)$ and the proof is complete.

(3.3) Corollary. $M_a(S,\omega)$ is a convolution measure algebra.

(3.4) Corollary. Let S be a C-distinguished k-space. Then $M_a^l(S,\omega)$ is a Banach algebra, left ideal and solid in $M_b(S,\omega)$.

Remark. The k-topology is coincided with the original topology for k-spaces. Thus $M_a^l(S,\omega)$ is a Banach algebra, when S is endowed with the original topology. In particular, every locally compact or complete metric space is k-space, (see [17]). As a consequence, we answer the question raised in [6].

(3.5) Corollary. Let S be either a locally compact or complete metric semitopological semigroup. Then $M_a^l(S,\omega)$ is a Banach algebra, left ideal and solid in $M_b(S,\omega)$.

In the following, we consider $M_a^l(S,\omega)$, when S is a subsemigroup of a group. Let m be the left Haar-measure on a locally compact group G, and S be a Borel subset of G. We denote

$L^1(S,\omega) = \{h: S \rightarrow \mathbb{R} | h$ is Borel-measurable and $\|h\|_\omega = \int_s |h|\,\omega dm$ is finite $\}$. If $f, g \in L^1(S,\omega)$, then $L^1(S,\omega)$, with following product, is a Banach algebra.

$$f * g(y) = \int_s f(x) g(x^{-1}y)\, dm(x), \text{ for } y \in S.$$

We will show that for each $\mu \in M_a^l(S,\omega)^+$ there exist a unique $f \in L^1(S,\omega)$ such that, $\mu(E) = \int_s f dm$, for $E \in B(S)$.

(3.6) Theorem. Let S be a subsemigroup of a locally compact group with positive Haar- measure. Then $M_a^l(S,\omega)$ is $L^1(S,\omega)$ as a Banach algebra.

Proof. (i) First we show that,

$$M_a^l(S) = \{\mu|s : \mu \in M_b(G) \text{ and } \mu \ll m\}.$$

Let $\mu \in M_b(G)$ such that $\mu \ll m$ and $\nu = \mu|s$. Since $m \in M_a^1(S)$ and $\mu\chi_s \ll m$, so $\mu\chi_s \in M_a^l(G)$, by (3.2). Hence the map $x \rightarrow \nu(x^{-1}K) = \mu\chi_s(x^{-1}K)$ is continuous for each compact set K. Thus $\nu \in M_a^l(S)$.

Conversely, since $m(S) > 0$, so supp $(m|s) \neq \emptyset$. Let $z \in$ supp $(m|s)$ and W be a relatively compact neighbourhood of z, clearly $m(W)$ is finite. Let also $\nu \in M_a^l(S)$ and $\mu(E) = \nu(E \cap S)$, for $E \in B(S)$. Then $\mu \in M_b(G)$ such that $\nu = \mu|s$. Suppose $m(K) = 0$, for some compact set $K \subseteq G$. Then $m'(F) = 0$, where m' is the right Haar measure of G and $F = zK$, by [10, p. 272]. Let $\lambda = m'\chi_w$. Then $\lambda(Fx^{-1}) \leq m'(F) = 0$, for all $x \in G$. Thus

$$0 = \int_G \lambda(Fx^{-1}) d\mu = \int_G\int_G \chi_F(yx)\, d\lambda(y)\, d\mu(x)$$
$$= \int_G\int_G \chi_F(yx)\, \chi_w(y)\, dm'(y)\, d\mu(x)$$
$$= \int_G \mu(y^{-1}F) \chi_w(y)\, dm'(y).$$

Thus $m'\{y \in W : \mu(y^{-1}F) > 0\} = 0$, so $\mu(K) = 0$. For, suppose $\nu(z^{-1}F) = \mu(K) > 0$. Since the map $x \rightarrow \mu(x^{-1}F) =$

$\nu(x^{-1}F)$ is continuous on S, so there exists an open neighborhood V of z in S such that $\nu(x^{-1}F) > 0$, for all $x \in V$. Thus $m'(V) = 0$, which is a contradiction. Hence $\mu \ll m'$, also $m' \ll m$, by [10, p. 272], so $\mu \ll m$ and the proof of (i) is complete.

(ii) Let $\mu \in M_a^l(S,\omega)^+$. Then by using (i) and the Radon-Nykodym Theorem and the fact that μ is σ-finite, there exist a unique $f \in L^1(S,\omega)$ such that, $\mu(E) = \int_E f d\mu$, for $E \in B(S)$. In general, let $\eta = [\eta^+, \eta^-] \in M_a^l(S,\omega)$ and f^+, f^- corresponds to η^+, η^-, respectively, as above. Then by a standard argument, one can show that the map $\eta \to f^+ - f^-$ is an isometric isomorphism from $M_a^l(S,\omega)$ onto $L^1(S,\omega)$.

(iii) Let $\mu, \nu \in M_a^l(S,\omega)$ and $\mu \mapsto f$, $\nu \mapsto g$. Then,

$$\mu * \nu(C) = \int_C f * g(z)\, dm(z), \text{ for } C \in K(S).$$

By inner-regularity of $\mu * \nu$ and m,

$$\mu * \nu(E) = \int_E f * g(z)\, dm(z), \text{ for } E \in B(S).$$

In general, let $\eta \mapsto f$ and $\xi \mapsto g$. Then,

$$\xi * \eta \mapsto f * g = (f^+ * g^+ + f^- * g^-) - (f^+ * g^- + f^- * g^+).$$

Therefore the proof is complete.

Remark. Prof. H.A.M. Dzinotyiweyi recalled that if $m(S) > 0$, then by using [7, p. 16] one can show that the interior of S^2 is non-empty. Also, every continuous function on an open subset of G can extend to a continuous function on G. Thus clearly, $M_a^l(S) = L_1(S)$.

The following corollary is the Theorem (19.18) in [10].

(3.7) Corollary. Let G be a locally compact group. Then $M_a^l(G) = L_1(G)$.

In the following, we find $M_a^l(S,\omega)$ for a subset S of IR in the Euclidean topology of IR, but with a different multiplication, related to the results of this paper. Their proofs can be obtained by using the definition of $M_a^l(S,\omega)$.

(3.8) Examples. (i) Let $S = ([0,1], .)$, where $x \,.\, y = \min\{x + y, 1\}$, for $x,y \in S$, and ω be a weight function on S. Then $\omega^{-1} \leq 1$ and

$$M_a^l(S,\omega) = L^1(S,\omega) \oplus \{\lambda\bar{1} : \lambda \in \mathbb{R}\}.$$

(ii) Let $S = [0, +\infty)$ with the usual multiplication [resp., addition]. Then $M_a^l(S,\omega) = \{\lambda\bar{0} : \lambda \in \mathbb{R}\}$ [resp. $L^1(S,\omega)$].

(iii) Let $S = ([0,1], .)$, where $x \,.\, y = y$ for $x,y \in S$. Then $M_a^l(S,\omega) = M_b(S,\omega)$ and $M_a^r(S,\omega) = \{0\}$, so $M_a^l(S,\omega) \neq M_a^r(S,\omega)$.

(iv) Let $S = ([0,1], .)$, where $x \,.\, y = \min\{x, y\}$ [resp. $\max\{x, y\}$] and ω be a weight function on S. Then S is an idempotent semigroup, so $\omega^{-1} \leq 1$ and $M_a^l(S,\omega) = \{\lambda\bar{0} : \lambda \in \mathbb{R}\}$ [resp. $\{\lambda\bar{1} : \lambda \in \mathbb{R}\}$].

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References

1. Baker, A.C. and Baker, J.W. Algebras of measures on a locally compact semigroup, *J. London Math. Soc.*, **1**, 249-259, (1969).
2. Baker, A.C. and Baker, J.W. Algebras of measures on a locally compact semigroup, II, *Ibid.*, **2**, 651-659, (1970).
3. Baker, A.C. and Baker, J.W. Algebras of measures on a locally compact semigroup, III, *Ibid.*, **4**, 685-695, (1972).
4. Berg, C., Christensen, J.P.R. and Ressel, P. Harmonic analysis on semigroups, *Springer Graduate Texts in Math.*, **100**, (1984).
5. Dzinotyiweyi, H. A. M. Algebras of measures on C-distinguished topological semigroups, *Math. Proc. Camb. Phil. Soc.*, **84**, 323-336, (1978).
6. Dzinotyiweyi, H.A.M. Some aspects of abstract harmonic analysis, *Semigroup Forum*, **29**, 1-12, (1984).
7. Dzinotyiweyi, H.A.M. The analogue of the group algebra for topological semigroups, *Pitman Pub. Lond. Research Notes in Math.*, **98**, (1984).
8. Ghahramani, F. Weighted group algebras as an ideal in its second dual space, *Proc. Amer. Math. Soc.*, **90**, (1), 71-76, (1984).
9. Glicksberg, I. Weak compactness and separate continuity, *Pacific J. Math.*, **11**, 205-214, (1961).
10. Hewitt, E. and Ross, K.A. *Abstract harmonic analysis*, Vol. I, Springer-Verlag, Berlin, (1963).
11. Janssen, A. Integration separat stetiger Funktionen, *Math. Scand.*, **48**, 68-78, (1981).
12. Kharaghani, H. The convolution of Radon measures, *Proc. Amer. Math. Soc.*, **103**, (4), (1987).
13. Knowles, J.D. Measures on topological spaces, *Proc. Lond. Math. Soc.*, **17** (3), 139-156, (1967).
14. Lashkarizadeh, M. Representation of foundation semigroup and their algebras, *Cand. J. Math.*, **37**, 29-47, (1985).
15. Rejali, A. The Arens regularity of weighted convolution algebras on semitopological semigroups, *Proc. 21st International Math. Conf. Iran*, (1986).

16. Sleijpen, G.L.G. Locally compact semigroups and continuous translations of measures, *Proc. London Math. Soc.*, **37** (3), 75-97, (1978).
17. Willard, S. *General topology*. Addison-Wesley Pub. Company, (1970).
18. Wong, J.C.S. Convolution and separate continuity, *Pacific J. Math.*, **75**, 602-611, (1978).