THE ANALOGUE OF WEIGHTED GROUP ALGEBRA FOR SEMITOPOLOGICAL SEMIGROUPS

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Abstract

In [1,2,3], A. C. Baker and J.W. Baker studied the subspace \( M_a(S) \) of the convolution measure algebra \( M_b(S) \) of a locally compact semigroup. H. Dzinotyiweyi in [5,7] considers an analogous measure space on a large class of \( C \)-distinguished topological semigroups containing all completely regular topological semigroups. In this paper, we extend the definitions to study the weighted semigroup algebra \( M_a(S, \omega) \), where \( \omega \) is a weight function on a \( C \)-distinguished semitopological semigroup \( S \). We will show that this subspace is a convolution measure algebra. As a corollary, this answers in the affirmative a question raised by J.W. Baker and H. Dzinotyiweyi in [6].

Definitions and Preliminary Results

Throughout the paper, \( S \) [resp. \( X \)] will denote a Hausdorff semitopological semigroup [resp. topological space]. Let \( k_t \) denote the strongest topology on \( X \) which agrees with the original topology on the compact subset of \( X \). The topological space \( X \) is said to be a \( k \)-space if \( k = T \). By \( T \) we mean the weakest topology on \( X \) such that whenever a bounded real valued function which is continuous with respect to the topology \( k \) is then it is continuous with respect to \( T \). All notations and terminology in the subject of measure theory are as in [4] and [7]. We denote by \( K(X) \) [resp. \( B(X) \)] the set of all compact [resp. Borel] subsets of \( X \). Also by \( C_b(X) \) [resp. \( C_b(X, k) \)] we mean the set of all real-valued bounded continuous functions on \( (X, T) \) [resp. \( (X, k) \)]. We denote \( C_b(X) \subseteq C_b(X, k) \) and denote \( \|f\|_\infty = \sup \{|f(x)| : x \in X\} \), for \( f \in C_b(X) \). If \( C_b(X) \) separate points of \( X \), we say \( X \) is \( C \)-distinguished. Clearly, the family of \( C \)-distinguished spaces contain all completely regular spaces.

Let \( M_b(X) \) [resp. \( M_b(X, k) \)] be the set of all bounded Radon measures on \( (X, T) \) [resp. \( (X, k) \)]. If \( \mu = \mu^* - \mu^* \) be the Hahn decomposition of \( \mu \in M_b(X) \), then we write \( \mu^* = \mu^* - (\mu) \), where \( \mu^* \) is the unique extension Radon measure of \( \mu^* \) on \( (X, k) \) which agrees on compacta, (see [4, p. 18]). We recall that \( K(X, T) = K(X, k) \) and \( B(X, T) \subseteq B(X, k) \), so \( \nu(E) = \sup \{ \nu(K) : K \subseteq E \} \), for \( \nu \in M_b(X) \) and \( E \subseteq B(X, k) \).

In the following we give an alternate proof to a Glicksberg's result for general case, (see [9], [11]), noting that Glicksberg's proof can be modified, by using this method, to get this extended version.

For easy reference, we mention the following consequence of [4, p. 20-21].

(1.1) Lemma. Let \( X \) be a Hausdorff space and \( f : X \to [0, + \infty] \) be an arbitrary function. Then,

(a) \( \) Let \( A_{\alpha} = \{x \in X : f(x) > \frac{1}{2^n}\} \) and \( f_n = \sum_{\alpha=1}^{\infty} \frac{1}{2^n} \chi_{A_{\alpha}} \). Then

(b) \( \) The sequence \( f_n \) converges to \( f \) in measure; that is, for any \( \epsilon \),

(c) \( \) The sequence \( f_n \) converges to \( f \) pointwise; that is, for any \( x \),
0 ≤ f_n ≤ f and f_n increases to f. In particular if f is lower semicontinuous function, then there exists a sequence \( s_{n,k} = \sum_{k=1}^{\infty} \frac{1}{2^k} X_{s_{n,k}} \) of simple lower semicontinuous functions such that \( \lim_{n \to \infty} \lim_{k \to \infty} s_{n,k} = f. \)

(ii) If a net \((f_n)\) of lower semicontinuous functions \(X \to [0, +\infty]\) is increasing with \(\sup_n f_n = f\) and \(\mu \in M_0(X)\), then

\[
\int_X f d\mu = \sup_{\alpha} \int_X f d\alpha = \lim_{n \to \infty} \int_X f_n d\mu.
\]

(iii) Let \(f: X \to [0, +\infty]\) be a Borel-measurable function and \(K(X)\) be directed by inclusion. If \(\mu \in M_0(X)\), then

\[
\int_X f d\mu = \sup \int_X f d\mu : C \in K(X)\).
\]

(1.2) Theorem (Glicksberg’s Extended Version).
Let \((X, T_X)\) and \((Y, T_Y)\) be Hausdorff topological spaces and \(F: X \times Y \to IR\) be a bounded separately continuous function. If \(\mu \in M_b(X)\) and \(\nu \in M_b(Y)\), then

(i) The map \(x \to \int_Y F(x,y) d\nu(y)\) [resp. \(y \to \int_X F(x,y) dx\)] is \(k\)-continuous.

(ii) \(\int_X F(x,y) d\mu(x) = \int_Y \int_X F(x,y) d\mu(x) d\nu(y)\).

Proof. Without loss of generality, we can assume that \(F, \mu,\) and \(\nu\) are positive.
(i) Let \(x^*\) denote the point mass at \(x \in X\). Then by (1.1),

\[
\int_X F(x, y) d\mu(y) = \nu(x, F), \text{ where } x, F(x, y) \equiv F(x, y) = \sup \{ F(x, y) : D \in K(Y)\}.
\]

But the map \(x \to \int_Y F(x, y) d\nu(y)\) is continuous on each compact subset \(C\) of \(X\), by the Glicksberg theorem, (see [9]). Hence the map \(x \to \int_Y F(x, y) d\nu(y)\) is \(k\)-continuous on \(X\), for each \(D \in K(Y)\). Since the family of functions

\[
\{x \to \int_Y F(x, y) d\nu(y) : D \in K(Y)\}
\]

is directed upward to \(x \to \int_Y F(x, y) d\nu(y)\), so the map \(x \to \int_Y F(x, y) d\nu(y)\) is \(k\)-lower semicontinuous, by (1.1) (ii).

Similarly, the map \(x \to \int_X F(x, y) d\mu(x)\) is \(k\)-lower semicontinuous. Since \(\{x \to \int_X F(x, y) d\mu(x) : D \in K(Y)\}\) is directed upward to \(x \to \int_X F(x, y) d\mu(x)\), so the map \(x \to \int_X F(x, y) d\mu(x)\) is \(k\)-lower semicontinuous.

Therefore, the map \(x \to \int_X F(x, y) d\mu(x)\) is \(k\)-continuous.

By the same argument, the map \(y \to \int_X F(x, y) d\mu(x)\) is \(k\)-continuous.

(ii) Since the family of \(K\)-continuous functions \(\{x \to \int_Y F(x, y) d\nu(y) : D \in K(Y)\}\) is directed upward to \(k\) continuous map \(x \to \int_Y F(x, y) d\nu(y)\), by (i), so \(\{x \to \int_X F(x, y) d\mu(x) : C \in K(X)\}\) is directed upward to the integral \(\int_X F(x, y) d\mu(x)\) by (1.1) (iii). But the measures \(\bar{\mu}\) and \(\mu\) are concentrated on a \(\sigma\)-compact set and \(\bar{\mu}\) agree with \(\mu\) on compacta.

Hence

\[
\int_X F(x, y) d\nu(y) d\mu(x) = \sup (\int_C \int_D F(x, y) d\nu(y) d\mu(x) : C \in K(X), D \in K(Y)) = \int_C \int_D F(x, y) d\nu(y) d\mu(x) : C \in K(X), D \in K(Y)) = \int_Y \int_X F(x, y) d\mu(x) d\nu(y).
\]

(1.3) Corollary [11]. Let \(X, Y\) be Hausdorff completely regular topological spaces and \(F: (X, T_X) \times (Y, T_Y) \to IR\) be a bounded separately continuous function. If \(\mu \in M_b(X, T_X)\), \(\nu \in M_b(Y, T_Y)\), then

(i) The map \(x \to \int_Y F(x, y) d\nu(y)\) [resp. \(y \to \int_X F(x, y) d\mu(x)\)] is \(T\) continuous.

(ii) \(\int_X F(x, y) d\mu(x) = \int_Y \int_X F(x, y) d\mu(x) d\nu(y)\).

Weighted Convolution Measure Algebras \(M_b(S, \omega)\)

In [9], I. Glicksberg showed that \(M_b(S)\) with the usual convolution is a Banach algebra, when \(S\) is compact. Later, C.J. Wong [18] studied the space \(M_b(S)\), where \(S\) is a locally compact semitopological semigroup. Also, H. Kharagani [12] considered \(M_b(S)\) on \(\mathbb{C}\)-complete spaces included in locally compact and complete metric semitopological semigroups \(S\). It is to be noted that H. Dzmitrovich [5] showed that \(M_b(S)\) is a convolution measure algebra on a large class of \(C\)-distinguished spaces containing all completely regular topological semigroups \(S\). Finally, A. Jansen [11] proved \(M_b(S)\) need not be a Banach algebra with usual convolution under the assumption that \(S\) is a completely regular semitopological semigroup. In this section, we will introduce a convolution \(\star\) for which \((M_b(S), \star)\) be a (non associative) Banach algebra.

Let \(\omega : S \to (0, +\infty)\) be a Borel measurable weight function, that is \(\omega(s) \leq \omega(s) \omega(t)\), where \(s, t \in S\) for which \(1/\omega\) is bounded on compacta. Various authors have considered the space of weighted measure algebra \(M(\omega)\) consisting of all complex measures \(\mu\) such that \(1/\omega \in M_b(S)\). (see for example [8], [14]). The space \(M(\omega)\) need not be complete and the norm-algebra \(M(\omega)\) is different
from \( I_1(S, \omega) = \{ f : S \to \mathbb{IR} \mid \sum_{x \in S} |f(x)| \omega(x) < \infty \} \), where 
\( S \) has discrete topology. For these reasons we have chosen a different definition for the weighted convolution measure algebra \( M_b(S, \omega) \).

Let \( C_b(S, \omega) = \{ f : S \to \mathbb{IR} \mid \frac{f}{\omega} \in C_b(S) \} \). Then \( C_b(S, \omega) \) with the usual addition and the following multiplication,

\[
f \cdot g(x) = \frac{f(x)g(x)}{\omega(x)}, \quad \text{for } x \in S \text{ and } f, g \in C_b(S, \omega)
\]

with the norm, \( \|f \|_{C_b} = \sup \{|\frac{f}{\omega}(x)| : x \in S\} \), is a Banach algebra. Also the map \( f \mapsto \frac{f}{\omega} \) from \( (C_b(S, \omega), +) \) onto \( C_b(S) \) with pointwise multiplication is an isometric isomorphism.

In [5], H. A. M. Dziowntyewiyo showed that \( M_b(S, \omega) = C_b(S, \omega)^* \) as Banach algebra, where \( C_b(S) \) is \( C_b(S) \) with the strict-topology. In the following, we define \( M_b(S, \omega) \) such that the identity \( M_b(S, \omega) = C_b(S, \omega) \).

Let \( M_b^*(S, \omega) \) be the set of all Radon measures \( \mu \) on \( S \), that is inner-regular and finite on compacta, such that \( \mu \omega \in M_b(S, \omega) \).

In general, \( \varphi \) need not be injection. Following [15], let 
"\( \sim \)" be an equivalence relation on \( M_b(S, \omega) \) defined by,

\[
(\mu, \nu) \sim (\mu', \nu') \text{ if and only if } \mu + \nu = \mu' + \nu' \text{ and } [\mu, \nu] \text{ is the equivalence class of } (\mu, \nu), \text{then we define,}
\]

\[
\mathcal{M}_b(S, \omega) = \{ [\mu, \nu] : \mu, \nu \in M_b(S, \omega) \}.
\]

Let also \( C_b(S, \omega) \) denote \( C_b(S, \omega) \) with the \( \omega \)-strict topology, in the obvious way. One can show that \( M_b(S, \omega) \) with the norm \( \|\mu, \nu\|_{\omega} = \|\mu \omega, \nu\|_{\omega} \) and regard \( M_b(S, \omega) \) as a norm space over \( \mathbb{IR} \) is a Banach space isometric isomorphism to \( C_b(S, \omega)^* \).

Let us turn our attention to make \( M_b(S, \omega) \) into a convolution measure algebra. Since \( (C_b(S, \omega), +) \) is a Banach algebra, thus one can define a multiplication on \( C_b(S, \omega)^* \) and so \( M_b(S, \omega) \) such that it be a Banach algebra. In the following, we define a convolution "\( * \)" on \( M_b(S, \omega) \), where \( S \) is C-distin guished semitopological semigroup, such that 

\[
\mu^* \nu = \int_S \chi_x(y) \mu(dy) (x) \nu(y), \quad \text{for each compact set } K \subseteq S \text{ and } \mu, \nu \in M_b(S, \omega).
\]

Since \( \frac{1}{\omega} \) is bounded on compacta and \( \mu = (\mu \cdot \omega) \frac{1}{\omega} \), so each measure \( \mu \in M_b^*(S, \omega) \) is \( \sigma \)-finite. Let \( \mu, \nu \in M_b^*(S, \omega) \) and,

\[
\lambda(C) := \int_S \chi_x(y) d\mu(x) d\nu(y), \quad \text{for } C \subseteq K(S).
\]

Then the family of \( k \)-continuous maps \( \{ y \to \int_S \chi_x(y) d\mu(x) d\nu(y) \} \) is directed downward to the map \( y \to \int_S \chi_x d\mu(x) d\nu(y) \), by [7, p. 174]. Hence, the map \( y \to \int \chi_x d\mu(x) d\nu(y) \) is \( k \)-upper semicontinuous function on \( S \). Thus the family \( \{ \int f(x) d\mu(x) d\nu(y) : f \in C_b(S) \text{ and } f \geq \chi_x \} \) is directed downward to \( \lambda(C) \), see (1.1). In other words, \( \lambda(C) = \inf \{ l(f) : f \in C_b(S) \text{ and } f \geq \chi_x \} \), where \( l(f) = \int_S f(x) d\mu(x) d\omega(y) \), for \( f \in C_b(S) \). But \( l \) is a positive linear functional on \( C_b(S) \). Thus by the same argument as is used in [4, p. 36], one can show that \( \lambda \) is a Radon-content, that is

\[
\lambda(C_1) \leq \lambda(C_2) = \sup \{ \lambda(C) : C \text{ is a compact subset of } C_2 \setminus C_1 \},
\]

where \( C_1 \subseteq C_2 \). It is to be noted that,

\[
\lambda(C) = \int_S \int_S \frac{1}{\omega} \chi_x(y) d\mu(x) d\omega(y) \leq
\]

\[
\|1\|_\omega C \cdot \|\mu\|_\omega \|\nu\|_\omega
\]

is finite, for each \( C \subseteq K(S) \).

Therefore, the Radon-part \( \lambda_0 \) of \( \lambda \) is a Radon measure, by [4, p. 18]. We define,

\[
\mu^* \nu(E) = \lambda_0(E) = \sup \{ \lambda(C) : C \text{ is a compact subset of } E \} \text{ and } \mu. \nu(E) = \int_S \int_S \chi_x(y) d\mu(x) d\nu(y), \text{ for } E \in B(S).
\]

\[(2.1) \text{ Definition. Let } \mu, \nu, \mu', \nu', M_b(S, \omega) \text{ and } \lambda \in \mathbb{IR} \text{. Then}
\]

\[(i) [\mu, \nu] + [\mu', \nu'] = [\mu + \mu', \nu + \nu']
\]

\[(ii) [\mu, \nu]^* [\mu', \nu'] = [\mu^* \mu' + \nu^* \nu', \mu^* \nu + \mu^* \nu^*]
\]

\[(iii) \lambda(\mu, \nu) = \begin{cases} \lambda(\mu \cdot \nu) & \text{if } \lambda \geq 0 \\ \lambda(\nu, \mu) & \text{otherwise} \end{cases}
\]

It is easy to show that \( M_b(S, \omega) \) is a vector space.
Let \( \{ G_n \} \) be a family of open sets directed upward to \( G \). Then the family of \( k \)-lower semicontinuous maps \( \{ y \mapsto \int_G \chi_x (xy) d\mu (x) \} \) is directed upward to the \( k \)-lower semicontinuous map \( x \mapsto \int_G \chi_y d\mu (y) \), by (1.1). Hence, by (1.1) (ii), the family \( \{ \mu, \nu (G_n) \} \) is directed upward to \( \mu, \nu (G) \), that is, \( \mu, \nu \) is a \( \tau \)-smooth measure. In general, \( \mu, \nu \) is not Radon measure, by [11], so \( \mu, \nu \neq \mu^{*} \nu \). But \( \mu^{*} \nu \) is the maximal Radon measure on \( S \) coincide \( \mu \) on compacta. If \( S \) is \( \mathcal{C} \) compact space, that is \( S \) is a \( \mathcal{C} \)-set in the Stone \( \mathcal{C} \) compactification of \( S \), then every \( \tau \)-smooth measure is Radon-measure, see [13]. In particular, if \( S \) is either a locally compact or complete metric space, then the inner-convolution \( "_{**} \) is equal to the usual-convolution \( "_{*} \). In the following we give an alternative proof for the equality of \( "_{**} \) and \( "_{*} \), in this case, without using the Stone \( \mathcal{C} \) compactification.

\( 2.2 \) Theorem. Let \( S \) be either a locally compact or complete metric semitopological semigroup. Then \( (\mu, \nu (S, \omega)) \) is a convolution measure algebra.

**Proof.** (i) Suppose \( S \) is a locally compact space. Then for each \( x \in S \) there exists a relatively compact neighborhood \( V_x \), say. Let \( G \) be the family of all finite union of these \( V_x \), where \( x \in S \). If \( G = \bigcap_{k=1}^{n} V_a \in \mathcal{G} \), then \( \overline{\bigcap_{k=1}^{n} V_a} = \overline{G} \) is compact. Let \( \mu, \nu \in M_{\mu}^{*} (S) \). Then,

\[
\mu, \nu (S) = \sup \{ \mu, \nu (G) : G \in \mathcal{G} \} \\
\leq \sup \{ \mu, \nu (\overline{G}) : G \in \mathcal{G} \} \\
\leq \sup \{ \mu, \nu (C) : C \in \mathcal{K} (S) \} = \mu, \nu (S).
\]

Thus \( \mu, \nu (S) = \mu^{*} \nu (S) \). Let \( E = \{ E \in B(S) : \mu, \nu (E) = \mu^{*} \nu (E) \} \). One can show that \( E \) is a \( \sigma \)-algebra containing closed sets, so \( \mu^{*} \nu = \mu, \nu \). Since each measure in \( M_{\mu}^{*} (S, \omega) \) is \( \sigma \)-finite, so it is easy to show that \( \mu^{*} \nu = \mu, \nu \) for all \( \mu, \nu \in M_{\mu}^{*} (S, \omega) \).

(ii) Suppose \( S \) is a complete metric space. Then for each \( n \in \mathbb{N} \), define \( G_n = \bigcap_{k=1}^{n} G_n \), where \( x \in S \). Then it is clear, \( G_n \uparrow S \), so \( \mu, \nu (S) = \sup \{ \mu, \nu (G) : G \in G_n \} \). Hence for each \( \varepsilon > 0 \) and \( n \in \mathbb{N} \), there exist \( G_n \in G_n \) such that \( \mu, \nu (S \backslash G_n) < \varepsilon / 2^n \).

Put \( G_0 = \bigcap_{n=1}^{\infty} G_n \). Then,

\[
\mu, \nu (S \backslash G_0) \leq \mu, \nu (S \backslash G_n) \leq \sum_{n=1}^{\infty} \mu, \nu (S \backslash G_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.
\]

Also \( G_0 \) is totally-bounded, so \( G_0 \) is compact. Hence,

\[
\mu, \nu (S) = \sup \{ \mu, \nu (C) : C \in \mathcal{K} (S) \} = \mu^{*} \nu (S).
\]

Therefore \( "_{**} \) coincides with \( "_{*} \). The rest of the proof is routine.

We now state the main theorem of this section.

\( 2.3 \) Theorem. Let \( S \) be a \( C \)-distinguished semitopological semigroup such that either

(i) \( K^2 \) is compact, for each compact \( K \) in \( S \), or

(ii) \( x^2 \) and \( Kx^2 \) are compact, for each compact \( K \subseteq S \).

Then \( (\mu, \nu (S, \omega), *) \) is a convolution measure algebra.

**Proof.** (i) By a similar argument as is used in [7, p. 6-7] is immediate.

(ii) Let \( \mu, \nu, \eta \in M_{\mu}^{*} (S) \) and \( K \subseteq S \) be compact. Then,

\[
(\mu^{*} \nu) \ast \eta (K) = \int_{K} \chi_{\eta} (z) X d\mu (y) d\eta (x)
\]

\[
= \int_{K} \mu, \nu (Kz) d\eta (x)
\]

\[
= \int_{K} \mu, \nu (Kx) d\nu (y) d\eta (z)
\]

\[
= \int_{K} \mu, \nu (Kx) d\nu (y) d\eta (z)
\]

\[
= \int_{K} \mu^{*} \eta (x^{-1} K) d\mu (x)
\]

\[
= \mu^{*} (\nu \ast \eta) (K).
\]

By inner-regularity, \( (\mu^{*} \nu) \ast \eta = \mu^{*} (\nu \ast \eta) \). Thus \( "_{**} \) is associative.

Let \( k = \sum_{i=1}^{n} a_i \chi_{k_i} \), where \( a_i \in \mathbb{R}^+ \), \( n \in \mathbb{N} \) and \( K_i \) be compact subset of \( S \), for \( 1 \leq i \leq n \). Let also \( \mu, \nu \in M_{\mu}^{*} (S, \omega) \).

Then,

\[
(\mu^{*} \nu) (\omega \chi_k) = \sup \{ \int \omega \chi_k X d\mu (x) d\nu (y) : k \leq \omega \chi_k \}, \text{ see [4, p. 36-37]}
\]

\[
= \sup \{ \mu, \nu (k) : k \leq \omega \chi_k \}
\]

\[
\leq \mu, \nu (\omega \chi_k) = \int \omega \chi_k (xy) d\mu (x) d\nu (y)
\]

\[
\leq \| \mu \|_\omega \| \nu \|_\omega \text{ for each compact } C \text{ in } S.
\]

Therefore \( \| \mu^{*} \nu \|_\omega \leq \| \mu \|_\omega \| \nu \|_\omega \). The rest of the proof is easy.

\( 2.4 \) Corollary. Let \( S \) be a \( C \)-distinguished topological semigroup, or semitopological group. Then \( (M_{\mu} (S, \omega), *) \) is a Banach algebra.

The following example shows that the measure algebra \( M(\omega) \) is not complete.
Example. Let \( S = (\mathbb{N}, +) \) with the discrete topology and \( \omega(\alpha) = e^{\alpha} \), for \( \alpha \in S \). Then \( M(\omega) = I_1(\mathbb{N}) \cap V_1(\mathbb{N}, \omega) \) is not complete.

Proof. Let \( \lambda_n = e^n / n^2 \), for \( n \in \mathbb{N} \), and \( f_k(n) = \begin{cases} \lambda_n & \text{if } n \leq k \\ 0 & \text{otherwise} \end{cases} \). Then \( (f_k) \) is Cauchy in \( (M(\omega), \|\cdot\|_\omega) \), which is not convergent [for, if \( f_k \to f \), then \( f = (\lambda_n) \) and
\[
\sum_{n=1}^{\infty} |f(n)| = \infty, \text{ so } f \notin I_1(\mathbb{N}).
\]

The following example shows that "**" need not be associative.

Example. Let \( S, \mu, \nu, \ast \) be as in [11, p. 77] and \( \eta' = e, \) where \( e : = (e_\alpha) \) and \( e_\alpha : [0, 1] \to [0, 1] \) be defined by \( e_\alpha(x) = 1 \), for \( x \in [0, 1] \). Then,
(i) Let \( C = \{\epsilon\} \). Then \( \mu \ast (\nu \ast \eta') (C) = \mu \ast (\nu \ast \eta') (S) \neq \mu \ast (\nu \ast \eta') (C) \).
(ii) Let \( l(f) = \int f(x) \, d\mu(x) \, d\nu(y) \), for \( f \in C_b(S) \). Then \( l \) is not strictly continuous, (c.f. [7], p. 6-7).

Weighted Convolution Measure Algebra \( M^*_\omega(S, \omega) \)

A.C. Baker and J.W. Baker in [1,2,3] introduced and studied the convolution measure algebra \( M^*_\omega(S) \), absolutely continuous measures, analogous to the group algebra \( L^1(G) \), for a locally compact topological semigroup \( S \). Later, several authors studied this algebra, for example [5] and [16]. In particular, in [6] Dzintiyiwek asked whether \( M^*_\omega(S) \) can be made into a convolution measure algebra, whenever \( S \) is a semitopological semigroup.

In this section, we give an affirmative answer to this question and show that this space has the advantage that if \( \mu \in M^*_\omega(S) \) and \( \nu \in M^*_\omega(S) \), then \( \mu \ast \nu = \mu \ast \nu \). For a suitable definition of \( M^*_\omega(S, \omega) \) analogous to \( M^*_\omega(S) \), see [7]. Also \( M^*_\omega(S, \omega) \) is a solid and left ideal of \( M^*_\omega(S, \omega) \).

Let \( \eta = [\mu, \nu] \in M^*_\omega(S, \omega) \). Then \( \eta \omega(\alpha) = \mu \omega(\alpha) - \nu \omega(\alpha) \in M^*_\omega(S, \omega) \), so by the Hahn decomposition theorem, there exist unique \( \xi, \xi \in M^*_\omega(S, \omega) \) such that \( \eta \omega = \xi + \xi \) and \( \xi \perp \xi \).

Put \( \eta^+ = (\xi^+) ^\frac{1}{2} \) and \( \eta^- = (\xi^-) ^\frac{1}{2} \). Then \( \eta = [\eta^+, \eta^-] \) such that \( \eta^+ \perp \eta^- \). Let \( l \eta = \eta^+ \ast \eta^- \) and \( A \subseteq M^*_\omega(S, \omega) \). If \( A \) satisfies the following conditions, then \( A \) is called (weighted) convolution measure algebra.

(i) \( A \) is a norm-closed subalgebra of \( M^*_\omega(S, \omega) \).
(ii) \( A \) is solid, that is if \( \eta \in M^*_\omega(S, \omega) \) and \( \xi \in A \) such that \( l \eta \in l \xi \) implies \( \eta \in A \).

We define, \( M^*_\omega(S, \omega) = \{ \eta \in M^*_\omega(S, \omega) : l \eta \in M^*_\omega(S, \omega) \} \), where \( M^*_\omega(S) = \{ \mu \in M^*_\omega(S) : \text{The map } x \to [\mu(x) \cdot C \) is continuous for each \( C \in K(S) \} \). Similarly, one can define \( M^*_\omega(S, \omega) \) and \( M^*_\omega(S, \omega) = M^*_\omega(S, \omega) \cap M^*_\omega(S, \omega) \).

Throughout this section, \( S \) is a C-distinguished semitopological semigroup endowed with the \( k \)-topology (or k-topology).

Lemma. Let \( h : S \to [0, +\infty) \) be a Borel-measurable function and \( \mu \in M^*_\omega(S) \), \( \nu \in M^*_\omega(S) \). Then,
(i) the map \( x \to \int_S h(x) \, d\nu(y) \) is \( k \)-lower semicontinuous (k-l.s.c.),
(ii) \( \mu \ast \nu(h) = \mu \ast \nu(h) = \int_S h(x) \, d\mu(x) \, d\nu(y) = \int_S h(x) \, d\mu(x) \, d\nu(y) \).

Proof. (i) Let \( E \subseteq S \) be a Borel set, \( x \in S \). Then by (1.1)(iii).
\[ \bar{x} \ast \nu(E) = \sup \{ v(x) K : K \text{ is a compact subset of } E \} \]

But the map \( x \to x(\cdot \cdot K) \) is \( k \)-continuous, so the map \( x \to x(\cdot \cdot E) \) is \( k \)-lower semicontinuous (k-l.s.c.).

Let \( (s_{\eta, \omega}(x)) \) be a sequence of positive, Borel measurable simple functions increasing to \( h \), pointwise. (see (1.1)). Then \( \int s_{\eta, \omega}(x) \, d\nu(y) \) increasingly converges to \( \int h(x) \, d\nu(y) \). But the map \( x \to \int s_{\eta, \omega}(x) \, d\nu(y) \) is \( k \)-lower semicontinuous. Hence (i) follows.

(ii) Let \( E \subseteq S \) be a Borel set and \( K(S) \) be directed by inclusion. Then the family of \( k \)-continuous functions \( x \to x(\cdot \cdot C) \) is \( k \)-lower semicontinuous (k-l.s.c.). Hence (1.1)(ii).

Therefore, \( \mu \ast \nu(E) = \sup \{ \mu \ast \nu(K) : K \text{ is a compact subset of } E \} = \mu \ast \nu(E) \).

By a standard argument and applying (1.1)(ii) is immediate.

We now state the main theorem of this paper. As a corollary this answers the open question raised in [6].

Theorem. \( M^*_\omega(S, \omega) \) is a Banach algebra, left ideal and solid in \( M^*_\omega(S, \omega) \).

Proof. (i) First of all we show that \( M^*_\omega(S, \omega) \) is solid. Let \( \nu \in M^*_\omega(S, \omega) \) and \( \mu \in M^*_\omega(S, \omega) \) such that \( l \mu \in l \nu \). Then
it is easy to show that \( l \mu \omega \ast \mu \omega \). Since the map, 

\[
y \to \int_{\mathcal{X}} (xy) d \mu \omega (x) = l \mu \omega \ast \mu \omega (K)
\]

is \( k \)-upper semicontinuous, by (1.2) and [7, p.174]. Thus by a similar argument as is used in [7, p. 10], one can show that \( l \mu \omega \in M^*_k (S) \). Thus \( \mu \in M^*_k (S, \omega) \).

(ii) Now we show that \( M^*_k (S, \omega) \) is a left ideal of \( M (S, \omega) \).

Let \( \mu \in M^*_k (S, \omega) \), \( \nu \in M^*_k (S, \omega) \), and \( K \subseteq S \) be compact, then,

\[
\nu \ast \omega = \int_{\mathcal{X}} \omega (x) (\omega \ast \omega) (x) d \mu \omega (x), \text{by (3.1),}
\]

and the map \( \omega \rightarrow \omega (x) \) is \( k \)-separately continuous bounded function.

Thus by (1.2), the map \( x \rightarrow \mu \ast \nu \) is \( k \)-continuous. But \( (\mu \ast \nu) \omega = (\mu \nu) \omega \leq (\mu \omega) \nu \) and \( (\nu \omega) \mu \ast \nu = (\nu \omega) (\mu \omega) \ast \nu \) and \( M^*_k (S) \) is solid, by (i), so \( (\mu \ast \nu) \omega \in M^*_k (S, \omega) \).

Thus \( \mu \ast \nu \in M^*_k (S, \omega) \). In general, let \( \xi \in M^*_k (S, \omega) \) and \( \eta \in M^*_k (S, \omega) \). Then \( \xi \ast \eta \in M^*_k (S, \omega) \), so \( \xi \ast \eta \in M^*_k (S, \omega) \), by (i).

(iii) \( M^*_k (S, \omega) \) is a closed subalgebra of \( M (S, \omega) \). For, let \( \xi, \eta \in M^*_k (S, \omega) \) and \( \lambda \in IR \). Then \( \lambda \xi + \eta \xi \in M^*_k (S, \omega) \), by (i). Let \( \eta_n = [\mu, \nu] \rightarrow \eta = [\mu, \nu] \) in \( M^*_k (S, \omega) \), \( f, g \in M^*_k (S, \omega) \), \( \mu \ast \nu \in M^*_k (S, \omega) \), and \( f(x) \in M^*_k (S, \omega) \), \( f(x) = (\mu \ast \nu)(x \ast \omega) \), for \( x \in S \) and compact set \( K \). Then \( \| f \ast g \| \leq \| f \| \mu \ast \nu \| \eta \| \), so \( f \) is \( k \)-continuous. That is, the map \( x \rightarrow \| f \ast g \| \) is \( k \)-continuous. Thus \( \eta \in M^*_k (S, \omega) \) and the proof is complete.

(3.3) Corollary. \( M (S, \omega) \) is a convolution measure algebra.

(3.4) Corollary. Let \( S \) be a \( C \)-distinguished \( k \)-space. Then \( M^*_k (S, \omega) \) is a Banach algebra, left ideal and solid in \( M (S, \omega) \).

Remark. The \( k \)-topology is coincided with the original topology for \( k \)-spaces. Thus \( M^*_k (S, \omega) \) is a Banach algebra, when \( S \) is endowed with the original topology. In particular, every locally compact or complete metric space is \( k \)-space, (see [17]). As a consequence, we answer the question raised in [6].

(3.5) Corollary. Let \( S \) be either a locally compact or complete metric semitopological semigroup. Then \( M^*_k (S, \omega) \) is a Banach algebra, left ideal and solid in \( M (S, \omega) \).

In the following, we consider \( M^*_k (S, \omega) \), when \( S \) is a subsemigroup of a group. Let \( m \) be the left Haar-measure on a locally compact group \( G \), and \( S \) be a Borel subset of \( G \). We denote \( L^1 (S, \omega) = (h : S \rightarrow IR \wedge h \) is Borel-measurable and \( \| h \| = \int_S | h | \omega d \mu m \) is finite \). If \( f, g \in L^1 (S, \omega) \), then \( L^1 (S, \omega) \), with following product, is a Banach algebra.

\[
f \ast g (y) = \int_S f (x) g (x \ast y) d \mu (x), \text{for} y \in S.
\]

We will show that for each \( \mu \in M^*_k (S, \omega) \) there exist a unique \( f \in L^1 (S, \omega) \) such that \( \mu (E) = \int_E f d \mu m \), for \( E \in B (S) \).

(3.6) Theorem. Let \( S \) be a subsemigroup of a locally compact group with positive Haar measure. Then \( M^*_k (S, \omega) \) is \( L^1 (S, \omega) \) as a Banach algebra.

Proof. (i) First we show that,

\[
M^*_k (S) = \{ \mu \omega : \mu \in M (S, \omega) \}.
\]

Let \( \mu \in M (S, \omega) \) such that \( \mu \in M (S, \omega) \) and \( \omega = \mu \omega \). Since \( m \in M^*_k (S) \) and \( \mu \omega = \mu \omega \), so \( \mu \omega = \mu \omega \). Hence the map \( x \rightarrow \omega (x \ast \omega) \) is \( k \)-continuous for each compact set \( K \). Thus \( \omega \in M^*_k (S) \).

Conversely, since \( m (S) > 0 \), so \( \supp (m \omega) = 0 \). Let \( \varepsilon \in \mu \omega \) and \( W \) be a relatively compact neighbourhood of \( \varepsilon \), clearly \( m (W) \) is finite. Let also \( \omega \in M^*_k (S) \) and \( \mu \omega = \mu \omega \). Let \( m' (F) = 0 \), where \( m' \) is the right Haar measure of \( G \) and \( F = m' (F) = 0 \), for all \( x \in G \). Thus

\[
0 = \int_{g \in G} \lambda (F \ast x d \mu = \int_{g \in G} \mu (F \ast x \ast \lambda (y) d \mu (x)
\]

\[
= \int_{g \in G} \mu (F \ast x \ast \lambda (y) d \mu (x)
\]

Thus \( m' (y \in W : \mu (y \ast F) > 0) = 0 \), so \( \mu (K) = 0 \). For, suppose \( \varepsilon (\omega) = \mu (K) > 0 \). Since the map \( x \rightarrow \mu (x \ast F) =
\]
\( \nu(x^{1} F) \) is continuous on \( S \), so there exists an open neighborhood \( V \) of \( z \) in \( S \) such that \( \nu(x^{1} F) > 0 \), for all \( x \in V \). Thus \( m^{*}(V) = 0 \), which is a contradiction. Hence \( \mu \neq m^{*} \), also \( m^{*} \neq m \), by [10, p. 272], so \( \mu \neq m \) and the proof of (i) is complete.

(ii) Let \( \mu \in M_{d}^{(S, \omega)}, \). Then by using (i) and the Radon-Nykodym Theorem and the fact that \( \mu \) is \( \sigma \)-finite, there exist a unique \( f \in L^{1}(S, \omega) \) such that, \( \mu(E) = \int_{E} f \, d\omega \), for \( E \in B(S) \). In general, let \( \eta = [\eta^{*}, \eta] \in M_{d}^{(S, \omega)} \) and \( f^{*}\) corresponds to \( \eta^{*}, \eta \), respectively, as above. Then by a standard argument, one can show that the map \( \eta \mapsto f^{*}\cdot f \) is an isometric isomorphism from \( M_{d}^{(S, \omega)} \) onto \( L^{1}(S, \omega) \).

(iii) Let \( \mu, \nu \in M_{d}^{(S, \omega)} \) and \( \mu \mapsto f, \nu \mapsto g \). Then,
\[
\mu \ast \nu(C) = \int_{C} f^{*} g(z) \, dm(z), \text{ for } C \in K(S).
\]
By inner-regularity of \( \mu \ast \nu \) and \( m \),
\[
\mu \ast \nu(E) = \int_{E} f^{*} g(z) \, dm(z), \text{ for } E \in B(S).
\]
In general, let \( \eta \mapsto f \) and \( \xi \mapsto g \). Then,
\[
\xi \ast \eta \mapsto f \ast g = (f^{*} g^{*} + f^{*} g) \cdot (f^{*} g + f \ast g^{*}).
\]
Therefore the proof is complete.

Remark. Prof. H.A.M. Dzintoyiweyi recalled that if \( m(S) > 0 \), then by using [7, p. 16] one can show that the interior of \( S \) is non-empty. Also, every continuous function on an open subset of \( G \) can extend to a continuous function on \( G \). Thus clearly, \( M_{d}^{(S)} = L^{1}(S) \).

The following corollary is the Theorem (19.18) in [10].

(3.7) Corollary. Let \( G \) be a locally compact group. Then \( M_{d}^{(G)} = L^{1}(G) \).

In the following, we find \( M_{d}^{(S, \omega)} \) for a subset \( S \) of \( IR \) in the Euclidean topology of \( IR \), but with a different multiplication, related to the results of this paper. Their proofs can be obtained by using the definition of \( M_{d}^{(S, \omega)} \).

(3.8) Examples. (i) Let \( S = (0, 1), \) where \( x, y = \min \{x + y, 1\} \), for \( x, y \in S, \) and \( \omega \) be a weight function on \( S \). Then \( \omega^{1} \leq 1 \) and
\[
M_{d}^{(S, \omega)} = L^{1}(S, \omega) \oplus \{\lambda \bar{\lambda} : \lambda \in IR\}.
\]

(ii) Let \( S = [0, +\infty) \) with the usual multiplication [resp.,

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