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ON COMMUTATIVE GELFAND RINGS

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Abstract

A ring is called a Gelfand ring (*pm* ring) if each prime ideal is contained in a unique maximal ideal. For a Gelfand ring R with Jacobson radical zero, we show that the following are equivalent: (1) R is Artinian; (2) R is Noetherian; (3) R has a finite Goldie dimension; (4) Every maximal ideal is generated by an idempotent; (5) *Max* (R) is finite. We also give the following results:an ideal of R is uniform , if and only if, it is a minimal ideal; *Ass (R)* is exactly the set of all maximal ideals which are generated by an idempotent element of R.

Introduction

A ring is called a Gelfand ring (*Pm* ring) if each prime ideal is contained in a unique maximal ideal. For a commutative ring R, Demarco and Orsatti [4] show: R is pm, if and only if, *Max* (R)is Hausdorff, and if *Spec* (R) is normal. Also Shu-Hao Sun [10] studied the above equivalence for a noncommutative ring (see [1] and [3]). The purpose of this paper is to study the structure of these rings.

Throughout this paper, R is assumed to be a commutative ring with an identity. We denote *Spec*(R), Max (R) *and Min*(R) for the species of prime and *Min (R)* ideals, maximal ideals and minimal prime ideals of R, [4], [5] and [6]). Also, we denote $\rho_0(R)$, $\mathcal{M}_o(R)$, and $I_o(R)$ for the set of the isolated points of *Spec* (R) , *Max* (R) and *Min* (R), respectively. For a semiprimitive Gelfand ring R, we show that

$$\rho_o(R) = \mathcal{M}_o(R) = I_o(R) = Ass(R).$$

For each $a \in R$ and $M \in Max(R)$, let $\mathcal{M}(a) = \{M \in Max(R): a \in M\}$ and $O_M = \bigcap_{P \subseteq M} P$, where P ranges over all prime ideals contained in M. One can easily see that in a semiprimitive Gelfand ring R, $O_M = \{a \in R : M \in int\ \mathcal{M}(a)\}$ and for any $P \in Spec(R)$, $P \subseteq M$, if and only if, $O_M P \subseteq$ (*intM (a)* is the interior of the set $\mathcal{M}(a)$ in the space *Max* (R)).

Let J (R)be the Jacobson radical (intersection of all maximal ideals) of R, if $J(R) = 0$ then R is called semiprimitive. We note that the spaces *Max* (R) and *Max* (R/J) are homeomorphic; hence, *Max* (R) is Hausdorff if *Max* (R/J) is Hausdorff. Therefore, we adopt the blanket assumption that $J = 0$, i. e. *Throughout this paper we suppose R is semiprimitive.*

A non-zero ideal in R is said to be essential if it intersects every non-zero ideal non-trivially and the intersection of all essential ideals or the sum of all minimal ideals, is called the *socle* (see [9]). We denote the socle of R by $S(R)$.

A set $\{B_i\}_{i \in I}$ of non-zero ideals in R is said to be independent if $B_i \cap (\Sigma_{i \neq j \in I} B_j) = (0)$, i. e., $\Sigma_{i \in I} B_i = \oplus_{i \in I} B_i$. Then we say R has a finite Goldie dimension if every independent set of non-zero ideals is finite. An ideal I in R is said to be uniform if any two non-zero ideal contained in I intersect non-trivially. These ideals are related to finite Goldie dimension, in fact, if R has a finite Goldie dimension,

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then there is a finite direct sum of uniform ideals which is essential in R. (see 9).

Let M be an R-module. A prime ideal P is called an associated prime ideal of M if P is an annihilator of some $m \in M$, i. e., $P = Ann(m) = \{ r \in R : rm = 0 \}$. (See [8]). The set of associated primes of M is written Ass (M), in fact

$Ass(M) = \{ Ann(m) : m \in M \} \cap Spec(R)$.

It is known that in a reduced ring R, $Ass(R)$ is exactly the set of all non-essential prime ideals of R.

Gelfand Rings

In this section, we give the following examples:

Example 1. Every local ring is a Gelfand ring. Direct summand and direct product of Gelfand rings are also Gelfand rings.

Example 2. Let X be a topological space and $C(X)$ be the ring of all continuous real functions on X, then $C(X)$ is a Gelfand ring.

Example 3. Every zero-dimensional ring (i. e, every prime ideal is maximal) is a Gelfand ring, in particular, Artin rings and Von Neumann regular rings are Gelfand rings.

The following lemma is probably well known (See proposition (1. 2) in [7].

Lemma2. 1. Let R be a semiprimitive ring and $M \in Max(R)$, then $M \in \mathcal{M}_o(R)$, if and only if, $M = (e)$, for some idempotent elements of R. Furthermore, I is a minimal ideal of R if and only if $I \oplus M = R$, for some $M \in \mathcal{M}_o(R)$.

Proof. Let $M = (e)$, where $e^2 = e$, then $\{M\} = Max(R) - \mathcal{M}(1-e)$. Conversely, if $\{M\}$ is open in $Max(R)$ then $I = \cap M'$. $M' \neq M\} \neq (0)$ implies $I \cap M = (0)$ and $I \oplus M = R$, i. e., M is generated by an idempotent.

The following shows a one - one correspondence between the minimal ideals of R and the isolated points of $Max(R)$.

Lemma 2. 2. Let R be a semiprimitive ring and I a non - zero ideal, then I is a minimal ideal of R, if and only if, I

$= \bigcap_{M' \in Max(R)-\{M\}} M'$, for a unique maximal ideal $M \in M_o(R)$.

Proof. If I is a minimal ideal of R then $I = (e)$, where e is an idempotent. So $M = (1-e)$ is a maximal ideal and we have $\mathcal{M}(e) = Max(R) - \{M\}$. Hence $I = \bigcap_{M' \in M(e)} M$. The converse follows from Lemma 1.

Proposition 3. Let R be a reduced ring, then

(1) If $S \subseteq Min(R)$ is dense in $Min(R)$, then $Ass(R) \subseteq S$.

(2) $P \in \rho_o(R)$ if and only if $P \in I_o(R)$ and P is not the intersection of the prime ideals which contains it strictly.

(3) $I_o(R) = Ass(R)$.

In particular, if R is semiprimitive we have

(4) $\rho_o(R) = \mathcal{M}_o(R)$.

Proof. (1) Suppose $P \in Ass(R)$, hence $P = Ann(a)$, for some $a \in R$, Therefore $P = Ann(a) \bigcap_{Q \in s\text{-}V(a)} Q$, where $V_{(a)} = \{P \in Spec(R): a \in P\}$. This implies that $P = Q$, for some $Q \in S$.

(2) Suppose $P \in \rho_o(R)$, clearly $P \in I_o(R)$. Now if $P = \bigcap_{Q \in V(P)-\{P\}} Q$, then $\bigcap_{Q \in Spec(R)-\{P\}} Q \subseteq P$, i.e., $P \notin \rho_o(R)$, a contradiction. Conversly, suppose $P \in I_o(R)$ and $P \neq \bigcap_{Q \in V(P)-\{P\}} Q$, there exists $a \in \bigcap_{Q \in Min(R)-\{P\}} Q - P$ and $b \in \bigcap_{Q \in V(P)-\{P\}} Q\text{-}P$, thus, we have $ab \in \bigcap_{Q \in Spec(R)-\{P\}} Q\text{-}P$, i. e., $P \in \rho_o(R)$.

(3) Assume that $P \in I_o(R)$, there exists $a \in \bigcap_{Q \in Min(R)-\{P\}} Q\text{-}P$, hence $P = Ann(a) \in Ass(R)$. Conversly, let $P \in Ass(R)$ so $P = Ann(a)$, for some $a \in R$. In contrast, suppose $P \notin I_o(R)$, put $S = Min(R) - \{P\}$. It is observed that S is dense in $Min(R)$ and (1) implies that $Ass(R) \subseteq S$, consequently $P \in S$ is a contradiction.

(4) Suppose $M \in \mathcal{M}_o(R)$, then $M = (e)$ where e is an idempotent element of R. Hence, for any $M \neq P \in Spec(R)$, $1-e \in p$ This means that $\bigcap_{Q \in Spec(R)-\{M\}} P \not\subseteq M$, i. e., $M \in \rho_o(R)$. conversely, trivial.

Theorem 4. Let R be a semiprimitive Gelfand ring, then

$\rho_o(R) = \mathcal{M}_o(R) = I_o(R) = Ass(R)$.

Proof. By propositon, it is sufficient to prove that $M_o(R)$

$= I_o(R)$. Let $P \in I_o(R)$, then for a unique maximal ideal, M' $\in Max(R)$, $P \leq M'$. Therefore $\bigcap_{M \in Max(R)-\{M'\}} O_M \not\subset P$.

This means that $\bigcap_{M \in Max(R)-\{M'\}} O_M \neq (0)$, Hence there exists $0 \neq e \in \bigcap_{M \in Max(R)-\{M'\}} M \neq (0)$. Observe that $e \notin M'$, thus M' is an isolated point of *Max* (*R*), consequently $P = M' \in \mathcal{M}_o(R)$, conversely, trivial.

Corollary 5. In a semiprimitive Gelfand ring *R*, every prime ideal is either an essential ideal or an isolated maximal ideal. In particular,
$Ass(R) = \{ M \in Max(R) : M = (e)$, *where e is an idempotent*$\}$.

Proof. Evident by lemma and proposition.

The following result shows that in a semiprimitive Gelfand ring, the set of uniform ideals and the set of minimal ideals coincide. This result is known in *C* (*X*), (See[2]).

Proposition 6. Let *R* be a semiprimitive Gelfand ring and *I* be an ideal in *R*, then the following are equivalent.

(1) *I* is a uniform ideal.

(2) For any two non-zero elements $a, b \in I$, $ab \neq 0$.

(3) *I* is a minimal ideal.

Proof. (1)⇒ (2) We note $(a) \cap (b) \neq 0$ for all non - zero elements $a, b \in I$.hence, there exists $c_1, c_2 \in R$ such that $ac_1 = bc_2 \neq 0$. This shows that $abc_1c_2 \neq 0$ and therefore $ab \neq 0$.

(2) ⇒(3) By Lemma, it is sufficient to show that there is a fixed isolated point $M \in \mathcal{M}_o(R)$ such that $Max(R) - \{M\} \subseteq \mathcal{M}(a) \forall a \in I$. Now, let $0 \neq a \in I$, M' and M'' be two distinct elements in $Max(R) - \mathcal{M}(a)$ and *G*, *H* be two disjoint open sets containing M', M'', respectively. Then there are $b_1 \in \bigcap_{M \in Max(R)-G} M'-M''$ and $b_2 \in \bigcap_{M \in Max(R)-H} M$ $-M''$. Clearly ab_1 and ab_2 are non-zero elements of *R* and $ab_1ab_2 \in \bigcap_{M \in Max(R)} M = 0$, a contradiction. Next suppose that for distinct non - zero elements or *R* $a_1, a_2 \in I$, there are distinct elements $M_1, M_2 \bigcap_{M \in Max(R)} Max(R)$ such that $Max(R) - \{M_1\} \subseteq \mathcal{M}(a_1)$ and $Max(R) - \{M_2\} \subseteq \mathcal{M}(a_2)$. Then we have $a_1a_2 = 0$ which contradicts (2).

(3)⇒ (1) Trivial.

Theorem 2.7. In a semiprimitive Gelfand ring *R*, the following are equivalent.

(1) *R* is Artinian.

(2) *R* is Noetherian.

(3) *R* has a finite Goldie dimension.

(4) *Max* (*R*) is finite.

In addition, these conditions on *R* imply that the Goldie dimension of *R* is equal to the number of maximal ideals, i.e., $\dim R = Max(R)$.

Proof. The pattern of proof will be (1)⇒ (2)⇒(3)⇒ (4)⇒(1).

(1)⇒ (2) ⇒(3) It is well- known.

(3)⇒ (4) Suppose that the Goldie dimension of *R* is finite, then there is an essential ideal which is a finite direct sum of uniform ideals . Since each uniform ideal is minimal in *R*, this essential ideal is the socle of *R*. Hence the number of minimal ideals of *R* is finite. On the other hand, Lemma implies that the cardinality of $\mathcal{M}_o(R)$ is the same as the cardinality of the set of minimal ideals in *R*, Thus, $\mathcal{M}_o(R)$ is finite. Now we show that $\mathcal{M}_o(R)$ is dense in *Mas* (*R*) is dense in *Max* (*R*) . We know that *S* (*R*) is essential, so $Ann(s(R)) = (0)$. Now if $a \in \bigcap_{M \in M_o(R)} M$ then for any minimal ideal *I* of *R*, $aI = (0)$, so $a \in Ann(s(R))$ and this implies $a = 0$. Therefore $Max(R) = \mathcal{M}_o(R)$ But *Max* (*R*) is discrete and compact, hence it is finite.

(4)⇒ (1) By Chinese remainders theorem, $R \cong \prod_{M \in Max(R)} R/M$, so *R* is Artinian.

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