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AN ALGORITHM FOR FINDING THE STABILITY OF LINEAR TIME-INVARIANT SYSTEMS

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Abstract

The purpose of this paper is to show that the ideas and techniques of the classical methods of finding stability, such as the criteria of Leonhard and Nyquist, can be used to derive simple algorithm to verify stability. This is enhanced by evaluating the argument of the characteristic equation of a linear system in the neighbourhood of the origin of the complex plane along the imaginary axis.

(1) Introduction

It is evident [1] that the essential properties of the possible modes of the transient response of a linear system having a single response-variable x(t) are determined by the nature of the roots of its characteristic equation F (s) =0. As a result the stability of such a system is determined by its characteristic roots. If the characteristic roots are all located in the left hand-side of the complex s-plane, the system is said to be assymptotically stable [2]. The dynamical problem of analysing the stability of a single or multi-variable linear system is reduced to the algebraic problem of investigating the roots of the appropriate characteristic equation. If all the roots of this equation can be computed, it is evident that the stability of the system can be decided simply by examining the location of the characteristic roots in the s-plane. However, the scope of stability investigations which rely on the actual computation of characteristic roots is limited. Therefore by means of the stability criteria avaliable it is possible to decide the stability of a linear system without actually calculating the characteristic roots. One such method which does not require the actual computation of the roots or graph plotting is presented in the following sections.

Keywords: Characteristic roots, Stability

(2) Theorems

In this section we present some theorems which serve as a theoretical basis for our algorithm.

Theorem (2.1). Consider a rational algebraic function of the complex variable s, having zeros at $s=z_1, z_2, \ldots z_m$ and poles at $s=p_1, p_2, \ldots p_n$ of the form

$$H(s)=\lambda\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \qquad (2.1)$$

where λ is a constant. If a contour C such as that shown in Fig. 2. 1(a) is traversed once in the clockwise, the change in the argument of the complex function H(s) is given by $2\pi N$, where N is the number of times the origin in the H(s)-plane is encircled in the anti-clockwise sense as the contour C is traversed once in the clockwise sense. (The encirclement theorem [1].)

Proof. It is evident from [1] that if a value is assigned to $s=\sigma+i\omega$, H(s) will itself be a complex number of the form

$$H(s) = U(\sigma,\omega) + iV(\sigma,\omega).$$

In geometric terms, this means that to any point (σ,w) in the s-plane there corresponds a point (U,V) in the H(s) plane. It follows that if the point (σ,ω)

traverses the contour C in the s-plane, the point (U,V) will describe a corresponding contour T in the H(s)-plane (see Fig 2.1b).

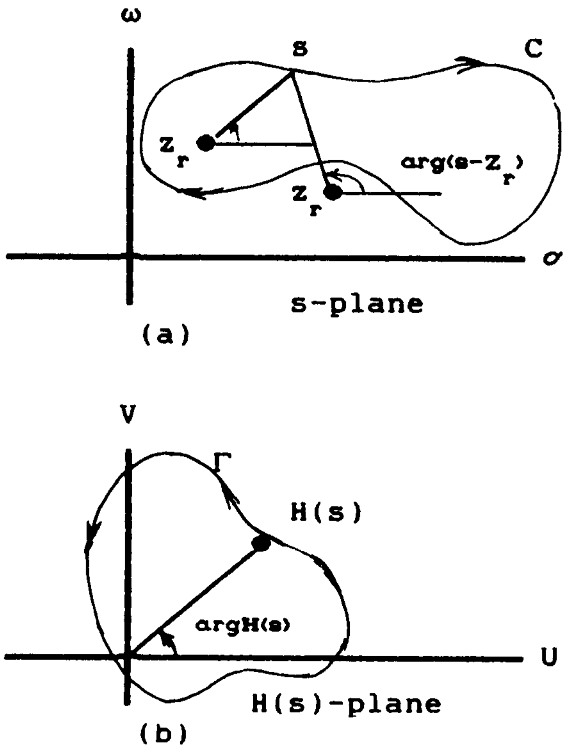


Fig. 2.1. Corresponding contours in the s-plane and the H(s)-plane

Now consider any simple closed contour C such as that shown in Fig. 2.1 (a). If C is traversed once in the clockwise sense, the change in the argument of the complex number represented by the point (U,V) in the H(s)-plane will be given by

$$\Delta \arg H(s) = \sum_{r=1}^{m} \Delta \arg (s-z_r) - \sum_{r=1}^{n} \Delta \arg (s-p_r) \quad (2.2)$$

It is evident that $\Delta \arg (s-z_r)$ is the change in the angle of a vector drawn from z to a point s on the contour C, thus it follows that $\Delta \arg (s-z_r)$ is equal to -2π or 0 according as the zero z_r is inside or outside C (see Fig. 2.1. (a)). Hence, if Z zeros of H (s) lie within C, it may be inferred that

$$\sum_{r=1}^{m} \Delta \arg (s-z_r) = -2\pi Z$$

Similarly, if P poles of H(s) lie within C, it is evident that

$$\sum_{r=1}^{n} \Delta \arg (s-p_r) = -2\pi P.$$

It may therefore be deduced from equation (2.2) that

$$\Delta \arg H(s) = 2\pi (P-Z), \quad (2.3)$$

i.e., that

$$N = P - Z, \quad (2.4)$$

where $N = \Delta \arg H(s)/2\pi$. Since N is clearly the number of times the origin in the H(s)-plane is encircled in the anti-clockwise sense as the contour C is traversed once in the clockwise sense, the result expressed by equation (2.4) or (2.3) is called the encirclement theorem. It is important to note that this theorem is valid only if no poles or zeros of H(s) lie on the contour C. Also, provided the last condition is satisfied, the validity of the theorem is not restricted to rational algebraic functions.

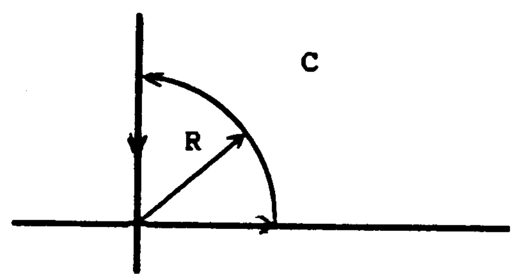
Theorem (2.2). Let F(s), the characteristic equation of a single or multi-variable linear system, be of the form

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$$

with constant real coefficients a_i. **F(s) is free of** roots in the right-hand side of the complex plane if the change in the argument of F(s) around a quadrant of radius R is zero, where

$$R = \mathrm{Sup}\left\{ |a_i| : i = 1, \dots, n \right\} + 1.$$

Proof. Consider [3]-[4] the contour C as shown in Fig. 2.2.



complex plane

Fig. 2.2. Contour C, a quadrant of a circle of radius R.

Since the coefficients of F(s) are assumed to be real, it is well known that all the zeros of F(s) will appear with conjugates [5]. Thus, for stability consideration it is sufficient to search for zeros in the upper right half plane. Since F(s) is analytic around C, for the case F(s) has no zeros on the real and the imaginary axis, by the theorem (2.1) it follows that

$$\Delta_C \arg \mathbf{F}(s) = 2\pi N \qquad (2.5)$$

where Δ_C denotes the change in the argument of **F**(s) around the contour C and N denotes the number of zeros of **F**(s) in C. For **F**(s) to have no zeros in C we require that:

$$\Delta_C \arg \mathbf{F}(s) = 0. \qquad (2.6)$$

Lemma. If **F**(s) satisfies the hypothesis of Theorem (2.2), then for the purpose of stability verification it is sufficient to evaluate the argument of **F**(s) along the imaginary axis only.

Proof. The argument of the characteristic polynomial **F**(s) remains zero along the real axis part of the contour C shown in Fig. 2.2. Therefore if there is any effective change in the argument of **F**(s), leading to multiples of 2π, it must appear while s is traversing the imaginary axis part of the contour C towards the origin of the complex plane. This means that for stability verification it is sufficient to calculate the argument of **F**(s) along the imaginary axis for y varying from R= Sup $\{|a_i| : i=1,\ldots,n\}$ +1 to zero.

The following algorithm obtains the argument of a given function **F**(s) for as many points as desired with y varying from R to zero. If the argument tends to zero as y tends to zero, then **F**(s) will represent a stable system, if it tends to 2π, then **F**(s) will represent an unstable system.

(3) Algorithm

To determine stability by finding argument of **F**(s) along the imaginary axis where s=iy, for a given R.

INPUT: R, the radius of contour C enclosing the roots of **F**(s);M, maximum number of iterations.

OUTPUT: Message of stability for zero argument or message of instability otherwise.

STEP 1 Set J=1.
STEP 2 While J ≤ M do steps 3-5.
STEP 3 Set y = R/J;
u = Re**F**(iy);
v = Im**F**(iy);
c = | u/v | ;
teta = arctan (c).
STEP 4 If u < 0 and v >0 then
teta = teta + π/2;
If u < 0 and v < 0 then
teta = teta + π;
If u > 0 and v < 0 then
teta = teta + 3π/2.
STEP 5 J = J + 1.
STEP 6 If teta < 2π then
OUTPUT ('System is stable.')
else OUTPUT ('System is unstable.
STOP.

(4) Examples

The following examples have been tested by the above algorithm.

Example 4.1

Consider. $\mathbf{F}(s)=s^3+5s^2+17s+13$

Results obtained by using the above algorithm show that the change in the argument of **F**(s) is zero. Thus **F**(s) represents a stable system.

Example 4.2

Consider $\mathbf{F}(s) = s^2-s+\exp(-0.5s)$

which represents the characteristic equation of a linear time-delayed system [6]. The results obtained by using the above algorithm show that the change in the argument of **F**(s) is 2π; thus **F**(s) represents an unstable system.

(5) Conclusion

The extent to which the Algorithm can be applied can now be assessed very simply on the basis of equation (2.5) and the stability condition (2.6). Thus, provided that the characteristic function is such that the conditions of Theorem (2.2) are satisfied, then the Algorithm presented in this paper indicates a simple method of determining stability.

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