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A CHARACTERIZATION OF EXTREMELY AMENABLE SEMIGROUPS

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Abstract

Let S be a discrete semigroup, m (S) the space of all bounded real functions on S with the usual supremum norm. Let $A \subset m(S)$ be a uniformly closed left invariant subalgebra of m (S) with $1 \epsilon A$. We say that A is extremely left amenable if there is a multiplicative left invariant mean on A. Let $P_A = \{h \epsilon A: h = |g - l_s g| \text{ for some } g \epsilon A, s \epsilon S\}$. It is shown that A is extremely left amenable if and only if there is a mean φ on A such that $\varphi(P_A) = 0$.

1. Introduction

Let S be a semigroup and m (S) the Banach algebra of all bounded real-valued functions on S with supremum norm. If $f \epsilon m(S)$ and $s \epsilon S$, let $l_s f(x) = f(sx)$ for any $x \epsilon S$.

Let $A \subset m(S)$ be a uniformly closed left invariant (i.e. $l_s f \epsilon A$ for any $f \epsilon A$ and $s \epsilon S$) subalgebra of m (S) with $1 \epsilon A$ (1 is the constant one function on S). A linear functional $\varphi \epsilon A^*$ (the continuous dual of A) is a mean if $\varphi(f) \geq 0$ for any $f \geq 0$ $f \epsilon A$ and $\varphi(1) = 1$. This is equivalent to the condition that

$$\text{Inf}\{f(x): x \epsilon S\} \leq \varphi(f) \leq \text{Sup}\{f(x): x \epsilon S\}$$

for all $f \epsilon A$.

We say that the subalgebra A is extremely left amenable (ELA) if there is a multiplicative left invariant mean on A, i.e. a mean φ on A such that $\varphi(l_s f) = \varphi(f)$ and $\varphi(fg) = \varphi(f)\varphi(g)$, for all $f, g \epsilon S$ and all $s \epsilon S$. Denote by P_A the set of all $h \epsilon m(S)$ of the form $h = |g - l_s g|$, for some $g \epsilon A$, $s \epsilon S$, also let H_A be the set of all $h \epsilon A$ which have a representation $h = \Sigma_{j=1}^{n} (f_j - l_{s_j} g_j)$, for some $f_j, g_j \epsilon A, S_j \epsilon S 1 \leq j \leq n$. In case $A = m(S)$ we denote P_A by P and if m (S) is ELA, we say that S is ELA.

Extremely left amenable semigroups were introduced for the first time by T. Mitchell [6] and later on studied by E. Granirer [3], [4], [5], and recently by J. C. S. Wong [7].

Key words: Invariant means, Extremely Amenable Semigroups

2. Basic Results

First we offer a Lemma.

Lemma 2. 1. Let A be a uniformly closed subalgebra of m (S).

(i) A is a lattice, if in addition A is left invariant then $P_A \subseteq P$.

(ii) If $f \epsilon A$ and $f \geq 0$, then $\sqrt{f} \epsilon A$.

Let $\varphi \epsilon A^*$ be a mean, then

(iii) $|\varphi(fg)|^2 \leq \varphi(f^2)\varphi(g^2)$, for all $f, g \epsilon A$.

(iv) $\varphi(|f|) = 0$ implies that $\varphi(f) = 0$, for all $f \epsilon A$.

Proof. (i) That A is a lattice is known by [2], hence if in addition A is left invariant, then P_A A.

(ii) Let $m_c(S)$ be the space of bounded complex-valued functions on S with supremum norm. With conjugate as involution, $m_c(S)$ is a C^*-algebra. Now A + iA is a closed subalgebra of $m_c(S)$. If we consider f as an element of the C^*-algebra A + iA, it is easy to see that the spectrum of f is contained in $[0.\infty)$, in fact if $\lambda \notin [0,\infty)$ then,

$$\frac{1}{|\quad|} \leq \frac{1}{\text{Im}\lambda} \qquad \text{if Im}\lambda \neq 0$$

$$\frac{}{|f-\lambda|} \leq -\frac{1}{\lambda} \qquad \text{if Im}\lambda \neq 0$$

So by [1, proposition 3.5], $f = g^2$ for some self-adjoint, hence real-valued function g. Therefore $\sqrt{f} = g \epsilon A$.

(iii) Similar to the proof of Cauchy - Schwarz in equality.

(iv) If $\varphi(|f|) = 0$, then $\varphi(f^+ + f^-) = 0$, so $\varphi(f^+) = \varphi(f^-) = 0$ i.e. $\varphi(f) = 0$.

Theorem 2.2 Let A be a uniformly closed left invariant subalgebra of m (S) with $1 \in A$. Then A is ELA if and only if there is a mean $\varphi \in A^*$ such that $\varphi(p_A) = 0$.

Proof. Suppose A is ELA and let φ be a multiplicative left invariant mean on A, then $\varphi(f - l_s f)^2 = 0$, for all $f \in A$, $s \in S$. So by Lemma 2.1, with f replaced by $|f - l_s f|$ and g replaced by 1, we obtain,

$$(\varphi(|f - l_s f|))^2 \le \varphi(f - l_s f)^2 = 0,$$

hence $\varphi(P_A) = 0$.

Conversely, suppose there is a mean $\varphi \in A^*$ such that $\varphi(P_A) = 0$. By parts (i) and (ii) of Lemma 2.1, we have $|g - l_s g|^{1/2} \in A$, for all $g \in A$, $s \in S$, therefore by Lemma 2.1 (iii) we have,

$$|\varphi(|g - l_s g|^{1/2} |g - l_s g|^{3/2})|^2 \le \varphi(|g - l_s g|^3)$$

i.e. $\varphi(g - l_s g)^2 = 0$. Now another application of Lemma 2.1 (iii) shows that

$$(\varphi(|f(g - l_s g)|))^2 \le \varphi(f^2)\,\varphi(g - l_s g)^2 = 0$$

for all $f \in A$. Hence by Lemma 2.1 (iv), $\varphi(f(g - l_s g)) = 0$ i.e. $\varphi(H_A) = 0$, therefore H_A is not dense in A, so by [4, Lemma 3], A is ELA.

Corollary 2.3. S is ELA if and only if there is a mean φ on m (S) such that $\varphi(p) = 0$.

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