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CHEBYSHEV SUBALGEBRAS OF JB-ALGEBRAS

A.K. Mirmostafaee[1] and A. Niknam[2]

[1]*Department of Mathematics, Damghan College of Sciences, Damghan 36715/364 Islamic Republic of Iran*
[2]*Department of Mathematics, Faculty of Sciences, Islamic Azad University of Mashad, Mashad, Islamic Republic of Iran*

Abstract

In this note, we characterize Chebyshev subalgebras of unital JB-algebras. We exhibit that if B is Chebyshev subalgebra of a unital JB-algebra A, then either B is a trivial subalgebra of A or $A= H \oplus \mathrm{R}.1$, where H is a Hilbert space.

1. Introduction

An important characterization for Chebyshev subspaces of $C(X)$ is due to Haar and characterizes finite dimensional Chebyshev subspaces of $C(X)$, where X is a compact Hausdorff space ([6], p. 215). Following his work on non-commutative cases, Chebyshev subspaces of C^*-algebras were studied in [3-5]. In this paper, we characterize Chebyshev subalgebras of JB-algebras. In section 3, we will show that if B is a Chebyshev subalgebra of a unital JB-algebra A, then B is also unital and either A is of the form $H \oplus \mathrm{R}.1$, where H is a Hilbert space, or B is a trivial subalgebra of A. In particular, Chebyshev subalgebras of alternative JB-algebras are trivials. The paper is divided into three sections. In section 2 we give a few preliminaries and section 3 contains the main results of the paper.

2. Preliminaries

Let (X, d) be a metric space, a subset $Y \subset X$ is called *Chebyshev* (*semi-Chebyshev*), if every point x in X admits a unique (at most one) nearest point in Y. This point is called the *best approximation* to x and is denoted by $P_Y(x)$.

A non-associative algebra A, over $\Phi(\Phi=\mathrm{R}$ or $\Phi=\mathbb{C})$ is called a *Jordan algebra* if for each $a,b\in A$, $a.b=b.a$ and $a^2.(a.b)= a.(a^2.b)$. If B is an associative algebra, we define the *Jordon product* of two elements $a,b\in B$ by the rule

$$a.b=\frac{1}{2}(ab+ba)$$

In terms of this product, B becomes a Jordan algebra denoted by B^J. A Jordan algebra which is isomorphic to a subalgebra of B is called a *special Jordan algebra*. The following theorem plays an important role in our computation ([2], p. 34).

Theorem 2.1 (The Shirshov-Cohn theorem). Any Jordan algebra generated by two elements (and 1, if unital) is special.

For a,b,c in a Jordan algebra A, define the *Jordan triple product* $U_{a,c}(b) = \{abc\}$ as follows:

$$\{abc\} = a.(b.c) - b.(a.c) + c.(a.b)$$

In any Jordan algebra A the identity:

$$\{abc\}.d = \{(a.d)bc\} + \{ab(c.d)\} - \{a(b.d)c\},$$

holds for each a,b,c and d in A ([2], p. 36). If an element

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a in a Jordan algebra A is invertible, then $U_{a,a}= U_a$ is invertible as an operator on A, and $(U_a)^{-1}= U_{a^{-1}}$ ([1], p. 19).
Let e be an idempotent in a unital Jordan algebra A, then A has a representation of the form

$$A= \{eAe\} \oplus \{eA(1-e)\} \oplus \{(1-e) A(1-e)\},$$

which is called the *Peirce decomposition* of A corresponding to e. Furthermore, we have the following (Peirce) multiplication rules ([2], p. 49):

$$\{eAe\}.\{(1-e)A(1-e)\}= 0 \ \& \ \{eA(1-e)\}.\{eA(1-e)\} \subset \{(1-e)A(1-e)\}\oplus\{eAe\}.$$

A *JB-algebra* is a Jordan algebra A with a complete norm $\|.\|$ satisfying the following properties:

$$\|a.b\| \leq \|a\| \|b\| \ \& \ \|a\|^2 \leq \|a^2 + b^2\|.$$

If A is a unital JB-algebra and $a \in A$, an element $a \in A$ is called *invertible* with b as an *inverse*, if $a.b = 1$ and $a^2.b = a$. This notion reduces to the customary one for special Jordan algebras, by virtue of the equivalence ([1], p. 17)

$$a.b = 1, a^2b = a \Leftrightarrow ab = ba = 1.$$

We denote by $C(a)$ the smallest norm-closed JB-algebra of A containing a and 1. Then $C(a)$ is associative. We define the *spectrum* of a, denoted by $sp(a)$, to be the set of $\lambda \in R$, such that $a-\lambda 1$ does not have an inverse in $C(a)$. The following conditions are equivalent ([1], Proposition 2.4):
(i) a is invertible with inverse b in the Jordan algebra A,
(ii) a is invertible with inverse b in the Banach algebra $C(a)$.
The reader may consult [1] or [2] for more information about Jordan algebras.

3. Results

The following result can be easily obtained from the analogue one for C^*-algebras.

Lemma 3.1 ([5] Lemma 1.1). Let A be a unital JB-algebra, and let B be a non zero JB-subalgebra of A. If B is semi-Chebyshev in A, then $1 \in B$.

Hereafter, we will assume that A is a JB-algebra and B is a Chebyshev subalgebra of A.

Lemma 3.2. If A is unital and $x \in B$ is not invertible, then $U_x A \subset B$.

Proof. Let $\varepsilon > 0$. Choose continuous functions f, g and h on $sp(x)$ such that f,g vanish in a neighbourhood of 0, $h(0) = 1, fg = f, gh = 0$ and $|f(t)-t|<\varepsilon$, for each $t \in sp(x)$. (Note that such functions exist, for example, let for each $t \in sp(x)$:

$$f(t)=\begin{cases} t & \text{if } |t| \geq \frac{\varepsilon}{2} \\ 3t \pm \varepsilon & \text{if } \frac{\varepsilon}{3} \leq \pm t \leq \frac{\varepsilon}{2}, \\ 0 & \text{if } |t| \leq \frac{\varepsilon}{3} \end{cases}$$

$$g(t)=\begin{cases} 1 & \text{if } |t| \geq \frac{\varepsilon}{3} \\ \frac{\pm 12t}{\varepsilon} - 3 & \text{if } \frac{\varepsilon}{4} \leq \pm t \leq \frac{\varepsilon}{3} \\ 0 & \text{if } |t| \leq \frac{\varepsilon}{4} \end{cases}$$

and h be a continous function, obtained by the Uryshon's lemma, such that $supp(h) \subset (-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})$, $h(0) = 1$ and $-1 \leq h \leq 1$, then f, g and h satisfy the required conditions). Put $y = f(x)$ $e = g(x)$, $z = h(x)$. Therefore $y.e = y$, $e. z = 0$ and $\|y - x\| < e$. Since $o \in sp(x)$, $z \neq 0$, we may assume that $\|e\| \leq \|z\| = 1$. If $\{yAy\} \not\subset B$, take $a = \{ya'y\} \in U_y(A)\backslash B$ and let $b \in B$ be the best approximation to a. Using Theorem 2.1 for the subalgebra generated by x and a', we see that $e.a= a$, $e^2.a=a$; hence

$$\|a-b.e\| = \|a.e - b.e\| \leq \|a-b\|$$

and

$$\|a-b.e^2\|= \|a.e^2-b.e^2\| \leq \|a-b\|$$

Since b is the best approximation to a, these inequalities imply that $b.e = b$ and $b.e^2 = b$, hence $b = \{ebe\}$ and $U_e(a-b) = a-b$. Choose $0<\lambda< \|a-b\|$. By considering the subalgebra generated by $a-b$ and z and applying Theorem 2.1, we see that $U_e(a-b).z = 0$ and, therefore, the subalgebra generated by $U_e\{a-b\}$ and z is associative, thus we have

$$\|a-(b+\lambda z)\|= \|(a-b)-\lambda z\|= \|U_e(a-b)-\lambda z\|= \|a-b\|.$$

This implies that $z = 0$, which is a contradiction, so $U_y A \subseteq B$. With $\varepsilon = \frac{1}{n}$, we can find a sequence $\{y_n\} \subset B$, such that $y_n \to x$ and $\{y_n A y_n\} \subset B$ for all n. Hence $\{xAx\} \subset B$.

Lemma 3.3. If $\{0,1\} \subset sp(x)$ for some $x \in B$, then $A.(x(1-x)) \subset B$.

Proof. Since $o \in sp(x)$, $U_x A \subset B$. The identity (see section 2)

$$\{xax\}.y = 2\{(x.y)ax\}-\{x(a.y)x\},$$

and Lemma 3.2 imply that $\{(x.y)ax\} \in B$, for each $y \in B$, $a \in A$. Hence $\{[x(1-x)]Ax\} \subset B$. Similarly, since $o \in sp(1-x)$, we have $\{(x(1-x))A(1-x)\} \subset B$. By the identity

$$\{(x(1-x))ax\} + \{(x(1-x))a(1-x)\} = a.(x(1-x))$$

we have $a.(x(1-x)) \in B$, for each $a \in A$. This proves the lemma.

Lemma 3.4. If $sp(x)$ contains more than two points for some $x \in B$, then $B = A$.

Proof. Take $\lambda_0 \in sp(x)$, by assumption there are at least two more points $\lambda_1, \lambda_2 \in sp(x)$. By the Uryshon's lemma there are continuous functions f_1, f_2 on $sp(x)$ such that $f_1(\lambda_0) = f_1(\lambda_2) = 1, f_1(\lambda_1) = 0, f_2(\lambda_1) = f_2(\lambda_0) = \frac{1}{2}$ and $f_2(\lambda_2) = 1$. Put $f = f_1 f_2$, then $f(\lambda_0) = \frac{1}{2}, f(\lambda_1) = 0$, and $f(\lambda_2) = 1$. Let $g = f(1-f)$ and note that $g(\lambda_0) = \frac{1}{4}$. By Lemma 3.3, $g(x).A \subset eB$. Applying this argument to every point in $sp(x)$, for each $\lambda \in sp(x)$, we can find $f_\lambda \in C(sp(x))$, such that $f_\lambda(\lambda)(1-f_\lambda(\lambda)) = g_\lambda(\lambda) > 0$, and $g_\lambda(x).A \subset B$. By compactness of $sp(x)$, one can find a finite set $\{g_i\}_{i=1}^{n} \subset C(sp(x))$ with $\sum_{i=1}^{n} g_i(\lambda) > 0$ for all $\lambda \in sp(x)$ such that $g_i(x).A \subset B$, then $y = \sum_{i=1}^{n} g_i(x)$ is an invertible element in B and $y.A \subset B$. Since

$$\{f_i(x)(f_j(x)(1-f_j(x)))A\} = \{[f_i(x)f_j(x)(1-f_j(x)]Af_j(x)\} + \{f_i(x)f_j(x)(1-f_j(x)A(1-f_j(x))\},$$

we have $(f_i(x)g_j(x)).A \subset B$, hence $(f_i(x)(y)).A \subset B$. Since $\{(yf_i(x)Af_i(x)\} \subset B$ and for every $a \in A$,

$\{[y.(1-f_i(x))]a(1-f_i(x)\} = y.a - \{yaf_i(x)\} - (y.f_i(x)).a + \{(yf_i(x))af_i(x)\}$, one can easily see that $\{yaf_i(x)\} \in B$. Also

$$\{[f_i(x)(1-f_i(x))]ay\} = \{f_i(x)ay\} - \{f_i^2(x)ay\}$$

By applying the same argument for f_i^2 as used for f_i, it follows that $\{g_i(x)ay\} \in B$ for each $a \in A$. Hence $\sum_i \{g_i(x)ay\} \in B$, i.e. $\{yay\} \in B$ for each $a \in A$. Hence

$$a = U_{y^{-1}} U_y(a) \in B, \forall a \in A, i.e. A = B.$$

Now, we are ready to state the main result of the paper:

Theorem 3.1. If A is a JB-algebra with unit, and if B is a Chebyshev JB-subalgebra of A, then either B is a trivial subalgebra of A or $A = H \oplus R.1$, for some Hilbert space H, with dim $(H) \geq 2$.

Proof. If $B \neq A$ and $B \neq R.1$, then Lemma 3.4 shows that there is an element $e \in B$, such that $sp(e)$ contains two points, we may assume that $sp(e) = \{0,1\}$, so that e is a non-trivial projection. Lemmas 3.2 and 3.1 show that $\{eAe\} \subset B$ and $\{(1-e)A(1-e)\} \subset B$. Thanks to the Shirshov-Cohn theorem, for every $y \in \{eAe\}$, the subalgebra generated by 1, y and e is associative and $y.e = y$. Since $sp(y)$ doesn't have more than two elements, using spectral theory, we see that there exists some $\alpha \in R$, such that $y = \alpha e$. So that $\{eAe\} = Re$. Similarly $\{(1-e)A(1-e)\} = R(1-e)$. Using the Peirce decomposition of A corresponding to e, we have

$$A = Re \oplus \{(1-e)Ae\} \oplus R(1-e).$$

Let $x \in \{(1-e)Ae\}$, then by Peirce multiplication rules, we see that

$$x^2 = \lambda e + \mu 1, (\lambda, \mu \in R).$$

If $\lambda \neq 0$, then $e = \lambda^{-1}(x^2 - \mu.1)$ belongs to the associative subalgebra generated by x and 1, but in this case, $x = \{(1-e)xe\} = 0$, contradiction. So that $x^2 = \mu.1$. The identity

$$x.y = \frac{1}{2}[(x+y)^2 - x^2 y^2], \quad (x, y \in A),$$

shows that $x.y = \lambda 1$ for each $x,y \in \{(1-e)Ae\}$. Thus, we may define an inner product on $\{(1-e)Ae\}$ by

$$\langle x,y \rangle 1 = x.y.$$

Since $s = 1-2e$ is a symmetry (i.e. $s^2 = s$) and $s.x = 0$, $\forall x \in \{(1-e)Ae\}$, this inner product can be extended to $H = \{(1-e)Ae\} \oplus Rs$. Thus $A = H \oplus R.1$. Note that dim $H \geq 2$, for if $\{(1-e)Ae\} = R.1$, then $B = A$.

Example 3.1. Let $A = H_2(R)$, the set of all Hermitian 2×2 matrices with entries in R. Then, with respect to the Jordan product, A is a non-associative JB-algebra. Following the proof of Theorem 3.1, one can easily see that A has a representation of the form $A = H \oplus Rl$. Routine calculations show that the unique best approximation to

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} \text{ is } \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

Thus B, the set of all diagonal matrices with entries in R, is a Chebyshev JB-subalgebra of A.

An *alternative Jordan algebra* is a Jordan algebra for which $a^2.b = a.(a.b)$. It is known that if x and y are in an alternative algebra, then the subalgebra generated by $\{1, x, y\}$ is associative ([2], p. 21).

Corollary 3.1. If B is a Chebyshev subalgebra of unital alternative JB-algebra, then either $B = A$ or $B = R.1$.

Proof. If $B \neq R.1$ and $B \neq A$, the above theorem asserts that A has a representation of the form

$$A = H \oplus R.1,$$

where $H = \{1\text{-}e)Ae\} \oplus R(1\text{-}2e)$ and $1, e \in B$. Since A is alternative, for each $a \in A$, the subalgebra generated by $\{1, a, e\}$ is associative, thus $\{(1\text{-}e)ae\} = 0$ and therefore $B = A$, contradiction.

The complex version of JB-algebras are called *JB^*-algebras* or *Jordan-C^* algebras*. Their formal definition is as follows. Let A be a complex Banach space which is a complex algebra equipped with an algebra involution*. Then A is a JB^*-algebra if the following three conditions are satisfied for all $a, b \in A$:

(i) $\|a.b\| \leq \|a\| \, \|b\|$,
(ii) $\|a^*\| = \|a\|$,
(iii) $\|\{aa^*a\}\| = \|a\|^3$.

It is easily verified that if B is a *-closed Chebyshev subset of a JB^*-algebra A, then $P_B(a^*) = P_B(a)^*$. Therefore if B is a Chebyshev subalgebra of A, then the self-adjoint elements of B are a Chebyshev subalgebra of self adjoint elements of A, which is a JB-algebra ([2], p. 91). Thus we have the following result.

Corollary 3.2. Let A be a unital JB^*-algebra which is also alternative. If B is a Chebyshev subalgebra of A, then B is a trivial subalgebra of A.

An element $b_0 \in B$ is said to be a best simultaneous approximation of the pair $a_1, a_2 \in A$ if and only if for each $b \in B$:

$$\max (\|a_1 - b_0\|, \|a_2 - b_0\|) \leq \max (\|a_1 - b\|, \|a_2 - b\|)$$

Corollary 3.3. If B is a JB-subalgebra of a unital JB-algebra A, and if each pair $a_1, a_2 \in A$ has unique best simultaneous approximation in B, then either $A = H \oplus R.1$, for some Hilbert space H or $B = R.1$.

Proof. Let $A_1 = A \times A$, with pointwise operations and with the norm

$$\|(a_1, a_2)\| = \max (\|a_1\|, \|a_2\|)$$

It is easy to see that A_1 is a unital JB-algebra. Identifying B with

$$B_1 = \{(b,b) : b \in B\}$$

one can see that B is a Chebyshev subalgebra of A_1 since A has more than one point $A_1 \neq B$. So the corollary is established by theorem.

References

1. Alfsen, E.M., Shultz, F.W. and Stormer, E. A Gelfand-Neumark theorem for Jordan algebras. *Adv. Math.*, **28**, 11-56, (1978).
2. Hanche-Olsen, H. and Stormer, E. *Jordan operator algebras.* Monographs Stud. Math. 21, Pitman, Boston-London-Melbourne, (1984).
3. Pedersen, G.K. Chebyshev subspaces of C^*-algebras. *Math Scand.*, **45**, 147-156, (1979).
4. Robertson, A.G. Best approximation in von Neumann algebras. *Proc. Cambridge Philos. Soc.*, **81**, 233-236, (1977).
5. Robertson, A.G. and Yost, D. Chebyshev subspaces of operator algebras. *J. London Math. Soc.*, **19**, (2), 523-531, (1979).
6. Singer, I. *Best approximation in normed linear spaces by elements of linear subspaces.* Springer Verlag, (1977).