SOME POINTS ON CASIMIR FORCES

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Abstract

Casimir forces of massive fermionic Dirac fields are calculated for parallel plates geometry in spatial space with dimension d and imposing bag model boundary conditions. It is shown that in the range of ma>>1 where m is mass of fields quanta and a is the separation distance of the plates, it is equal to massive bosonic fields Casimir force for each degree of freedom. We argue this equality exists for any massive anyonic field in two-dimensional spatial space. Also the ratio of massless fermionic field Casimir force to its bosonic correspondent in d-dimensional spatial space is $(1 - \frac{1}{2^d})$.

Introduction

Casimir effect of vacuum fluctuations has been extensively investigated in various contexts [1-8]. The calculation for massive bosonic field between two parallel and confining plates has been carried out [9-11].

In the range of ma>>1, the corresponding Casimir force is attractive and is equal to $\frac{m^{d/2+1}}{(4\pi a)^{d/2}}$ exq(-2ma) per degree of freedom, where d stands for the spatial space dimension. The calculation of this effect has also been carried out for zero-mass fermonic fields by Ken Johnsen and others for the first time [12-13]. It turns out that the contribution of each degree of freedom of zero-mass fermionic field in Casimir force is $\frac{7}{8}$ the contribution of the corresponding photonic field. It is well known in statistical physics that the pressure due to free fermion and boson degree of freedom at finite temperature is related by the same $\frac{7}{8}$ factor [14]. In the first part of this

article, we prove that in the range of ma>>1, the contribution of each degree of freedom of massive fermionic field in Casimir force is the same as the corresponding massive bosonic field in any spatial space dimension.

The second part of the discussion is restricted to two-dimensional spatial spaces and concludes that this equality is conserved by any massive anyonic field which exists only in a two-dimensional spatial space. The third part of the calculation is confined to massless fields in d-dimensional spatial spaces and the ratios of these forces are calculated in this case.

Massive Fermionic Fields

The Hamiltonian for a quantized massive fermionic field between two parrallel plates in d-dimensional spatial space is

$$H = (\frac{DRF}{4}) \sum_{\lambda,k} \int \frac{d^{d-1}k_T}{(2\pi)^{d-1}} [b^+_{\lambda k k_T} b_{\lambda k k_T} - d_{\lambda k k_T} d^+_{\lambda k k_T}] \omega_{k k_T}$$

(1)

where $\lambda = \pm 1$ is a spin index, k is quantized momentum normal to plates; k_T is a (d-1)-dimensional continuous transverse momentum; and $\omega_{kk_T} = (k^2 + k_T^2 + m^2)^{1/2}$ and $\hbar = 1$ is assumed. DRF stands for dimensional regularization factor [15]. Here the operator $b_{\lambda kk_T}^{+} b_{\lambda kk_T}$ counts the number of fermions in the mode (λ, k, k_T) while $d_{\lambda kk_T}^{+} d_{\lambda kk_T}$ counts the number of antiparticles in the mentioned mode. Since the $d_{\lambda kk_T}^{+} d_{\lambda kk_T}$ operator appears in the Hamiltonian, the vacuum state will have a divergent zero-point energy which gives rise to the Casimir effect. On the two confining plates, the field is required to satisfy the MIT boundary conditions [16], leading to

$$k = \frac{n\pi}{2a} \text{ with } n = 1,3,5,\dots \qquad (2)$$

The expectation value of H in vacuum state is

$$\langle 0|H|0\rangle = -\left(\frac{DRF}{4}\right) \sum_{\lambda,k} \left| \frac{d^{d-1}k_T}{(2\pi)^{d-1}} \omega_{kk_T} \right. \qquad (3)$$

In a homogeneous and isotropic d-dimensional space we have [9].

$$\int f(k)d^d k = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int k^{d-1} f(k) \, dk \qquad (4)$$

Since the (d-1)-dimensional plates are homogeneous and isotropic, (4) may be used to cast (3) into

$$\langle 0|H|0\rangle = -\frac{(DRF/4)}{2^{(d-2)}\pi^{(d-1)/2}\Gamma((d-1)/2)} \sum_{\lambda,k} \int_0^\infty k_T^{d-2} \omega_{kk_T} \, dk_T \qquad (5)$$

Replacing the integration parameter k_T by $[(k^2+m^2)t]^{1/2}$, (5) becomes

$$\langle 0|H|0\rangle = -\frac{(DRF/4)}{2^{(d-1)}\pi^{(d-1)/2}\Gamma((d-1)/2)} \sum_{\lambda,k} (k^2+m^2)^{d/2} \int_0^\infty t^{(d-3)/2} (1+t)^{1/2} dt \qquad (6)$$

Now using the integral form of beta function

$$B(1+r, -s-r-1) = \int_0^\infty t^r (1+t)^s \, dt$$

where $B(q,p) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$ $\qquad (7)$

(6) will be reduced to

$$\langle 0|H|0\rangle = -\frac{(DRF/4)\Gamma(-d/2)}{2^{d-1}\pi^{d-1/2}\Gamma(-1/2)} \sum_{\lambda,k} (k^2+m^2)^{d/2} \qquad (8)$$

$$\langle 0|H|0\rangle = -\frac{(DRF/4)\Gamma(-d/2)}{2^{d-1}\pi^{(d-1/2)}\Gamma(-1/2)} \sum_{n=1}^{\infty} \left[\frac{(\pi(2n-1))^2}{4a^2} + m^2 \right]^{d/2} \qquad (9)$$

Since

$$\sum_{n=1}^{\infty} \left[\frac{(\pi(2n-1))^2}{4a^2} + m^2 \right]^{d/2} = \sum_{n=-\infty}^{n=0} \left[\frac{(\pi(2n-1))^2}{4a^2} \, m^2 \right]^{d/2} \qquad (10)$$

(9) can be arranged in the form

$$\langle 0|H|0\rangle = -\frac{(DRF/4)\Gamma(-d/2)}{2^{d-1}\pi^{(d-1)/2}\Gamma(-1/2)} \frac{1}{2} \sum_{n=-\infty}^{+\infty} \left[\frac{(\pi(2n-1))^2}{4a^2} + m^2 \right]^{d/2} \qquad (11)$$

For determining the infinite series in (11) we compute it as follows.

Let us consider the theta function $v_2(x)$ which is defined as [17-18]

$$v_2(x) = \sum_{n=-\infty}^{+\infty} \exp\left[-\pi \left(n - \frac{1}{2}\right)^2 x \right] \qquad (12)$$

and define S as

$$S = \int_0^\infty dx \, x^{-(d+2)/2} \exp\left(-\frac{m^2}{\pi} x\right) v_2(x/a^2) \qquad (13)$$

By using the integral form of gamma function, S can be written as

$$S = \pi d/2 \, \Gamma\left(-\frac{d}{2}\right) \sum_{n=-\infty}^{+\infty} \left[\left(\frac{2n-1}{2a}\right)^2 + \left(\frac{m}{\pi}\right)^2 \right]^{d/2} \qquad (14)$$

Comparing (14) and (11) gives

$$\langle 0|H|0\rangle = 2^{-d} S \, (DRF/4) \qquad (15)$$

Thus calculating $\langle 0|H|0\rangle$ turns to finding S. Changing the integration parameter x into y^{-1} in (13), S becomes

$$S = \int_0^\infty dy \, y^{(d-2)/2} v_2\left(\frac{1}{a^2 y}\right) \exp\left(-\frac{m^2}{\pi y}\right) \qquad (16)$$

$v_2(x)$ and $v_4(x)$ are related by the relation

$$v_2\left(\frac{1}{a^2y}\right) = a\sqrt{y}\; v_4\,(a^2y)$$ (17)

where $v_4(x)$ is

$$v_4(x) = \sum_{n=-\infty}^{+\infty} (-1)^n \exp(-\pi\, n^2 x)$$ (18)

Now (16) may be rearranged by using (17) and (18) to

$$S = a\int_0^\infty dy\; y^{(d-1)/2} \exp\left(-\frac{m^2}{\pi y}\right)\left[1+\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty} (-1)^n \exp(-\pi\, n^2\, a^2 y)\right]$$ (19)

$$S = a\left(\frac{m^2}{\pi}\right)^{(d+1)/2}\int_0^\infty dt\; t^{-(d+3)/2} \exp(-t)\left[1+\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty} (-1)^n\right.$$

$$\left.\exp\left(\frac{-a^2 n^2 m^2}{t}\right)\right]$$ (20)

The integral form of gamma function may be used for the first term

$$S = a\left(\frac{m^2}{\pi}\right)^{(d+1)/2}\left[\Gamma\left(-\frac{d+1}{2}\right)+\right.$$

$$\left.\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty} (-1)^n\int_0^\infty dt\; t^{-(d+3)/2} \exp\left[-\left(t+\frac{a^2 n^2 m^2}{t}\right)\right]\right]$$ (21)

Then changing $t \to |anm|\,t'$ (21) can be written as

$$S = a\left(\frac{m^2}{\pi}\right)^{(d+1)/2}\left[\Gamma\left(-\frac{d+1}{2}\right)+\right.$$

$$\left.\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty} (-1)^n |anm|^{(d+1)/2}\int_0^\infty dt'\; t'^{-(d+3)/2} \exp\left[-|anm|\left(t'+\frac{1}{t'}\right)\right]\right]$$ (22)

The integral form of modified Bessel functions may be used to simplify (22), we have

$$K_v(x) = \frac{1}{2}\int_0^\infty \exp\left[-\frac{x}{2}\left(t+\frac{1}{t}\right)\right] t^{-(v+1)}dt$$ (23)

and with (23) we get

$$S = a\left(\frac{m^2}{\pi}\right)^{(d+1)/2}\left[\Gamma\left(-\frac{d+1}{2}\right)+\sum_{\substack{n=-\infty\\n\neq 0}}^{+\infty} (-1)^n \frac{K_{(d+1)/2}(2|anm|)}{|anm|^{(d+1)/2}}\right]$$ (24)

By applying (24) to (15) we obtain

$$<0\,|H|\,0> = \frac{a}{2^d}\left(\frac{m^2}{\pi}\right)^{(d+1)/2}\left[\Gamma\left(-\frac{d+1}{2}\right)+4\sum_{n=1}^{+\infty} (-1)^n\right.$$

$$\left.\frac{K_{(d+1)/2}(2|anm|)}{|anm|^{(d+1)/2}}\left(\frac{DRF}{4}\right)\right]$$ (25)

The first term in (25) gives rise to a force independent of "a" and can be dropped.

$$<0\,|H|\,0> = \frac{a}{2^{d-2}}\left(\frac{m^2}{\pi}\right)^{(d+1)/2}\sum_{n=1}^{+\infty} (-1)^n$$

$$\frac{K_{(d+1)/2}(2|anm|)}{|anm|^{(d+1)/2}}\left(\frac{DRF}{4}\right)$$ (26)

The asymptotic form of K_v is

$$K_v(z) \to \sqrt{\frac{\pi}{2z}}\exp(-z)\quad\text{for } z\gg 1$$ (27)

In the range $ma\gg 1$, (27) may be used in (26) and only the first term has a significant contribution and the others can be ignored. Then we have

$$<0\,|H|\,0> = -\frac{(DRF)\, m^{d/2}}{2^{d+1} a^{d/2}\pi^{d/2}}\exp(-2ma)$$ (28)

By using $F = -\dfrac{\partial E}{\partial a}$ and the approximation $ma\gg 1$, Casimir force can be obtained

$$F = -\frac{(DRF)\, m^{d/2+1}}{2^d a^{d/2}\pi^{d/2}}\exp(-2ma)$$ (29)

Since we have four degrees of freedom for fermionic fields, for finding Casimir force per each degree of freedom we should divide (29) by (DRF)

$$F_f = -\frac{m^{d/2+1}}{4\pi a^{d/2}}\exp(-2ma)\quad\text{per each degree}$$ (30)

This force is attractive and has the exact form of Casimir force of massive bosonic fields in the range $ma\gg 1$.

Casimir Force in Two Space

It is well known that planar physical systems in two space and one time dimensions display a peculiar phenomena. There exist quantum states that carry angular momentum which are not quantized in half-integer units and whose statistics are neither bosonic nor fermionic. The excitations with fractional statistics are often referred to as anyons [19]. Any non-relativistic system of anyons may be represented by a system of bosons or fermions plus an interaction which is characterized by a Chern-Simon potential. Since the corresponding Lagrangian of Chern-Simon fields is metric independent, its contribution in energy-stress tensor $T^{\mu\nu}$ vanishs. Any quantized massive field may be considered as an equivalent system of non-relativistic gas for which the Chern-Simon theory may be used to described its physical behaviour.

Due to the lack of energy-stress tensor in Chern-Simon potentials, we can conclude that it does not contribute in any way to the Casimir force. This can explain why in two dimensions we get the same result for bosonic and fermionic massive fields Casimir forces.

Now, as regards the Casimir force of massive anyonic fields, in the range of $ma \gg 1$ the quantized massive anyonic field may be considered a non-relativistic system of bosons or fermions, plus a Chern-Simon interaction term which has no contribution to the Casimir force. Thus, we can conclude that we arrive at the same result for any massive anyonic field we have obtained for bosonic and fermionic massive fields in the range $ma \gg 1$.

Massless Fermionic Field

Now, first of all we would like to calculate the Casimir force of massless fermionic field in an arbitrary spatial space dimension and in this way find the ratio of fermionic to bosonic massless Casimir force in general case, which is 7/8 in three dimensions.

Using (19) and putting $m = 0$ we get

$$S = a \int_0^\infty dy\, y^{(d-1)/2}\, [1+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} (-1)^n \exp(-\pi\, n^2\, a^2 y)] \qquad (31)$$

Again the first term may be dropped because it gives rise to a force independent of a. Changing the integration parameter to $t = \pi n^2 a^2 y$, (31) gives

$$S = a \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{(-1)^n}{(\pi\, n^2 a^2)^{(d+1)/2}} \int_0^\infty dt\, t^{(d-1)/2} \exp(-t) \qquad (32)$$

By using the integral form of gamma function, (32) may be rearranged to

$$S = \frac{2a\, \Gamma((d+1)/2)}{(\pi a^2)^{(d+1)/2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{d+1}} \qquad (33)$$

The infinite series in (31) can be replaced by the following closed form, let

$$S' = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{d+1}} = -\sum_{\substack{n \\ odd}} \frac{1}{n^{d+1}} + \sum_{\substack{n \\ even}} \frac{1}{n^{d+1}}$$

$$S' = -\sum_{\substack{n \\ odd}} \frac{1}{n^{d+1}} + \frac{1}{2^{d+1}} \sum_{n=1}^{\infty} \frac{1}{n^{d+1}} \qquad (34)$$

Now we use the zeta function which gives

$$\sum \frac{1}{n^{d+1}} = \zeta(d+1) \qquad (35)$$

and (34) becomes

$$S' = -\sum_{\substack{n \\ odd}} \frac{1}{n^{d+1}} + \frac{\zeta(d+1)}{2^{d+1}} \qquad (36)$$

On the other hand $\zeta(d+1)$ may be rearranged as

$$\zeta(d+1) = \sum_{\substack{n \\ odd}} \frac{1}{n^{d+1}} + \sum_{\substack{n \\ even}} \frac{1}{n^{d+1}} = \sum_{\substack{n \\ odd}} \frac{1}{n^{d+1}} + \frac{\zeta(d+1)}{2^{d+1}} \qquad (37)$$

So (37) gives

$$\sum_{\substack{n \\ odd}} \frac{1}{n^{d+1}} = (1 - \frac{1}{2^{d+1}})\, \zeta(d+1) \qquad (38)$$

S' can be found by putting (38) in (36), we obtain

$$S' = (1 - \frac{1}{2^d})\, \zeta(d+1) \qquad (39)$$

S can be calculated from (33) by using (39) for S'

$$S = -\frac{2a\, \Gamma((d+1)/2)}{(\pi a^2)^{(d+1)/2}} (1 - \frac{1}{2^d})\, \zeta(d+1) \qquad (40)$$

(40) and (15) give $<0|H|0>$ as

$$<0|H|0> = -\frac{a\Gamma((d+1)/2)}{2^{d-1}(\pi a^2)^{(d+1)/2}} (1 - \frac{1}{2^d})\, \zeta(d+1) (\frac{DRF}{4}) \qquad (41)$$

Casimir force may be obtained by $F = -\dfrac{\partial E}{\partial a}$

$$F = -\frac{d\,\Gamma\,((d+1)/2)}{2^{d-1}(\pi a^2)^{(d+1)/2}}\,(1-\frac{1}{2^d})\,\zeta\,(d+1)\;(DRF) \qquad (42)$$

For finding Casimir force per degree of freedom we should divide (42) by (DRF) and we get

$$F_{f\text{-massless}} = -\frac{d\,\Gamma\,((d+1)/2}{(4\pi a^2)^{(d+1)/2}}\,(1-\frac{1}{2^d})\,\zeta\,(d+1) \qquad (43)$$

Casimir force of massless bosonic field has been calculated [6]. It is equal to

$$F_{b\text{-massless}} = -\frac{d\,\Gamma\,((d+1)/2}{(4\pi a^2)^{(d+1)/2}}\,\zeta\,(d+1) \qquad (44)$$

Dividing (43) by (44) we get the final result, the ratio of fermionic to bosonic massless Casimir force in an arbitrary spatial space with dimension d

$$\frac{F_{f\text{-massless}}}{F_{b\text{-massless}}} = (1-\frac{1}{2^d}) \qquad (45)$$

In three dimensions, we get the same $\dfrac{7}{8}$ factor as mentioned before and in two dimensions this ratio is $\dfrac{3}{4}$.

Conclusion

We have shown that the Casimir forces of massive fields have the same form and the ratio of massless

fermionic to bosonic field Casimir force is $(1-\dfrac{1}{2^d})$. This number now needs only a physical explanation. We think this work will help us to find a proper relativistic theory of anyons. It is interesting to note that this ratio depends on d and is close to one for larger d. This is surprising as one would expect it to be a pure spin dependent property and independent of spatial space parameters.

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