THE STEFAN PROBLEM WITH KINETIC FUNCTIONS AT THE FREE BOUNDARY

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Abstract

This paper considers a class of one-dimensional solidification problem in which kinetic undercooling is incorporated into the temperature condition at the interface. A model problem with nonlinear kinetic law is considered. The main result is an existence theorem. The mathematical effects of the kinetic term are discussed.

1. Introduction

Mathematical models of solidification including interface kinetics effects have been considered for quite some time (see[1], and references therein). This class of free boundary problems, which arises in a number of physical situations, is that of nonequlibrium problems, in which the phase-change temperature is dependent on the velocity of the front at which the phase-change occurs (for more physical problems, see [3]-[5]). Here, we study a model problem with nonlinear kinetic law at the interface in the one-dimensional case.

Specifically, let the curve $x= s(t)$ be defined as the interface that separates the liquid and solid phases. We may write the system of equations as

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t} \text{ in } Q_1= \{(x,t)/0 <x<s(t),\ 0<t \leq T\}, \quad (1)$$

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{\partial u_2}{\partial t} \text{ in } Q_2= \{(x,t)/s(t) <x<1, 0<t \leq T\}, \quad (2)$$

and on the interface $x= s(t)$ as

$$u_1= u_2= g_1(V(t)), \quad (3)$$

$$\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x} = g_2(V(t)), \quad (4)$$

$$s(0)= b,\ 0<b<1, \quad (5)$$

These equations are subject to the initial and boundary conditions

$$u_1(x,0)= \varphi_1(x)\ 0 \leq x \leq b, \quad (6)$$

$$u_2(x,0)= \varphi_2(x)\ \ b\leq x \leq 1, \quad (7)$$

$$u_i(i-1, t)= f_i(t)\ t\geq 0\ (i= 1,2), \quad (8)$$

where

$$V(t)= \frac{ds\,(t)}{dt} \quad (9)$$

is the propagation velocity of the free boundary.

For the discussion below, we will also denote problem (1-9) as problem (p). This problem has been widely studied and the mathematical results are fairly well understood. The model in which $g_1(V(t))=\varepsilon V(t)$, ($\varepsilon$ is

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constant) and $g_2(V(t))= -LV(t)$ has been consideredc by Xie Weiging in [1]. The one-phase and one-dimensional problems are considered by Frankel and Roytburd in [2]. Our objective in this paper is to understand the mathematical effects of the kinetic term for the above model problem in the classical framework. The results here parallel completely the results that have been proved for the standard Stefan problem.

In the next section we collect the assumptions on kinetic functions and formulate the main result. In the following section we derive an integral equation for the problem (p) which is the analogous equation for the classical Stefan problem [6]. Finally we prove that the integral operator is a contraction for small times and thus we establish the local existence for a solution of the problem (p).

2. Statement of the Problem

We now introduce a set of rather general requirements on kinetic functions g_1, g_2 and initial and boundary conditions.

We assume that

(H1) g_1 is a continuously differentiable function on $(0,1)$, and $|g_1| \le M_2$, $|g'_1| \le M'_2$, $|g''_1| \le M''_2$

(H2) $g_2^{-1}(u)$ is a continuously differentiable and increasing positive function and for any given u, $|g_2^{-1}(u)| \le K$ and $|g_2^{'-1}(u)| \le \delta_1 \sqrt{\sigma_1}$, where σ_1 is a very small number in the interval $(0,1)$;

(H3) $f_i(t) \in C^1(R^1) \cap L^\infty(R^1) (i= 1,2)$, $\varphi_1(x) \in C^1[0,b]$, $\varphi_2(x) \in C^1[b, 1]$.

The main result of this paper is the following global existence theorem.

Theorem 2.1. Consider the problem (p). Suppose that the kinetic functions g_1, g_2, and initial and boundary data satisfy the assumptions in (H1)-(H3). Then there exists a solution of the free boundary problem (p).

We say that $u_1(x,t)$, $u_2(x,t)$, $s(t)$ form a classical solution of (1-9). It satisfies

(i) $s \in C^1(0,T)$.

Denoting by $Q_T \equiv Q \equiv (0,1) \times (0,T)$ and by u_i the restrictions to Q_i of $u(x,t)$,

(ii) $u_i(x,t) \in C(\overline{Q_i}) \cap C^{2,1}(Q_i)$,

$u_{ix} \in C(\overline{Q_i} \setminus \{x= i-1\}$, $i= 1,2$,

and (1-9), the functions $\varphi_i(x)$ and $f_i(t)$ $(i= 1,2)$ in (6-8) satisfying

$f_i(t) \in C^1(R^1) \cap L^\infty(R^1)$, $\varphi_1(x) \in C^1[0,b]$, $\varphi_2(x) \in C^1[b, 1]$,

and the consistency conditions

$$\varphi_1(b)= \varphi_2(b), f_2(0)= \varphi_2(1), f_1(0)= \varphi_1(0).$$

The proof of the theorem 2.1 contains two major ingredients; in the next section we reduce the problem (p) to an equivalent problem of solving a nonlinear integral equation of Volterra type for $u_{ix}(x,t)$ $(i= 1,2)$, and in the last section we prove that the integral operator is a contraction for small times.

Reducing the Problem (p) to an Integral Equation

In this section we reduce the problem (p) to an equivalent problem of solving two nonlinear integral equations of Volterra type for $u_{ix}(s(t),t)$ $(i= 1,2)$.

Lemma 1. Let $p(t)$ $(0 \le t \le \sigma)$ be a continuous function and let $s(t)$ $(0 \le t \le \sigma)$ satisfy a Lipschitz condition. Then, for every $0<t\le\sigma$

$$\lim_{x\to s(t)+0} \frac{\partial}{\partial x} \int_0^t \rho(\tau)\, K(x,t;\, s(\tau),\, \tau)d\tau =$$

$$-\frac{1}{2}\rho(t) + \int_0^t \rho(t) \left[\frac{\partial}{\partial x} K(x,\, t;\, s(\tau),\, \tau)\right]_{x=s(t)} d\tau, \quad (10)$$

where

$$K(x,t;\, \zeta,\, \tau)= \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}} \exp\left\{\frac{-(x-\zeta)^2}{4(t-\tau)}\right\}.$$

Proof. We shall first prove that for any fixed positive $\delta>t$, the integral

$$I \equiv \int_{t-\delta}^t \frac{x-s(t)}{2(t-\tau)} K(x,\, t;\, s(\tau),\, \tau)\, d\tau$$

$$-\int_{t-\delta}^t \frac{s(t)-s(\tau)}{2(t-\tau)} K(s(t),\, t;\, s(\tau),\, \tau)\, d\tau \quad (11)$$

satisfies the inequality

$$\lim_{x\to s(t)+0} \sup \left|I - \frac{1}{2}\right| \le A\delta^{1/2}; \quad (12)$$

here and in what follows, various constants which are independent of x,t,δ will be denoted by A (A may depend on σ). Writing $I = I_1 + I_2$, where

$$I_1 = \int_{t-\delta}^{t} \frac{x - s(t)}{2(t-\tau)} K(x, t; s(\tau), \tau)\, d\tau$$

$$I_2 = \int_{t-\delta}^{t} \frac{s(t) - s(\tau)}{2(t-\tau)} [K(x, t; s(\tau), \tau) - K(s(t), t; s(\tau), \tau)]\, d\tau.$$

Then by the assumption $|s(t) - s(\tau)| < A|t-\tau|$, we get

$$|I_2| \le \int_{t-\delta}^{t} \frac{d\tau}{(t-\tau)^{1/2}} \le A\delta^{1/2}. \tag{13}$$

To evaluate I_1, we introduce

$$|J_1| = \int_{t-\delta}^{t} \frac{x - s(t)}{2(t-\tau)} K(x, t; s(\tau), \tau)\, d\tau. \tag{14}$$

Then

$$J_1 - I_1 = \int_{t-\delta}^{t} \frac{x - s(t)}{2(t-\tau)} K(x, t; s(\tau), \tau) \left\{1 - \exp - \left[- \frac{(x - s(\tau))^2 - (x - s(t))^2}{4(t-\tau)}\right]\right\} d\tau.$$

Similarly by [6], we obtain

$$|J_1 - I_1| \le A\delta^{1/2}. \tag{15}$$

Now by substituting $z = (t-\tau)/(x-s(t))^2$ in (14) and noting that $x-s(t) > 0$, we get

$$J_1 = \frac{1}{4\sqrt{\pi}} \int_0^{\delta'} z^{-3/2} \exp\left[\frac{-1}{4z}\right] dz \text{ where } \delta' = \delta/(x-s(t))^2. \tag{16}$$

As $x \to s(t) + 0$, $\delta' \to \infty$ and consequently, $J_1 \to 1/2$. Combining this result with (15) and (13), and recalling that $I = I_1 + I_2$ the relation (12) follows. We shall now complete the proof of Lemma 1 with the aid of (12) and [6]. Putting

$$L_1 = \int_{t-\delta}^{t} \rho(\tau) \frac{x - s(t)}{2(t-\tau)} K(x, t; s(\tau), \tau)\, d\tau - \int_{t-\delta}^{t} \rho(\tau) \frac{s(t) - s(\tau)}{2(t-\tau)} K(s(t), t; s(\tau), \tau)\, d\tau, \tag{17}$$

we claim that

$$\lim_{x \to s(t)+0} \sup \left|L_1 - \frac{1}{2}\rho(t)\right| \le A\delta^{1/2} + A\, \underset{t-\delta \le \tau < t}{l.u.b.} |\rho(t) - \rho(\tau)| \tag{18}$$

Indeed, this follows by writing in (17), $\rho(\tau) = \rho(t) + [\rho(\tau) - \rho(t)]$ and using (12) and [6]. Observe next that the function

$$L_2 = \int_0^{t-\delta} \rho(\tau) \frac{x - s(t)}{2(t-\tau)} K(x, t; s(\tau), \tau)\, d\tau - \int_0^{t-\delta} \rho(\tau) \frac{s(t) - s(\tau)}{2(t-\tau)} K(s(t), t; s(\tau), \tau) d\tau \quad (0<\delta<t)$$

Satisfies the relation

$$\lim_{x \to s(t)} L_2 = 0$$

Combining this remark with (18) we get

$$\lim_{x \to s(t)+0} \sup \left|(L_1 + L_2) - \frac{1}{2}\rho(t)\right| \le A\delta^{1/2} + A\, \underset{t-\delta \le \tau < t}{l.u.b.} |\rho(t) - \rho(\tau)|.$$

Since the left-hand side is independent of δ, and the right-hand side can be made arbitrarily small if δ is sufficiently small, we get

$$\lim_{x \to s(t)+0} \sup \left| (L_1 + L_2) - \frac{1}{2}\rho(t) \right| = 0,$$

which is precisely the jump relation (10).

We shall now reduce the problem (p) to a system of integral equations.

We introduce Green's function for the half-plane $x>0$,
$G(x, t; \zeta, \tau) = K(x, t; \zeta, \tau) - K(-x, t; \zeta, \tau)$.

Suppose that u_1, u_2, s form a solution of (p) and let

$$v_2(t) = \frac{\partial u_2}{\partial x}(s(t), t),\ v_1(t) = \frac{\partial u_1}{\partial x}(s(t), t) \tag{19}$$

Using [6], we get

$$u_1(x, t) = \int_0^t G(x, t; s(\tau), \tau)\left[v_1(\tau) - g_1(V(\tau))\, V(\tau)\right] d\tau$$

$$- \int_0^t g_1(V(\tau)) \frac{\partial G}{\partial \zeta}(x, t; s(\tau), \tau)\, d\tau \tag{20}$$

$$+ \int_0^t f_1(\tau) \frac{\partial u_1}{\partial \zeta}(x, t; 0, \tau)\, d\tau + \int_0^b \varphi_1(\zeta)\, G(x, t;, \zeta, 0)\, d\zeta$$

Integrating Green's identity

$$\frac{\partial}{\partial \zeta}\left(G \frac{\partial u_2}{\partial \zeta} - u_2 \frac{\partial G}{\partial \zeta}\right) - \frac{\partial}{\partial \tau}(G u_2) = 0,$$

over the domain $s(\tau) < \zeta < 1, 0 < \varepsilon < \tau < t - \varepsilon$ and letting $\varepsilon \to 0$, upon using (2), (3), (7) and (8), we get

$$u_2(x, t) = \int_0^t G(x-1, t; s(\tau)-1, \tau)\left[g_1(V(\tau))V(\tau) - v_2(\tau)\right] d\tau$$

$$+ \int_0^t g_1(V(\tau)) \frac{\partial G}{\partial \zeta}(x-1, t; s(\tau)-1, \tau)\, d\tau$$

$$+ \int_b^1 \varphi_2(\zeta)\, G(x-1, t, \zeta-1, 0)\, d\zeta \tag{21}$$

$$- \int_b^t f_2(\tau) \frac{\partial G}{\partial \zeta}(x-1, t; 0, \tau)\, d\tau$$

$$= R_1 + R_2 + R_3 + R_4.$$

We proceed to differentiate both sides of (21) with respect to x and then take $x \to s(t) + 0$. Using Lemma 1, we find

$$\lim_{x \to s(t)+0} \frac{\partial R_1}{\partial x} = -\frac{1}{2}\left[g_1(V(t))\, V(t) - v_2(t)\right] \tag{22}$$

$$+ \int_0^t \left[g_1(V(\tau))\, V(\tau) - v_2(\tau)\right] \frac{\partial G}{\partial x}(s(t)-1, t; s(\tau)-1, \tau)\, d\tau$$

In order to evaluate $\lim_{x \to s(t)} \frac{\partial R_i}{\partial x}$ $(i = 2,3,4)$ we use

$N(x-1, t; \zeta-1, \tau) = K(-(x-1), t; \zeta-1, \tau) + K(x-1, t; \zeta-1, \tau)$.

Using the relation $G_x = -N_\zeta$ we get

$$\frac{\partial R_2}{\partial x} = -g_1(V(0))\, N(x-1, t; s(\tau)-1, 0)$$

$$- \int_0^t g_1'(V(\tau))\, N(x-1, t; s(\tau)-1, \tau)\, d\tau. \tag{23}$$

Similarly,

$$\frac{\partial R_3}{\partial x} = -\varphi_2(1)\, N(x-1, t; 0, 0)$$

$$+ \int_b^t \varphi_1'(\zeta)\, N(x-1, t; \zeta-1, 0)\, d\zeta, \tag{24}$$

and

$$\frac{\partial R_4}{\partial x} = f_2(0)\, N(x-1, t; 0, 0) + \int_0^t f_2'(\tau)\, N(x-1, t;, 0, \tau)\, d\tau. \tag{25}$$

Combining (22), (23), (24) and (25), we obtain from (21)

$$v_2(t)= 2[f_2(0) - \varphi_2(1)]\, N(s(t)-1, t; 0, 0) - g_1(V(t))V(t)$$

$$-2g_1(V(0))\, N(s(t)-1, t; s(\tau)-1, 0)$$

$$+2\int_0^t g_1(V(\tau))\, V(\tau) \frac{\partial G}{\partial x}(s(t)-1, t; s(\tau)-1, \tau)d\tau$$

$$-2\int_0^t v_2(\tau) \frac{\partial G}{\partial x}(s(t)-1, t; s(\tau)-1, \tau)d\tau \tag{26}$$

$$-2\int_0^t g_1'(V(\tau))\, N(s(t)-1, t; s(\tau)-1, \tau)d\tau$$

$$+2\int_0^t \varphi_2'(\zeta)\, N(s(t)-1, t; \zeta-1, 0)\, d\zeta$$

$$+\int_0^t f_2'(\tau)\, N(s(t)-1, t; 0, \tau)\, d\tau.$$

Similarly, we get [6]

$$v_1(t)= 2[\varphi_1(0) - f_1(0)]\, N(s(t), t; 0, 0) - g_1(V(t))V(t)$$

$$+2\int_0^b \varphi_1'(\zeta)\, N(s(t), t; \zeta, 0)\, d\zeta$$

$$-2\int_0^t f_1'(\tau)\, N(s(t), t; 0, \tau)\, d\tau$$

$$+2g_1(V(0))\, N(s(t), t; s(\tau), 0) \tag{27}$$

$$+2\int_0^t g_1'(V(\tau))\, N(s(t), t; s(\tau), \tau)\, d\tau$$

$$+2\int_0^t v_1(\tau) \frac{\partial G}{\partial x}(s(t), t; 0, \tau)\, d\tau$$

$$-2\int_0^t g_1(V(\tau))\, V(\tau) \frac{\partial G}{\partial x}(s(t), t; s(\tau), \tau)\, d\tau$$

and by (4), and (5) we obtain

$$s(t) = b + \int_0^t g_2^{-1}(v_2(\tau) - v_1(\tau))\, d\tau. \tag{28}$$

We have thus proved that for every solution u_1, u_2, s of the system (p) for all $t<\sigma$, the functions $v_1(t)$, $v_2(t)$, defined by (19), (for $0<t<\sigma$) satisfy the nonlinear integral equations of Volterra type (26) and (27), where $s(t)$ is given by (28).

Conversely, suppose that for $\sigma>0$, $v_1(t)$ and $v_2(t)$ are continuous solutions of the integral equations (26), (27), for $0\le t\le\sigma$, which $s(t)$ given by (28). We shall prove that $u_1(x,t)$, $u_2(x,t)$, $s(t)$ form a solution of (1-9) for all $t<\sigma$, where $u_1(x,t)$, $u_2(x,t)$ are defined by (20) and (21), respectively.

First one can (see also [7]) easily verify (1), (2) and (5-8). We next differentiate $u_1(x,t)$ with respect to x and take $x\to s(t)-0$. Using [6] we find that $\frac{\partial u_1}{\partial x}(s(t), t)= v_1(t)$, we also differentiate $u_2(x,t)$ with respect to x and take $x\to s(t)+0$. Using Lemma 1, the previous evaluations of $\frac{\partial R_i}{\partial x}$ for $i= 2,3,4$, and the integral Equation (28), we find that $\frac{\partial u_2(s(t), t)}{\partial x}= v_2(t)$. Since, by (28), $v_2(t) - v_1(t)= g_2(V(t))$, (1.4) follows. Thus it remains to prove that $u_1(s(t), t)= u_2(s(t), t)= g_1(V(t))$.

Integrate Green's identity (with $G(x, t; \zeta, \tau)$ and u_1) in the domain $0<\zeta<s(\tau)$, $0<\varepsilon<\tau<t-\varepsilon$ and let $\varepsilon\to 0$. Comparing the integral representation obtained of $u_1(x,t)$ with the original definition of $u_1(x,t)$ by (20) (with $\frac{\partial u_1}{\partial x}(s(\tau), \tau)= v_1(\tau)$), we conclude that:

$$\int_0^t [u_1(s(\tau), \tau) - g_1(V(\tau))]\, G_\zeta(x, t; s(\tau), \tau)\, d\tau= 0$$

if $0<x<s(t)$, $0<t<\sigma$, by [6], we get, $u_1(s(t), t)= g_1(V(t))$.

We also integrate Green's identity (with $G(x-1, t, \zeta-1, \tau)$ and u_2) in the domain $s(\tau)<\zeta<1$, $0<\varepsilon<\tau<t-\varepsilon$ and let $\varepsilon\to 0$. Comparing the integral representation obtained for $u_2(x,t)$ with the original definition of $u_2(x,t)$ by (21) (with $\frac{\partial u_2}{\partial \zeta}(s(\tau), \tau)= v_2(t)$), we conclude that:

$$\int_0^t [u_2(s(\tau), \tau) - g_1(V(\tau))]\, G_\zeta(x-1, t; s(\tau)-1, \tau)\, d\tau= 0$$

if $s(\tau)<x<1, 0<t<\sigma$.

Taking $x \to s(t)+0$ and using Lemma 1 and [6], we find that $u_2(s(t), t) = g_1(V(t))$. We have thus proven:

Lemma 2. The problem (p) for $t<\sigma$ is equivalent to the problem of finding a continuous solution $v_1(t)$ and $v_2(t)$ for the integral Equations (26) and (27) (for $0 \le t<\sigma$) where $s(t)$ is given by (28).

4. Existence of a Solution of Problem (p)

In this section, we establish the existence of problem (p). We shall show that a version of the integral Equations (26) and (27), where s(t) is given by (28), defines a contraction mapping for $0<t<\sigma$ if σ is sufficiently small.

4.1. Integral Operator

We now introduce the integral operator

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} \begin{array}{c} -g_1(V(t))V(t) + 2\int_0^b \varphi_1'(\zeta)N(s(t), t; \zeta, 0)d\zeta - 2\int_0^t f_1'(\tau)N(s(t), t, 0, \tau)d\tau \\ + 2g_1(V(0))\, N(s(t), t; s(\tau), 0) + 2\int_0^t g_1'(V(\tau))N(s(t), t; s(\tau), \tau)d\tau \\ +2\int_0^t v_1(\tau)\frac{\partial G}{\partial x}(s(t), t; s(\tau), \tau)d\tau - 2\int_0^t g_1(V(\tau))V(\tau)\frac{\partial G}{\partial x}(s(t), t; s(\tau), \tau)d\tau \\ \\ -g_1(V(t))V(t) - 2g_1(V(0))N(s(t)-1, t; s(\tau)-1, 0) \\ + 2\int_0^t g_1(V(\tau))V(\tau)\frac{\partial G}{\partial x}(s(t)-1, t; s(\tau)-1, \tau)d\tau \\ - 2\int_0^t v_2(\tau)\frac{\partial G}{\partial x}(s(t)-1, t; s(\tau)-1, \tau)d\tau \\ - 2\int_0^t g_1'(V(\tau))N(s(t)-1, t; s(\tau)-1, \tau)d\tau \\ + 2\int_b^1 \varphi_1'(\zeta)\, N(s(t)-1, t;, \zeta-1, 0)\, d\zeta + 2\int_0^t f_1'(\tau)\, N(s(t)-1, t; 0, \tau)d\tau \end{array} \end{bmatrix} \qquad (29)$$

where $s(t)$ is given by (28).

Let Tv denote the nonlinear operator on the right-hand side of (29). We shall show that the transformation

$$\omega = Tv$$

is a contraction of an appropriate subset of $C[0, \sigma]$ for some σ and therefore has a unique fixed point $v = Tv$.

4.2. A Ball Mapped into Itself

In the Banach space $C_\sigma = C[0, \sigma]$ with uniform norm, we consider the closed ball

$$B_{M,\sigma} = \{v = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} | v_i \in C_\sigma, \|v\| = \sup_{0<t<\sigma} |v_i(t)| \le M,$$

$i = 1,2\}$ with M to be specified later on. We will estimate separate term (29), by H1, H2, we find

$$|V(t)g_1(V(t))| \le \delta_1 M_2 \sqrt{\sigma_1} = C_1,$$

and the other terms in (29) are estimated by H1, H2 and using the elementary inequality $ye^{-y} \leq$ const. for $y > 0$, we note only that $0<\zeta<b$ and $b<s(t)$. Thus, we obtain

$$|v_1| \leq (C_1 + C_2 \|\varphi_1'\| \sqrt{\sigma} + C_3 \|f_1'\| \sqrt{\sigma} + C_4\sqrt{\sigma} + C_5\sqrt{\sigma} + C_6 M\sqrt{\sigma} + C_7\sqrt{\sigma}),$$

where the constants C_2, C_3, C_4, C_5, C_6, C_7 are simple combination of $\pi, b, \frac{1}{b}, M, M'_2, M_2, K$.

In a similar fashion, we find that

$$|v_2| \leq (C_1 + D_2\sqrt{\sigma} + D_3\sqrt{\sigma} + D_4 M\sqrt{\sigma} + D_5\sqrt{\sigma} + D_6 \|\varphi_2'\| \sqrt{\sigma} + D_7 \|f_2'\| \sqrt{\sigma}),$$

where the constants D_i, $i=2,3,\ldots,7$ are simple combinations of $b, M_2, \pi, M_2', M', \frac{1}{b}$.

If M is now taken to be

$$M = 2C_1$$

then for

$$\sqrt{\sigma} < \min\left\{(C_2 \|\varphi_1'\| + C_3 \|f_1'\| + C_4 + C_5 + C_6 M + C_7)^{-1}\frac{M}{2},\ (D_2 + D_3 + D_4 M + D_5 + D_6 \|\varphi_1'\| + D_7 \|f_2'\|)^{-1}\frac{M}{2}\right\}$$

the ball $B_{M,\sigma}$ is mapped into itself.

4.3. T is contraction on $B_{M,\sigma}$.

Let, $\omega = Tv$, $\omega' = Tv'$, where $v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ and $v'(t) = \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix}$, then

$$\omega - \omega' = \left[\begin{array}{c}
-(g_1(V(t))V(t) - g_1(V'(t))V'(t)) + 2\int_0^b \varphi_1'(\zeta)(N(s(t),t;\zeta,0) - N(s'(t),t;\zeta,0))d\zeta \\
-2\int_0^t f_1'(\tau)(N(s(t),t;0,\tau) - N(s'(t),t;0,\tau))d\tau + 2(g_1(V(0))N(s(t),t;,s(\tau),0) \\
-g_1(V'(0))N(s'(t),t;s'(\tau),0)) + 2\int_0^t (g_1'(V(\tau)N(s(t),t;s(\tau),0) \\
-g_1'(V'(\tau))N(s'(t),t;s'(\tau),\tau))d\tau + 2\int_0^t (v_1(\tau)\frac{\partial G}{\partial x}(s(t),t;s(\tau),\tau) \\
-v_1'(\tau)\frac{\partial G}{\partial x}(s'(t),t;s'(\tau),\tau))d\tau - 2\int_0^t (g_1(V(\tau)V(\tau)\frac{\partial G}{\partial x}(s(t),t;s(\tau),\tau) \\
-g_1(V'(\tau))V'(\tau)\frac{\partial G}{\partial x}(s'(t),t;s'(\tau),\tau))d\tau \\
\\
-(g_1(V(t))V(t) - (g_1(V'(t))V'(t)) - 2(g_1(V(0))N(s(t)-1,t;s(\tau)-1,0) \\
-g_1(V'(0))N(s'(t)-1,t;s'(\tau)-1,0))
\end{array}\right.$$

$$+2\int_0^t (g_1(V(\tau))V(\tau)\frac{\partial G}{\partial x}(s(t)-1, t; s(\tau)-1, \tau)$$

$$-g_1(V'(\tau))V'(\tau)\frac{\partial G}{\partial x}(s'(t)-1, t; s'(\tau)-1, \tau))\, d\tau$$

$$-2\int_0^t (v_2(\tau)\frac{\partial G}{\partial x}(s(t)-1, t; s(\tau)-1, \tau) - v_2'(\tau)\frac{\partial G}{\partial x}(s'(t)-1, t; s'(\tau)-1, \tau))d\tau$$

$$-2\int_0^t (g_1'(V(\tau))N(s(t)-1, t; s(\tau)-1, \tau) - g_1'(V'(\tau))N(s'(t)-1, t; s'(\tau)-1, \tau))\, d\tau$$

$$+2\int_0^t (\varphi_1'(\zeta)(N(s(t)-1, t; \zeta-1, 0) - N(s'(t)-1, t; \zeta-1, 0))\, d\zeta$$

$$+2\int_0^t (f_2'(\tau)(N(s(t)-1, t; 0, \tau) - N(s'(t)-1, t; 0, \tau))\, d\tau$$

$$= \begin{bmatrix} W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 \\ Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 \end{bmatrix}$$

The estimations are based on the mean value theorem and H1, H2, (28). First we note that

$$|\Delta N| \equiv |N(s(t), t; \zeta, 0) - N(s'(t), t; \zeta, 0)| =$$

$$\left|\frac{\partial N(\bar{s}, t; \zeta, 0)}{\partial x}\right| |s(t)-s'(t)| \le N_1' |b +$$

$$\int_0^t g_2^{-1}(v_2(\tau) - v_1(\tau))d\tau - b - \int_0^t g_2^{-1}(v_2'(\tau) - v_1'(\tau))d\tau|$$

$$= N_1' \left|\int_0^t g_2^{-1}(v_2(\tau) - v_1(\tau)) - g_2^{-1}(v_2'(\tau) - v_1'(\tau))\, d\tau\right|$$

$$\le N_1' \int_0^t \left|\frac{dg_2^{-1}(\bar{V})}{dt}\right| |v_2(\tau) - v_1(\tau) - v_2'(\tau) + v_1'(\tau)|\, d\tau$$

$$\le N_1' \delta_1\sqrt{\sigma} \int_0^t (|v_2(\tau) - v_2'(\tau)| + |v_1(\tau) - v_1'(\tau)|\, d\tau$$

$$\le N_1' \delta_1 \|v - v'\| \sigma^{3/2}$$

where $\bar{s}$ is between $s(t)$ and $s'(t)$ and $\bar{v}$ is between $v_2(t) - v_1(t)$ and $v_2'(\tau) - v_1'(\tau)$. Now we are able to estimate three terms in (30):

$$|W_2| = 2\left|\int_0^b \varphi_1'(\zeta)(N(s(t), t; \zeta, 0) - N(s'(t), t; \zeta, 0))\, d\zeta\right|$$

$$= 2\left|\int_0^b \varphi_1'(\zeta)\Delta N\, d\zeta\right| \tag{31}$$

$$\le 2\|\varphi_1'\| N_1'\delta_1 b \|v - v'\| \sigma^{3/2} = N_2 \|\varphi_1'\| \|v - v'\| \sigma^{3/2}$$

Similarly,

$$|W_3| \le N_3 \|f_1'\| \|v - v'\| \sigma^{5/2} \tag{32}$$

Now, we obtain

$$|W_1| = |g_1(V(t))V(t) - g_1(V'(t))V'(t)|$$
$$= |\Delta g_1 V(t) - g_1(V'(t))[V'(t) - V(t)]|$$
$$\le \left|\frac{dg_1(\zeta)}{dV}\right| \|V(t) - V'(t)\| V(t)| + |g_1(V'(t)\| V(t) - V'(t)|$$

$$\le (M_2 M' + M_2)\|v - v'\| \sigma^{1/2} = N_1 \|v - v'\| \sigma^{1/2}. \tag{33}$$

Similarly,

$$|W_4| = 2|\Delta N g_1(V(0)) - N(s'(t), t; s'(\tau), 0)\,[g_1(V'(0)) - g_1(V(0))]| \le (N_4\sigma + N_5)\,\|v - v'\|\,\sigma^{1/2} \qquad (34)$$

and

$$|W_5| = 2\left|\int_0^t \Delta N g_1'(V(\tau))\,d\tau - \int_0^t N(s'(t), t; s'(\tau), \tau)\,[v_1'(\tau) - v_1(\tau)]\,d\tau\right| \le (N_6\sigma^2 + N_7\sigma)\,\|v - v'\|\,\sigma^{1/2} \qquad (35)$$

In a similar fashion, we find that

$$|W_6| = 2\left|\int_0^t \Delta G_x v_1(\tau)\,d\tau - \int_0^t G_x(s'(t), t; s'(\tau), \tau)\,[v_1'(\tau) - v_1(\tau)]\,d\tau\right| \le (N_8\sigma^2 + N_9\sigma^{1/2})\,\|v - v'\|\,\sigma^{1/2} \qquad (36)$$

and

$$|W_7| \le (N_{10}\sigma^2 + N_{11}\sigma)\,\|v - v'\|\,\sigma^{1/2}, \qquad (37)$$

where the constant N_i, $i=4,5,\ldots,11$ are simple combinations of $2, \pi, L_1, \delta_1, M_2, M_2', K_1, M_2''$, where L_1 and K_1 are the upper bounds of K_x and N respectively.

Similarly, we obtain the same results for z_i, $i=1,2,\ldots,7$, such that

$$|z_1| \le T_1\,\|v - v'\|\,\sigma^{1/2}, \ldots, |z_7| \le (T_{10}\sigma_2 + T_{11}\sigma)\,\|v - v'\|\,\sigma^{1/2}. \qquad (38)$$

The result in (31-38) yield the following contraction estimate $\|Tv - Tv'\| \le L\|v - v'\|$, where

$$L = \max\{(N_1 + N_2\|\varphi_1'\| + N_3\|f_1'\|\sigma^2 + N_4\sigma + N_5 + N_6\sigma^2 + N_7\sigma + N_9\sigma^{1/2} + N_8\sigma^2 + N_{10}\sigma^2 + N_{11}\sigma)\sigma^{1/2}, (T_1 + T_2\|\varphi_2'\|\sigma + T_3\|f_2'\|\sigma^2 + T_4\sigma + T_5 + T_6\sigma^2 + T_7\sigma + T_8\sigma^2 + T_9\sigma^{1/2} + T_{10}\sigma^2 + T_{11}\sigma)\sigma^{1/2}\}.$$

If $\sigma < 1$ and such that

$$\max\{(N_1 + N_2\|\varphi_1'\| + N_3\|f_1'\| + N_4 + N_5 + N_6 + N_7 + N_8 + N_9\sigma^{1/2} + N_{10} + N_{11})\sqrt{\sigma}, (T_1 + T_2\|\varphi_1'\| + T_3\|f_1'\| + T_4 + T_5 + T_6 + T_7 + T_8 + T_9\sigma^{1/2} + T_{10} + T_{11})\sqrt{\sigma}\} < 1, \qquad (39)$$

Then T is a contraction on $B_{M,\sigma}$. Therefore, it has a fixed point $v(t)=\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$ in $B_{M,\sigma}$ which is unique. Thus the existence is proved.

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