SOME RESULTS FOR SOLUTION AND FOCAL POINTS OF NONSELFADJOINT SECOND ORDER SYSTEMS

S. Fariabi

Department of Mathematics, Faculty of Science, Shahid Beheshti University, Tehran, Islamic Republic of Iran

Abstract

Consider y" (t) + A (t)y (t) + 0, y is a real n-dimensinal vector and A(t) is a real n×n matrix, continuous on some interval. Some positivity properties of solutions and conjugate points of y"(t) + A(t)y (t) = 0 appeared in literature. We prove similar results for focal points.

Introduction

The differential equations to be considered in this paper have the form

(1) $$x''(t) + A(t)\, x(t) = 0$$

where **x** is a real n-dimensional vector, A (t) is a real n×n matrix continuous on some interval.

Ahmad in [1] and Ahmad and Lazer in [2] have proved some results for conjugate points relative to (1), as where we prove the corresponding results for focal points relative to (1).

Preliminary Notations and Results

Definition 2.1. A number b, b>a, is called a focal point of **a** relative to (1) if there exists a nontrivial solution x (t) of (1) with property that x' (**a**) = x (**b**) = 0.

Definition 2.2 A point b is said to be the first focal point of a point **a** if and only if **b** is a focal point of **a** and there is no focal point of **a** smaller than **b**.

Definition 2.3. Equation (1) is said to be disfocal on an interval **I** if any nontrivial solution of it which has a derivative of zero at some point of **I** has no zero to the right of that point on **I**.

Definition 2.4. Matrix A (t) = (a_{ij} (t)) is called irreducible if it is impossible to have {1, 2, ..., n} = I∪J, I∩J = ∅, I ≠ ∅ ≠ J and $a_{ij} = 0$. for all i ∈ I, j ∈ J.

Keywords: Differential sytems; Focal points

Throughout this paper, we make extensive use of Green's function for the boundary value problem

$$x''(t) = -f(t)$$
$$x'(a) = x(b) = 0,$$

where **a<b**. Recall that

$$G(s,t) = \begin{cases} b-t, & a \le s \le t \le b, \\ b-s, & a \le t \le s \le b. \end{cases}$$

The function G is continuous on the square a≤s≤b, a≤t≤b. If f (t) is a continuous real valued function defined for a≤t≤b and if $x(t) = \int_a^b G(s,t)\, f(s)\, ds$ then x (t) is of class C^2 on [a, b], x"(t) = -f(t) and x'(a) = x (b)=0.

Given two vectors x = col ($x_1, \dots, x_n$) and y= col ($y_1, \dots, y_n$) in R^n, we write $x \le y$ ($x < y$) if for each k, k= 1, ..., n, $x_k \le y_k$ ($x_k < y_k$). Let **a** be a fixed number, for any b>a we let

K(b) = {continuous functions

u: $[a, b] \to R^n$ | u' (a)=0=u(b) and $0 \le u(t)$ for all t in (a, b)}. Let A(t) = (a_{ij} (t)) be an n×n continuous matrix defined on [a,b]. Assume that

$a_{ij}(t) > 0$ for $1 \le i \le n, 1 \le j \le n$ and $t \in [a, b]$ except possibly on a set of measure zero. If $u: [a, b] \to R^n$ is continuous, we define (Tu) (t) by

$$(Tu)(t) = \int_a^b G(s,t)\, A(s)\, u(s) ds.$$

It follows immediately that

$T(u+v) = Tu+Tv$,

$T(cu) = cTu, c \in R$

$u \in K$ implies $Tu \in K$,

$u \in K$, $u(t) \not\equiv 0$ implies $0 < (Tu)(t)$, $t \in (a, b)$

Let $\Lambda(b) = \{$real numbers $\lambda |$ there exists $u \in K(b)$, $u \not\equiv 0$, and $u(t) \leq \lambda T(u)(t)$ for $t \in (a, b)\}$.

Lemma 2.1.

$\Lambda(b) \neq \emptyset$. If $\lambda_0(b) = \inf\{\lambda | \lambda \in \Lambda(b)\}$, then $\lambda_0(b) > 0$.

Lemma 2.2.

There exists $u \in K(b)$, $u \not\equiv 0$, such that $u(t) = \lambda_0(b)\,(Tu)(t)$ on $[a, b]$.

Lemma 2.3. If there exists $\lambda_1 \in \Lambda(b)$ and $w \in K(b)$; $w(t) \not\equiv 0$, such that

$w(t) = \lambda_1 (Tw)(t)$ for $t \in [a, b]$

then $\lambda_1(b) = \lambda_0(b)$.

Lemma 2.4. If $a<b_1<b_2$, then $\lambda_o(b_2) < \lambda_o(b_1)$.

Lemma 2.5. The function $\lambda_o(b)$ is continuous on (a, ∞) and $\lambda_o(b) \to \infty$ as $b \to a$.

Lemma 2.6. Let $A(t)=(a_{ij}(t))$ and $\hat{A}(t)=(a_{ij}(t))$ be $n\times n$ matrices which are continuous on $[a,b]$ and for $1\leq i \leq n$, $1 \leq j \leq n$, $0<a_{ij}(t) \leq \hat{a}ij\ (t)$ on (a,b). For $u \in K(b)$ let

$$(Tu)(t) = \int_a^b G(s,t)\, \hat{A}(s)\, u(s) ds.$$

and $\hat{\Lambda}$ be the set of numbers $\hat{\lambda}$ such that $u(t)<\lambda(Tu)(t)$, $t \in (a,b)$ for some $u \in K(b)$, $u \neq 0$. If $\hat{\lambda}_0(b) = \inf\ \{\hat{\lambda} |\ \hat{\lambda} \in \hat{\Lambda}\}$, then $\hat{\lambda}_o(b) \leq \lambda_o(b)$.

Note: The above results have been proven in [5].

Main Theorems

Theorem 3.1. Let $A(t)=(a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ be two continuous $n\times n$ matrices defined on $[a, b]$ such that

$0 \leq b_{ij}(t) \leq a_{ij}(t)$, $t \in [a, b]$, $1 \leq i \leq n$, $1 \leq j \leq n$

and for some

$\bar{t} \in (a, b)$, $0 \leq b_{ij}(\bar{t}) < a_{ij}(\bar{t})$, $1 \leq i \leq n$, $1 \leq j \leq n$. Suppose $x'' + B(t)x=0$, $x(t) \not\equiv 0$, $x'(a) = x(b) = 0$.

Assertion. There exists a solution of $u''+ A(t)u = 0$, $u'(a) = u(c) = 0$, $u(t) \not\equiv 0$ with $a<c<b$, and $u \in K(c)$.

Proof. For $t \in [a, b]$, we have

$$x(t) = \int_a^b G(s, t)\, B(s)\, x(s) ds.$$

If $x(t)=col(x_1(t), \ldots, x_n(t))$, let $w(t) = col(|x_1(t)|, \ldots, |x_n(t)|)$. Then $w \in K(b)$ and $w \not\equiv 0$. For $k=1, \ldots, n$,

$$w_k(t) = |x_k(t)| = \left| \int_a^b G(s,t) \sum_{j=1}^n b_{kj}(s)\, x_j(s) ds \right|$$

$$\leq \int_a^b G(s,t) \sum_{j=1}^n b_{kj}(s)\, |x_j(s)|\, ds$$

$$= \int_a^b G(s,t) \sum_{j=1}^n b_{kj}(s)\, w_j(s) ds.$$

Now by the uniqueness theorem for differential equation, the components of $w(t)$ cannot vanish simultaneously on any subinterval of $[a, b]$ since $x(t) \not\equiv 0$. Thus since $b_{kj}(s) \leq a_{kj}(s)$, $s \in (a, b)$, and $b_{kj}(\bar{t}) < a_{kj}(\bar{t})$, we have

$$\int_a^b G(s,t) \sum_{j=1}^n b_{kj}\, w_j(s) ds < \int_a^b G(s,t) \sum_{j=1}^n a_{kj}(s)\, w_j(s) ds$$

for

$t \in [a, b)$. Hence, we have (2) $0 \leq w(t) < \int_a^b G(s,t)\, A(s)\, w(s)\, ds$

for $t \in [a,b)$. Let $A_m(t) = (a_{ij}(t) + \frac{1}{m})$.

As the element of A_m are strictly positive on $[a, b]$, for $m \geq 1$, we have,

(3) $0 \leq w(t) < \int_a^b G(s,t)\, A_m(s)\, w(s) ds$,

for $t \in (a, b)$. For each $m\geq 1$ and $d \in (a,b]$, define

$$(T_m^d u)(t) = \int_a^d G(s,t,d)\, A_m(s)\, u(s) ds$$

for $u \in k(d)$; let $A_m(d)$ be the set of numbers λ such that $u(t) \leq \lambda (T_m^d u)(t)$ for $t \in [a, d]$, and let $\lambda_{0m}(d) = \inf\{\lambda | \lambda \in \Lambda_m(d)\}$.

If $m_1<m_2$ then each element of $A_{m_1}(t)$ is greater than

the corresponding element of $A_{m_2}(t)$, so by Lemma 2.6

(4) $m_1<m_2$ implies $\lambda_{0m_1}(d) \le \lambda_{0m_2}(d)$.

From (3) we see that $1 \in A_m(b)$ for all m, and hence $\lambda_{0m}(b) \le 1$ for all m. As $\lambda_{om}(d)$ is continuous, decreasing in d, and $\lambda_{0m}(d) \to +\infty$ as $d \to a$, there exists a unique $d_m \in (a, b]$ such that $\lambda_{0m}(d_m) = 1$.

Moreover by (4) it follows that

$$a<d_{m_1} \le d_{m_2} \text{ if } m_1<m_2.$$

Hence, $\lim_{m\to\infty} d_m = c$ for some $c \in (a, b]$. By Lemma 2.2 there exists $u_m \in K(d_m)$, $u_m \not\equiv 0$, such that

$$u_m(t) = \lambda_{om}(d_m) \int_a^{d_m} G(s,t,d_m)\, A_m(s)\, u_m(s)ds$$

$$= \int_a^{d_m} G(s,t,d_m)\, A_m(s)\, u_m(s)ds.$$

Hence $\ddot{u}_m + A_m u_m = 0$, $u_m'(a) = u_m(d_m) = 0$. Without loss of generality as in the proof of Lemma 2.5 $\lim_{m\to\infty} u_m(a) = k \neq 0$. As $A_m(t) \to A(t)$ uniformly on [a, b) it follows that if u(t) is a solution of the initial value problem $u'' + A(t)u = 0$, $u'(a) = 0$, $u(a) = k$, then $u_m(t) \to u(t)$ uniformly on compact subinterval of $[a, \infty)$.

Hence

$$u(c) = \lim_{m\to\infty} u_m(d_m) = 0;$$

obviously $u \in K(c)$. To complete the proof we must show that c<b. Assume on the contrary that c=b, so that

(5) $$u(t) = \int_a^b G(s,t)\, A(s)\, u(s)ds.$$

let $$v(t) = \int_a^b G(s,t)\, A(s)\, w(s)ds.$$

Then v is of class C^2 on [a, b]. According to (2), $0 \le w(t) < v(t)$, $t \in [a, b)$. Hence, by the nonnegativity of the elements A(s), $s \in (a, b)$, the strict positivity of $A(\bar{t})$, and the strict positivity of G(s,t) for a<s<b, a<t<b, it follows that for $t \in (a,b)$,

(6) $$v(t) = \int_a^b G(s,t)\, A(s)\, w(s)ds < \int_a^b G(s,t)\, A(s)\, v(s)ds.$$

Similarly,

(7) $$-\int_a^b A(s)\, v(s)ds < -\int_a^b A(s)\, w(s)ds = v'(b).$$

Since, by the uniqueness theorem, the components of u(t) cannot vanish simultaneously on any open subinterval of (a,b), the same type of reasoning shows that

$$0<u(t),\ t \in [a, b)$$

$$u'(b) = -\int_a^b A(s)\, u(s)ds.$$

As $v(b) = u(b) = 0$, if $\alpha>0$ is sufficiently small, then

(8) $0<u(t) - \alpha v(t)$, $t \in [a, b)$

and

(9) $u'(b) - \alpha v'(b) < 0$,

If $\bar{\alpha} > 0$ is the least upper bound of the number α such that (8) and (9) hold then by continuity

(10) $0 \le u(t) - \bar{\alpha} v(t)$, $t \in [a,b)$.

and

(11) $u'(b) - \bar{\alpha} v'(b) \le 0$

and such that for some k, $1 \le k \le n$, one of the following two possibilities must hold:

If $u = \text{col}(u_1, \dots, u_n)$, $v = \text{col}(v_1, \dots, v_n)$, either

(12) $u_k(\bar{t}) - \bar{\alpha}\, v_k(\bar{t}) = 0$ for some $\bar{t}$, $a \le \bar{t} < b$,

or

(13) $u_k'(b) - \bar{\alpha}\, v_k'(b) = 0$.

However, as $\bar{\alpha} > 0$ we see from (5), (6) and (10),

$$u(t) = \int_a^b G(s,t)\, A(s)\, u(s)ds,$$

and by (6)

$$-\bar{\alpha}\, v(t) > -\bar{\alpha} \int_a^b G(s,t)\, A(s)\, u(s)ds$$

therefore.

$$u(t) - \bar{\alpha}\, v(t) > \int_a^b G(s,t)\, A(s)\, u(s)ds - \bar{\alpha} \int_a^b G(s,t)\, A(s)\, v(s)ds$$

$$= \int_a^b G(s,t)\, A(s)\, [u(s) - \bar{\alpha}\, v(s)]ds,$$

hence (11) is impossible.

Similarly, by (5), (7) and (11)

$$u'(b) = -\int_a^b A(s)\, u(s)ds,$$

$$-\bar{\alpha}\, v'(b) < \bar{\alpha} \int_a^b A(s)\, v(s)ds, \text{ hence}$$

$$u'(b) - \bar{\alpha} v'(b) < -\int_a^b A(s)\, u(s)ds + \bar{\alpha}\int_a^b A(s)\, v(s)ds$$

$$= -\int_a^b A(s)\, [u(s) - \bar{\alpha}\, v(s)]ds \leq 0,$$

which rules out (13). This contradiction gives the result.

Theorem 3.2. Assume that the $n\times n$ matrix $B(t) = (b_{ij}(t))$ is continuous on $[a,b]$ and that $b_{ij}(t) \geq 0$, $1\leq i,j\leq n$. And let b be the first focal point of a. There exists a nontrivial solution $u(t) = col\,(u_1(t), \ldots, u_n(t))$ of

$$x''(t) + B(t)\, x(t) = 0$$

such that $u'(a) = u(b) = 0$ and $u_k(t) \geq 0$, $k = 1, 2, \ldots, n$ and $t\in [a, b]$.

Proof. For each integer $m=1, 2, \ldots$, let $B_m(t) = (b_{ij}(t) + \frac{1}{m})$. Let $x(t)$ be a nontrivial solution of the boundary value problem $x''(t) + B(t)\, x(t) = 0$, $x'(a) = x(b) = 0$, and assume there exists no nontrivial solution of the boundary value problem $x''(t) + B(t)\, x(t) = 0$, $x'(a) = x(c) = 0$, if $a<c<b$. As every element of $B_m(t)$, is strictly greater than the corresponding element of $B(t)$, it follows from Theorem 3.1 that there exists a nontrivial solution of the boundary value problem $u''_m(t) + B_m(t)\, u_m(t)=0$, $u'_m(a)=u_m(c_m)=0$, such that $a<c_m<b$ and such that $u_m(t)\in K(c_m)$. As

$$u_m(t) = \int_a^{c_m} G(s,t,c_m)\, B_m(s)\, u_m(s)ds, \tag{14}$$

for $a\leq t\leq c_m$. Let

$$\| B_m(s) \| = \max_{1\leq i\leq n} \sum_{j=1} (b_{ij}(s) + \frac{1}{m})$$

and $u_m(t) = col\,(u_{m1}(t), \ldots, u_{mn}(t))$, let $1\leq k\leq n$ and $\bar{t}\in [a, c_m]$ be such that

$$u_{mk}(\bar{t}) = \max_{1\leq i\leq n}\ \max_{a\leq t\leq c_m} u_{mi}(t)$$

From (14) it follows that

$$u_{mk}(\bar{t}) \leq \int_a^{c_m} G(s, \bar{t}, c_m) \sum_{j=1}^{n} (b_{kj}(s) + \frac{1}{m})\, u_{mj}(s)ds$$

$$\leq u_{mk}(\bar{t}) \int_a^{c_m} G(s, \bar{t}, c_m) \sum_{j=1}^{n} (b_{kj}(s) + \frac{1}{m})ds$$

$$\leq u_{mk}(\bar{t})\, (c_m-a) \int_a^{c_m} \| B_m(s) \| ds$$

and hence

$$1 \geq \frac{1}{(c_m-a) \int_a^{c_m} \| B_m(s) \| ds} \tag{15}$$

thus, since $\| B_m(t) \| = n/m + \| B(s) \|$ is bounded independently of m, we infer the existence of a number $\delta>0$ such that

$$a + \delta\leq c_m<b \quad m\geq 1.$$

As in proof of Theorem 3.1 we may assume, without loss of generality, that

$$u_m(a)\to k\neq 0 \text{ as } m\to\infty$$

and that $\lim_{m\to\infty} c_m = c$ with $a+\delta\leq c\leq b$. If

$$u''(t) + B(t)\, u(t) = 0,$$
$$u'(a) = 0 \text{ and } u(a) = k$$

then the sequence $\{u_m(t)\}_1^\infty$ converges uniformly to $u(t)$ on $[a,b]$ and hence $u(c) = 0$. If $c<b$ we would have a contradiction to the previous assumption concerning b. If $a<\bar{t}<b$ then $\bar{t}<c_m$ for sufficiently large m and as $u_m \in K(c_m)$, $0\leq u_m(\bar{t})$. Hence $0\leq u(\bar{t})$ so $u\in K(b)$ and the theorem is proved.

Theorem 3.3. Let $A(t) = (a_{ij}(t))$ be an $n\times n$ matrix which is continuous on $[a,b]$ with $a_{ij}(t) > 0$ on (a,b); $i,j = 1, \ldots, n$. If there exists a nontrivial solution $v(t) = col\,(v_1, \ldots, v_n)$ of

$$y'' + A(t)y = 0 \tag{16}$$

such that $v'(a) = v(b) = 0$ and $v_k(t) \geq 0$, $k=1, \ldots, n$, then **b** is the first focal point of **a** relative to (16).

Proof. First we note that if **a** has a focal point relative to (16), then the first focal point of **a** relative to (16) exists. Since

$$\int_a^b G(s,t,b)\, A(s)\, v(s)ds$$

is a unique solution of the boundary value problem

$$x'' = - A(t)\, x(t),$$
$$x'(a) = x(b) = 0,$$

we must have

$$v(t) = \int_a^b G(s,t,b)\, A(s)\, v(s)ds. \tag{17}$$

Let $\bar{t}\in [a,b]$ be such that $v_k(\bar{t}) = \max_{1\leq j\leq n}\ \max_{t\in [a,b]} v_j(t)$

By the same argument that was used to establish the inequality (15) it follows that

$$b-a \geq \frac{1}{\int_a^b ||A(s)|| ds},$$

where **b** is any focal point of **a** relative to (16). If **a** did not have first focal point relative to (16) then the left side of the preceding inequality could be made approaching zero with the right side approaching infinity, a contradiction. We note that by (17) and Lemma 2.3, $\lambda_0(b) = 1$.

Suppose **b** is not the first focal point of **a** relative to (16). Then there exists a point **b'** in (a,b) such that **b'** is the first focal point of **a** relative to (16). By Theorem 3.2, there exists $u \in K(b')$, $u \neq 0$, satisfying

$$u'' + A(t)u = 0$$

therefore

$$u(t) = \int_a^{b'} G(s,t,b')\, A(s)\, u(s) ds.$$

By Lemma 2.3 $\lambda_0(b') = 1$. But this contradicts the strict monotonicity of $\lambda_0(b)$, established in Lemma 2.4. The proof is complete.

Theorem 3.4. Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix which is continuous on $[a, \infty)$ with $a_{ij}(t) \geq 0$. If

(18) $y'' + A(t)y = 0$

is disfocal on $[a, \infty)$, then there exists a nontrivial solution u(t) of (18) such that $u'(a) = 0$ and $0 \leq u(t)$ for $t \geq a$. Furthermore, if $A(t_0)$ is irreducible for some t_0, $t_0 > a$, then $0 < u(t)$ for $t > a$.

Proof. For each number m, let $A_m = (a_{ij}(t) + \frac{1}{m})$.

We first show that for each m, **a** has a focal point, and hence first focal point relative to

(19) $y'' + A_m y = 0.$

Let $\gamma > 1$ and let B_m be the diagonal matrix given by $B_m = \text{diag}\,(\frac{1}{m_\gamma}, \ldots, \frac{1}{m_\gamma})$. Clearly, each element of A_m is greater than the corresponding element of B_m.

Furthermore, $z(t) = \text{col}\,(\text{Cos}\,\frac{1}{\sqrt{m_\gamma}}.\,(t-a), 0, \ldots, 0)$ is a solution of

$$z'' + B_m z = 0$$

satisfying $z'(a) = 0 = z(a + \frac{\Pi}{2}\sqrt{m_\gamma})$. Therefore, by Theorem 3.1 **a** has a focal point to the left of $a + \pi\sqrt{m\gamma}$ relative to (19). This shows that the first focal point of **a** relative to (19) exists (see the proof of theorem 3.3). For each integer m, let c_m denote the first focal point of **a** relative to (19). If $m_1 < m_2$, then the elements of A_{m_1} are strictly greater than the corresponding elements of A_{m_2}. Hence by Theorem 3.1, $c_{m_1} < c_{m_2}$. By Theorem 3.2, there exists $y_m \in K(c_m)$, $y_m \neq 0$, satisfying

$$y_m'' + A_m(t)\, y_m = 0.$$

Multiplying the preceding equation by a suitable constant, we can assume without loss of generality, that $y_m(a) \to \zeta$ as $m \to \infty$, where $||\zeta|| = 1$. By continuity with respect to initial condition and parameters, if y(t) satisfies

$$y'' + A(t)y = 0,\; y'(a) \text{ and } y(a) = \zeta,$$

then $y_m \to y$ uniformly on compact subintervals of $[a, \infty)$. Now, for the strictly increasing sequence $\{c_m\}_{m=1}^{\infty}$, one of the possibilities holds. (1) $\lim_{m\to\infty} c_m = c < \infty$, (2) $\lim_{m\to\infty} c_m = \infty$. Suppose (1) holds. Then $y(c) = \lim_{m\to\infty} y_m(c_m) = 0$, contradicting the assumption that (18) is disfocal on $[a, \infty)$. Therefore, (2) must hold. For any fixed t, $a < t < \infty$, we have $y(t) = \lim_{m\to\infty} y_m(t)$. Since $y_m \in K(c_m)$, $0 \leq y_m(t)$ if $c_m > t$. Hence $0 \leq y(t)$, and the first part of our theorem is proved.

To prove the last part of our theorem, assume that $A(t_0)$ is irreducible for some $t_0 > a$. For each k, k=1, ..., n, u_k satisfies the equation

$$u_k'' + \sum_{j=1}^{n} a_{kj}(t)\, u_j(t) = 0.$$

Hence $u_k''(t) \leq 0$: Since $u'(a) = 0$, $u'_k(a) = 0$, and since $u_k''(t) \leq 0$ for all $t \geq a$, u'_k is decreasing. If at $t^* > a$, $u_k(t^*) = 0$, then $u'_k(t^*)$ is equal to zero. This implies that $u'_k \equiv 0$. Therefore $u_k \equiv 0$. Suppose it is false that $0 < u(t)$ for $t > a$. Let $I = \{i, i=1, \ldots, n \mid u_i(t) = 0\}$, and let $j = \{1, \ldots, n\} - I$. Then $\{1, 2, \ldots, n\} = I \cup J$, $J \cap I = \emptyset$. For each $j \in J$, $u_j(t) > 0$ for $t > a$. For each $i \in I$ and $s > a$, we have

$$0 = u_i^{\bullet}(s) + \sum_{k=1}^{n} a_{ik}(s)\, u_k(s) = \sum_{k=1}^{n} a_{ik}(s)\, u_k(s) = \sum_{j\in J} a_{ij}(s)\, u_j(s).$$

Since $u_j(s) > 0$ and $a_{ij}(s) \geq 0$, it follows $a_{ij}(s) = 0$. This shows that $a_{ij}(s) = 0$ on (a, ∞) for $i\in$ and $j\in J$. contradiction $A(t_0)$ is irreducible.

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