# ON TE EXISTENCE OF PERIODIC SOLUTION FOR CERTAIN NONLINEAR THIRD ORDER DIFFERENTIAL EQUATIONS

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We consider nonlinear third order differential equations.

$$x''' + f(t,x,x',x'') = 0 \qquad (1)$$

where $f(t,x,x',x'')$ is a continuous real-valued function with domain $[0,T] \times R^3$, $T > 0$. Further, we shall assume that all solutions of initial value problems for (1) extend to $[0,T]$. Using the above assumption, we shall establish the following theorem.

**Theorem 1.** Let there exist constants $k > 0$ and $C > 0$ such that

$$2M \leq Ck^3 \qquad (2)$$

where

$M = \{\max |k^2x' - f(t,x,x',x'')| : t \in [0,\omega],$
$|x| \leq C, |x'| \leq Ck, |x''| \leq Ck^2\}$

Then there exists $\omega_0, 0 < \omega_0 < \frac{\pi}{2k}$ such that for every $\omega, 0 < \omega \leq \omega_0$ equation (1) has a solution $x(t)$ satisfying the boundry conditions

$$x^{(i)}(0) + x^{(i)}(\omega) = 0, \qquad i = 0,1,2 \qquad (3)$$

**Proof.** Let $\omega \in (0, \frac{\pi}{2k}]$ and let $G(t,s)$ be the Green's function

$$G(t,s) = \begin{cases} \frac{1}{2k^2}\left[1 - \frac{\cos k(\frac{\omega}{2} - s + t)}{\cos k\frac{\omega}{2}}\right]; 0 \leq t \leq s \leq \omega \\ \frac{1}{2k^2}\left[-1 + \frac{\cos k(\frac{\omega}{2} + s - t)}{\cos k\frac{\omega}{2}}\right]; 0 \leq s \leq t \leq \omega \end{cases}$$

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Then equation (1) with boundary conditions (3) is equivalent to the integral equation

$$x(t) = \int_0^\omega G(t,s)\{k^2x'(s) - f(s,x(s),x'(s),x''(s))\}\,ds \qquad (4)$$

(See [1])

Let

$B = [x(t) \in C^2[0,\omega] : |x(t)| \leq C, |x'(t)| \leq Ck, |x''(t)| \leq Ck^2]$

and define the operator U on B by

$$(Ux)(t) = \int_0^\omega G(t,s)\{k^2x'(s) - f(s,x(s),x'(s),x''(s))\}\,ds. \qquad (5)$$

Then

$$|(Ux)(t)| \leq \frac{1}{2k^2}[\omega + \frac{2\sqrt{2}}{k}]M,$$

$$|(Ux)'(t)| \leq \frac{1}{2k}[\omega + \frac{2\sqrt{2}}{k}]M,$$

and

$$|(Ux)''(t)| \leq \frac{1}{2}[\omega + \frac{2\sqrt{2}}{k}]M$$

Hence U maps B continuously into itself provided that

$$\frac{1}{2k^2}[\omega + \frac{2\sqrt{2}}{k}]M \leq C \qquad (6)$$

$$\frac{1}{2k}[\omega + \frac{2\sqrt{2}}{k}]M \leq kC \qquad (7)$$

$$\frac{1}{2}[\omega + \frac{2\sqrt{2}}{k}]M \leq k^2C \qquad (8)$$

Clearly (6), (7) and (8) are equivalent to

$$\omega \leq 2\frac{k^3C - \sqrt{2}M}{kM} \qquad (9)$$

by (2) the right-hand side of (9) is non-negative. Therefore, if $\omega > 0$ is chosen so that

$$\omega \leq \min \{ \frac{\pi}{2k}, 2\frac{k^3 C - \sqrt{2} M}{kM} \}$$

it follows from Schauder's theorem that (5) has a solution $x(t)$ such that

$$|x(t)| \leq C, |x'(t)| \leq Ck, |x''(t)| \leq Ck^2,$$

Hence (1) has a solution $x(t)$ satisfying boundary conditions (3).

**Corollary 1.** If, in addition to all hypotheses of theorem 1, we further assume

i) $f(t,x,x',x'')$ is $2\omega$-periodic in $t$ that is

$f(t+2\omega,x,x',x'')=f(t,x,x',x'')$

ii) $f(t+\omega,-x,-x',-x'')=-f(t,x,x',x'')$

iii) $f(t,x,x',x'')$ is locally Lipschitzian with respect to $(x,x',x'')$. Then (1) has a $2\omega$-periodic solution $x(t)$ with the property that

$$\int_0^{2\omega} x(t)\, dt = O$$

**Proof.** Let us define $z(t)$ as follows

$$z(t) = \begin{cases} x(t) & ;0 \leq t \leq \omega \\ -x(t+\omega) & ;-\omega \leq t \leq 0 \end{cases}$$

It is obvious from boundary conditions (3) that $z(t)$ is continuous with its first and second derivatives and from condition (ii) $z(t)$ satisfies equation (1) with periodic boundary conditions

$z(-\omega)= z(\omega)$, $z'(-\omega)= z'(\omega)$, $z''(-\omega)= z''(\omega)$.

We now extend $z(t)$ periodically with period $2\omega$ to obtain a periodic solution of (1) (see [1]). Obviously

$$\int_0^{2\omega} x(t)\, dt = \int_0^{\omega} x(t)\, dt + \int_0^{2\omega} x(t)\, dt = \int_0^{\omega} x(t)\, dt + \int_0^{\omega} x(t+\omega)\, dt$$

$$= \int_0^{\omega} x(t)\, dt - \int_0^{\omega} x(t)\, dt = 0$$

Let us now consider a few applications of theorem 1.

$(A_1)$ Consider the third-order differential equation which is given by Reissig [2], in its general form

$$x''' + \phi(x')x'' + k^2 x' + f(x) = \mu p(t) \qquad (10)$$

**Theorem 2.** Equation (1) admits $2\omega$-periodic solution if we further assume

$$O \leq \phi(x) \leq b < \frac{k}{2}, \quad for\, all\ x \qquad (11)$$

$$\frac{|f(x)|}{|x|} \to 0 \quad (|x| \to \infty) \qquad (12)$$

$$x.f(x) \geq 0 \qquad (13)$$

$$p(t+\omega) = -p(t) \qquad (14)$$

**Proof.** We note that condition (12) implies that for any $\varepsilon > 0$, there exists a number $L(\varepsilon)$ such that

$$|f(x)| < \varepsilon C \quad if \quad C > L(\varepsilon) \quad and \quad |x| \leq C \qquad (15)$$

indeed, if $r(\varepsilon)$ is such that $|f(x)| < \varepsilon |x|$, $for \quad |x| \leq r(\varepsilon)$, $if$

$$M_1 = \max \{ |f(x)|, |x| \leq r(\varepsilon) \},$$

and

$$L(\varepsilon) = \max \left\{ r(\varepsilon), \frac{M_1}{\varepsilon} \right\}.$$

Then $L(\varepsilon)$ satisfies (15). On the other hand

$$M = \max \{ |k^2 x' - f| : t \in (0,\omega], |x| \leq C, |x'| \leq Ck, |x''| \leq Ck^2 \}$$
$$\leq \varepsilon C + Ck^2 b + |\mu| P$$

where $P = max|p(t)|, t \in [0,\omega]$. We need to show that (2) is satisfied for some small value of $|M|$, i. e.

$$\varepsilon C + Ck^2 b + |\mu| P \leq \frac{1}{2} Ck^2$$

or

$$\varepsilon C + |\mu| P \leq Ck^2 (\frac{1}{2} k - b),$$

by (11). The right-hand side of the above inequality is positive, and if we take $\varepsilon = |\mu|$ $and$ $|\mu|$ small enough (2) is satisfied and hence (10) possesses a $2\omega$-periodic solution $x(t)$ for which $|x(t)| \leq C, |x'(t)| \leq Ck^2$.

$(A_2)$ We consider the equation

$$x''' + \psi(x')x''^{(2n+1)} + a^2 x' + x^{2n+1} = \mu p(t) \quad n > 1 \qquad (16)$$

**Theorem 3.** Equation (16) admits $2\omega$-periodic solutions if the following conditions are satisfied

$$0 \leq \psi(x') \leq b \quad \text{for all } x' \qquad (17)$$

$$p(t,\omega) = -p(t) \qquad (18)$$

$|\mu|$ is sufficiently small (19)

**Proof.** Let $\frac{k}{\sqrt{2}} < a \leq k$, $\omega \in (o, \frac{\pi}{k}]$, and note that

$$M = \max \{ |k^2 x' - f(t,x,x',x'')| : t \in (0,\omega], |x| \leq C, |x''| \leq Ck^2 \}$$
$$\leq (k^2 - a^2) kC + bC^{2n+1} + C^{2n+1} + |\mu| P$$

where

$$P = max|p(t)|, t \in [0,\omega]$$

We only need to show that (2) is satisfied for some C. That is

$$(k^2 - a^2) kC + bC^{2n+1} + k^{2(2n+1)} + C^{2n+1} + |\mu| p \leq \frac{1}{2} k^3 C$$

or

$$bC^{2n} + k^{2(2n+1)} + C^{2n} + \frac{|\mu|}{C} p \leq k\, (a^2 - \frac{k^2}{2})$$

let $C = |\mu|^{\frac{1}{n}}$ then

$$b\,k^{2(2n+1)}|\mu|^2+|\mu|^2+|\mu|^{1-\frac{1}{n}}p=|\mu|^{1-\frac{1}{n}}\{P+|\mu|^{1+\frac{1}{n}}[1+b\,k^{2(2n+1)}]\}$$
$$\leq k\,(a^2-\frac{k^2}{2})$$

It is obvious that for sufficiently small $|\mu|$, we can make the above inequality to be true.
Hence by corollary 1, (16) has a $2\omega$-periodic solution $x(t)$ that

$$|x(t)|\leq|\mu|^{\frac{1}{n}},|x'(t)|\leq|\mu|^{\frac{1}{n}}k,|x''(t)|\leq|\mu|^{\frac{1}{n}}k^2$$

**(A₃)** Consider the equation

$$x''+x'+x^3=\frac{1}{8}\text{Sin }4t \qquad (20)$$

In this example we take $k=1$, $\omega=\frac{\pi}{4}$ and

$$M=\{|x'-f(t,x,x',x'')|:t\in(0,\omega],\ |x|\leq C,\ |x'|\leq Ck,|x''|\leq Ck^2\}$$
$$\leq C^3+\frac{1}{8}.$$

Hence for the condition (2) to be satisfied we must have

$$C^3+\frac{1}{8}\leq\frac{1}{2}C.$$

Obviously it is true if we take $C=\frac{1}{2}$. Therefore, by corollary 1, equation (21) has a $\frac{\pi}{2}$ periodic solution for which

$$|x|\leq C,|x'|\leq Ck,|x''|\leq Ck^2.$$

## References


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