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# THE LEFT REGULAR REPRESENTATION OF A COMMUTATIVE SEPARATIVE SEMIGROUP

B. Tabatabaie Shourijeh

*Department of Mathematics, University of Shiraz, Shiraz, Islamic Republic of Iran*

**Abstract**

In this paper, a commutative semigroup will be written as a *disjoint union* of its *cancellative* subsemigroups. Based on this fact we will define the *left regular representation* of a commutative separative semigroup and show that this representation is *faithful*. Finally concrete examples of commutative separative semigroups, their decompositions and their *left regular representations* are given.

## Introduction

A semigroup $\Sigma$ is called separative if for every s, t in $\Sigma$

$$s^2 = st = t^2$$

implies s=t.

Throughout this paper, unless otherwise specified, $\Sigma$ will denote a commutative separative semigroup (C.S.S.).

In this paper, we will define an equivalence relation on $\Sigma$ and show that each equivalence class is a cancellative subsemigroup of $\Sigma$. By the decomposition of $\Sigma$ under this equivalence relation we will define the left regular representation of $\Sigma$ and it will be shown that this representation is faithful.

## Results

**1.1 Definition.** For each s in $\Sigma$, the set $h_s=\{t \in \Sigma: t^n = su \text{ and } s^m=vt, \text{ for some } u,v \text{ in } \Sigma \text{ and } m,n \text{ in } N\}$ is called an *archimedean* component of $\Sigma$.

Now we state an important theorem on which this section is based. Note that other versions of this theorem can be seen in [1] and [2].

**1.2 Theorem.** (a) The relation

$$s \sim t \text{ if and only if } t \in h_s$$

is an equivalence relation on $\Sigma$ [3, Proposition 4:3.3];

(b) If $s \in \Sigma$, then $h_s$ is a subsemigroup of $\Sigma$ [3, Proposition 4:3.4];

(c) If $s \in \Sigma$, then $h_s$ is a *cancellative subsemigroup* of $\Sigma$ [3, Theorem 4:3.5].

Note that by part (a) of the above theorem, for every $\Sigma$ we can write

$$\Sigma = \bigcup_{s \in \Sigma} h_s.$$

Moreover, the cancellative property of each $h_s$ is among the necessary tools in considering the properties of the left regular representation of $\Sigma$.

At this stage we proceed to introduce a candidate for the left regular representation of $\Sigma$.

Let $\{\delta_t : t \in \Sigma\}$ be the standard orthonormal basis for $\ell^2(\Sigma)$. To each $s \in \Sigma$, we associate the linear operator $\lambda(s)$ on $\ell^2(\Sigma)$ such that

$$\lambda(s)\,\delta_t = \begin{cases} \delta_{st} & \text{if } st \in h_t \\ 0 & \text{otherwise} \end{cases} \quad (\phi)$$

We consider the properties of $\lambda$ in the following lemmas.

**1.3 Lemma.** If we correspond to each $s \in \Sigma$, the linear operator $\lambda(s)$ such that $(\phi)$ holds, then

$$\lambda(rs) = \lambda(r)\,\lambda(s)$$

for every r,s in $\Sigma$.

**Proof.** If $t \in \Sigma$, then

$$\lambda(rs)\,\delta_t = \delta_{rst}$$

if and only if $rst \in h_t$.

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And,

$$\lambda(r)\lambda(s)\delta_t = \lambda(r)\delta_{st} = \delta_{rst}$$

if and only if

$$st \in h_t \text{ and } rst \in h_{st}$$

Therefore in order to show that

$$\lambda(rs) = \lambda(r)\lambda(s)$$

it is enough to prove that,

$$rst \in h_t \text{ if and only if } st \in h_t \text{ and } rst \in h_{st}.$$

To see this, from $st \in h_t$ and $rst \in h_{st}$, we have

$$t \sim st \text{ and } st \sim rst$$

Now since $\sim$ is an equivalence relation on $\Sigma$ we have

$$t \sim rst, \text{ i.e. } rst \in h_t.$$

Conversly let $rst \in h_t$. Hence

$$(rst)^m = ut \text{ and } t^m = v(rst)$$

for some u,v in $\Sigma$ and m,n in N. Since

$$t^m = v(rst) = (vr)(st) \quad (1)$$

we have

$$(ts)^m = t^m s^m = s^m v(rst) = (s^{m+1} vr)t. \quad (2)$$

From (1) and (2) we see that

$$st \in h_t.$$

Now from $rst \in h_t$ and $st \in h_t$ it is easily seen that

$$rst \in h_{st}.///$$

The following lemma shows that each $\lambda(s)$ is a partial isometry.

**1.4 Lemma.** For $s \in \Sigma$, the linear operator $\lambda(s)$ which is defined in ($\phi$) is a partial isometry.

Proof. Let $s \in \Sigma$ and

$$D_s = \{t \in \Sigma : st \in h_t\}.$$

We will prove that each $\lambda(s)$ is a partial isometry with the initial space $\ell^2(D_s)$. It suffices to show that the map

$$t \rightarrow st$$

is injective on $D_s$.

Let $t_1, t_2 \in D_s$ and $st_1 = st_2$. Since $st_1 \in h_{t_1}$ and $st_2 \in h_{t_2}$, from

$$t_1 \sim st_1 = st_2 \sim t_2$$

we have $t_1 \in h_{t_2}$. Therefore $h_{t_1} = h_{t_2}$. Hence

$$st_1 = st_2 \in h_{t_1}.$$

Since $h_{t_1}$ is a cancellative semigroup from $st_1 = st_2$ we have $t_1 = t_2$. ///

Now we have the following definition.

**1.5 Definition.** Let $\Sigma$ be a commutative separative semigroup. For each $s \in \Sigma$, the linear operator $\lambda(s)$ on $\ell^2(\Sigma)$ defined by,

$$\lambda(s)\delta_t = \begin{cases} \delta_{st} & \text{if } st \in h_t \\ O & \text{Otherwise,} \end{cases}$$

is the left regular representation of $\Sigma$.

Before giving some examples, it should be noted that by [4], PI (H), the partial isometries on a Hilbert space H, from a semigroup.

**1.6 Examples.**

(a) Let $\Sigma$ be the additive semigroup

$$Z^+ = \{0,1,2,3,\ldots\}.$$

Obviously $Z^+$ is separative, $h_o = \{o\}$, $h_1 = \{1,2,3,\ldots\}$ and

$$Z^+ = h_o \cup h_1.$$

$$\lambda : Z^+ \rightarrow PI(\ell^2(Z^+)) \text{ by}$$

$$\lambda(m)\delta_n = \begin{cases} \delta_{m+n} & \text{if } m+n \in h_n \\ o & \text{Otherwise} \end{cases}$$

is the left regular representation of $Z^+$. It is easily seen that $\lambda(o) = I$, $\lambda(1)$ is the unilateral shift operator, S, and

$$\lambda(m) = \lambda(1)^m = S^m.$$

(b) Let $\Sigma$ be the additive semigroup $N^+ = \{1,2,3,\ldots\}$. Obviously $h_1 = N$ and,

$$\lambda : N \rightarrow PI(\ell^2(N)) \text{ defined by}$$

$$\lambda(m)\delta_n = \delta_{m+n}$$

is the left regular representation of $N^+$. Clearly $\lambda(1)$ is a shift of multiplicity one and

$$\lambda(m) = \lambda(1)^m$$

is a shift of multiplicity m.

We close this section by proving that, the left regular representation of a commutative separative semigroup is faithful.

**1.7 Theorem.** If $\Sigma$ is a commutative separative semigroup and $\lambda$ its left regular representation, then $\lambda$ is faithful.

**Proof.** Let $s_1, s_2 \in \Sigma$ and $\lambda(s_1) = \lambda(s_2)$. Since each $\lambda(s)$ is a partial isometry with the initial space $\ell^2(D_s)$ where

$$D_s = \{t \in \Sigma : st \in h_t\},$$

from $\lambda(s_1) = \lambda(s_2)$ we have $D_{s_1} = D_{s_2}$.

Since $s_1^2 \in h_{s_1}$ we see that

$$s_1 \in D_{s_1} = D_{s_2}.$$

From $s_1 \in D_{s_2}$ we have

$$s_2 s_1 \in h_{s_1}.$$

Therefore

$$\lambda(s_1)\,\delta_{s_1} = \lambda(s_2)\,\delta_{s_1}$$

or,

$$s_1^2 = s_1 s_2.$$

Since $h_{s_1}$ is a cancellative semigroup and

$$s_1^2 = s_1 s_2 \in h_{s_1}$$

we have $s_1 = s_2$ i.e., $\lambda$ is faithful. ///

G.K. Pedersen in [5, 7.2.1] defined the reduced C*-algebra, $C_r^*(G)$, of a group. Here in a similar way we can associate a C*-algebra to a semigroup.

## Conclusion

(a) For a given commutative separative semigroup $\Sigma$ we can define its reduced C*-algebra, $C_r^*(\Sigma)$ as follows:

$$C_r^*(\Sigma) = C^*(\{\lambda(s), \lambda(s)^*\}),\ s \in \Sigma.$$

where $\lambda$ is the left regular representation of $\Sigma$.

(b) If D is a commutative cancellative semigroup, it is obviously a separative semigroup, therefore we can consider its reduced C*-algebra, $C_r^*(D)$.

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