SOME RESULTS OF CONTINUITY OF ω_f

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Abstract

The dynamical behavior of a map on the unit interval has been the subject of much contemporary research. In this paper, we will consider the relation between the continuity of the map ω_f and ω_f for some positive integer k, where f is a continuous map from the unit interval to itself, and ω_f is a function which takes any element of the unit interval to the set of all subsequential limits of the orbit of x under f. Also, it is shown that for any k, the continuity of ω_f implies the equicontinuity of the iterates $\{f^n\}$.

Introduction

Suppose we are studying a social or physical system on which we make measurements at regular intervals. For example, suppose we are measuring the population of a simple species each year. Suppose that the population changes at a rate that is directly proportional to the population at the given time. Let $P(t)$ denote the population at time t, then we have:

$$\frac{dP}{dt} = \lambda P.$$

With the assumption of $P_0 = P(0)$ the solution to this differential equation is $P(t) = P_0 e^{\lambda t}$. Note that in this extremely simple situation, we did not take into account the obvious factors such as immigration, deathrate, overcrowding, etc. Let P_n denote the population after n generation, and use the most highly simplified method, that is, the population in the $n+1^{st}$ generation is directly proportional to the population of the n^{th} generation with the constant of proportionality λ. Hence,

$$P_{n+1} = \lambda P_n.$$

Let $f(x) = \lambda x$. If $x = P_0$, then $f(x) = P_1, f(f(x)) = P_2$, $f(f(f(x))) = P_3$, and so on. Experience shows these models are highly idealized. Therefore, to get a better reflection of reality we incorporate an additional constraint or parameter. That is, we assume there is some limiting value P for the population. A reasonable model would then be a generalized logistic equation

$$P_{n+1} = \lambda P_n(1 - P_n/P).$$

The dynamical behavior of this equation and even its more simplified version $f(x) = \lambda x(1-x)$, which again is known as the logistic equation, have been the subject of much contemporary research, and lead to one of the most complicated dynamical systems (for more details see [5], [4] and [9]), namely, the orbits of relatively close points may be far apart. Indeed, there is a set of points S (countable or even having a positive measure [10]) such that for any $x, y \in S, x \neq y$,

$$\limsup \left| f^n(x) - f^n(y) \right| > 0,$$
$$\liminf \left| f^n(x) - f^n(y) \right| = 0.$$

In other words f is chaotic in the sense of [6] (see also [11], [12] and [1]).

The datailed dynamical behavior of function $f(x) = \lambda x\ (1-x)$ for different λ can be found in [5]. Indeed, it is proved that for $\lambda = 4$, f is chaotic on the entire interval [0, 1]. For behavior of non-periodic flows and infinite limit set of iterated maps on an interval one may see [7] and [8].

Let $f: I \to I$ be a continuous function where I is the unit interval. For $x \in I$ we define the orbit of x to be the set of points $x, f(x), f^2(x), \ldots$, and we show this

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set by $O(x)$, where $f^{n+1}=f(f^n(x))$ for $n=1, 2,\ldots$. The attractor set of x under f is defined to be the set of all subsequential limits of $O(x)$, and we denote this set by $\omega(x,f)$. In the next section, we will consider the behavior of the map which takes an element $x \in I$ to $\omega(x,f)$. This map is denoted by ω_f. Also, we will find conditions under which continuity of this map and the map ω_{f^k} for an integer k is related to the equicontinuity of the family of iterates $\{f^n\}$.

Regularity

The following lemma will connect the concepts of continuity of ω_f and ω_{f^k}. In the following $f: I \to I$ is assumed to be a continuous function.

Lemma (2,1). Let k be any positive integer, then $\omega(x,f)=\cup_{j=0}^{k-1}\omega(f^j(x),f^k)$.

Proof. For $x_0 \in \omega(x,f)$, choose A= $\{n_1, n_2,\ldots\}$, such that $\{f^{n_i}(x)\}$ converges to x_0. For $0 \le j < k$, define $A_j = \{m \in A$: there exists an $l \in N$ such that $m=kl+j\}$. That is, we partition A, and hence $A=\cup_{j=0}^{k-1}A_j$. Suppose for some $0 \le j_0 < k$, A_{j0} is infinite. Enumerate A_{j0} and suppose that $A_{j0}=\{m_i^{j0}: i \in N\}$, then $\left\{f^{m_i^{j0}}\right\}$ will converge to x_0. Hence $x_0 \in \omega(f^{j0}(x),f^k)$. The rest is clear.

Let W denote the set of all nonempty compact subsets of I. For $A, B \in W$, we define $\rho(A,B)$ to be equal the distance between these sets. That is $\rho(A,B)=dist(A,B)$. Then (W,ρ) is a compact metric space.

Proposition (2, 2). If for some k, ω_{f^k} be continuous, then ω_f is continuous.

Proof. Observe that by lemma (2, 1), we have,

$$\rho(\omega(x,f),\omega(y,f)) \le \rho(\omega(x,f^k),\omega(y,f^k)).$$

In [3], Bruckner and Hu showed that on a compact metric space (X,ρ), if f is a surjective map on X, whose sequence of iterates $\{f_n\}$ is equicontinuous, then f is a homeomorphism. In particular if X is a closed interval, then f^2 is identity on X.

The connection of equicontinuity of $\{f^n\}$ and Γ-compactness of f is described in [2]. We state the following definition and proposition for later use.

Definition (2, 3). $f: I \to I$ is said to be Γ-compact if every sequence of iterates $\{f^n\}$ has a subsequence which is uniformly convergent on compact subsets of I.

On the compact set $I=[0,1]$, the criterion for Γ-compactness of a map f have been seen ([2]) to be dependent only on the connectedness of the fixed point set of the function f^2. For proof of the following proposition see [2].

Proposition (2, 4). $f: I \to I$ is Γ-compact if and only if the sequence $\{f^n\}$ is equicontinuous.

Proposition (2, 5). Let $f: I \to I$ be given, and F_2 be the set of fixed points of f^2. Then $F_2=\cap_{n=1}^{\infty}f^n(I)$ if and only if $\{f^n\}$ is equicontinuous.

Proof. First suppose $F_2=\cap_{n=1}^{\infty}f^n(I)$. Then F_2 is connected. Hence f is Γ-compact. Thus, by proposition (2, 4) the sequence of iterates $\{f^n\}$ is equicontinuous. Conversely, suppose $\{f^n\}$ is equicontinuous, then f is a homeomorphism on the interval $\cap_{n=1}^{\infty}f^n(I)$, and hence f^2 is identity on this interval. Therefore $F_2=\cap_{n=1}^{\infty}f^n(I)$.

Theorem (2, 6). The family of iterates $\{f^n\}$ is equicontinuous if and only if there exists a positive integer k such that ω_{f^k} is continuous.

Remark. Equicontinuity of $\{f^n\}$ implies the continuity of ω_{f^k} for any k.

Proof. Let $\{f^n\}$ be equicontinuous. It is clear that for $k \in N$, the family of iterates $\{f^{kn}\}$ is equicontinuous. Hence, for the given positive ε, there exists a positive δ, such that $|x-y|<\delta$ implies $|f^{kn}(x)-f^{kn}(y)|<\frac{\varepsilon}{4}$ for all n. Suppose x_0 be a given element of $\omega(x,f^k)$. Then there exists a sequence $\{n_i\}$ such that $|f^{kn_i}(x)-x_0|<\frac{\varepsilon}{4}$, for all i. Thus, if $|x-y|<\delta$, then $|f^{kn_i}(y)-x_0|<\frac{\varepsilon}{2}$. Therefore,

$$\rho(\{x_0\},\omega(y,f^k))<\varepsilon/2.$$

Likewise for any y_0 in $\omega(y,f^k)$ we will have

$$\rho(\{y_0\},\omega(x,f^k))<\varepsilon/2.$$

for all x with $|x-y|<\delta$. Thus, $|x-y|<\delta$ implies

$$\rho(\omega(x,f^k),\omega(y,f^k))<\varepsilon.$$

Suppose there exists $k \in N$ such that ω_{f^k} is continuous. Thus, by proposition (2,2) ω_f is continuous. Without loss of generality we may assume that $\cap_{n=1}^{\infty}f^n(I)$ is a non-degenerate closed interval, say [a, b]. Let F_i be the set of fixed points of the function f^i for i=1,2,.... Suppose there exist two different points $\alpha, \beta \in F_1$ such that $(\alpha,\beta) \cap F_1 = \phi$. Without loss of generality, we may assume that $f(x) > x$ on (α,β). Define $I_1=\{x \in (\alpha,\beta): f(x)=\beta\}$. If $I_1 \ne \phi$, we choose an element $x_1 \in I_1$, and define $I_2=\{x \in (\alpha,\beta): f(x)=x_1\}$. Next pick up an element $x_2 \in I_2$, and so on. By induction, we will have a sequence (x_n) such that $f^n(x_n)=\beta$ for n =1,2,..., and (x_n) is a decreasing sequence converging to α. Thus, by continuity of ω_f, $\alpha = \beta$. If $I_1 = \phi$, then for any $x \in (\alpha, \beta)$, $\omega(x, f) = \{\beta\}$, which again by continuity of ω_f, we must have $\alpha = \beta$. Therefore, F_1 is a connected set.

Now suppose $\{f^n\}$ is not equicontinuous, then by proposition (2,5) there exists an $x \in [a, b]$ such that $f^2(x) \neq x$. Since F_1 is connected and $f([a, b]) = [a, b]$, there must be at least two different points $\gamma, \delta \in [a, b]$ such that $F_2 \cap (\gamma, \delta) = \phi$. Hence, there exists a sequence $\{x_n\}$ converging to γ such that $\omega(x_n, f^2) = \{\delta\}$. But continuity of ω_f, implies $\{\gamma, f(\gamma)\} = \{\delta, f(\delta)\}$. Thus, $\gamma = f(\delta)$, $\delta = f(\gamma)$. So f^2 has a fixed point in (γ, δ), which is a contradiction.

As an application of theorem (2,6) in the following we shall see an "inverse" version of proposition (2,2).

Corollary (2,7). If ω_f is continuous, then ω_{f^k} is continuous for any positive integer k.

Proof. The continuity of ω_f implies the equicontinuity of $\{f^2\}$. Thus, for any k, $\{f^{kn}\}$ is equicontinuous, hence by theorem (2,6) and proposition (2,5) ω_{f^k} is continuous.

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(a) A. M. Bruckner. Three forms of chaos and their associated attractors, preprint.

(b) A. M. Bruckner and J. Ceder. Chaos in terms of map

$$x \to w(x, f),$$

to appear in the Pacific Journal of Mathematics.

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