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A COMMUTATIVITY CONDITION FOR RINGS

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Abstract

In this paper, we use the structure theory to prove an analog to a well-known theorem of Herstein as follows: Let R be a ring with center C such that for all $x,y \in R$ either $[x,y]=0$ or $x-x^n[x,y] \in C$ for some non negative integer $n=n(x,y)$ depending on x and y. Then R is commutative.

Introduction

Throughout this paper, R represents an associative ring with center C, and $J(R)$ denotes the Jacobson radical of R. As usual for $x,y \in R$ the commutator $xy-yx$ is denoted by $[x,y]$.

The Jacobson structure theory is one of the most useful in proving that appropriately conditioned rings are commutative or anticommutative [1,3]. Using this theory, Herstein [1] proved the following theorem:

Let R be a ring with center C such that for a fixed integer $n>1$, $x-x^n \in C$ for all $x \in R$ then R is commutative. This is one of the finest results in ring theory.

The objective of this paper is to prove an analog to the above-mentioned result. Indeed, we prove the following:

Theorem 1.1. Let R be a ring with center C such that for a fixed integer $n>1$ either $x-x^n[x,y] \in C$, or $[x,y]=0$ for all $x,y \in R$. Then R is commutative.

Materials and Methods

Preliminary Lemmas

We first establish the following Lemmas for a ring R satisfying the hypothesis of Theorem 1.1.

Lemma 2.1. If R is a division ring it is commutative.

Proof. First note that,

$[x,y]$ commutes with both x and y, for all $x,y \in R$. (1)

If not, for some $x,y \in R$, $[[x,y],y] \neq 0$, hence $x \neq 0$, $y \neq 0$ and $[x,y] \neq 0$. Being in a division ring we deduce that

$$0 \neq x^n[[x,y],x]=[x^n[x,y],x]=-[x-x^n[x,y],x],$$

that is $x-x^n[x,y] \notin C$; contrary to the hypothesis. Since R, as a division ring, has no nonzero nil ideal; we can, at this point, conclude that R is commutative by Herstein [2]. But we will finish the proof of this Lemma as follows:

Let $x,y \in R$, by (1) we have:

$$x^2y[x,y]=x(xy)[x,y]=x[x,y]xy \quad (2)$$

But $x[x,y]=-x[y,x]=-(xyx-x2y)=-[xy,x]$, thus by (2),

$$x^2y[x,y]=-[xy,x]xy \quad (3)$$

By (1), we know that xy commutes with $[xy,x]$, therefore (3) implies that

$$x^2y[x,y]=-xy[xy,x] \quad (4)$$

Since $[xy,x]=x[y,x]=-x[x,y]$, hence (4) yields that

$$x^2y[x,y]=xyx[x,y]$$

that is $(x^2y-xyx)[x,y]=0$, or $x[x,y]^2=0$. At any rate, since R is a division ring we must have $[x,y]=0$. Thus R is commutative.

Lemma 2.2. If R is semisimple it is commutative.

Proof. As is well-known, R is a subdirect sum of rings R_i which are primitive. As a homomorphic image of R, each R_i satisfies the hypothesis placed on R. Thus, to show that R is commutative, it suffices to

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prove that each R_i is commutative, in other words we may assume that R is primitive. As a primitive ring, either $R \approx D$ for some division ring D, or for some $k>1$ D_k is a homomorphic image of a subring of R. We wish to show that this latter possibility does not arise. If it did, then D_k, the complete matrix ring over D, satisfies the hypothesis placed on R. This is cleary false for the elements

$$x=\begin{bmatrix} 1 & 1 \dots & 0 \\ 0 & 0 \dots & 0 \\ 0 & 0 \dots & 0 \end{bmatrix}, \quad y=\begin{bmatrix} 0 & 1 \dots & 0 \\ 0 & 0 \dots & 0 \\ 0 & 0 \dots & 0 \end{bmatrix}$$

in D_k satisfy $x^2=x$, $[x,y]=y \neq 0$, and $x\text{-}x^n\,[x,y] \notin C$ for all positive integers n. Thus, R must be a division ring, hence it is commutative by Lemma 2.1. In this way, R is seen as a subdirect sum of commutative rings and so it must be commutative.

For general R satisfying the hypothesis of Theorem 1.1, since $R/J(R)$ is semisimple, we have

Corollary. For all $x,y \in R$, $[x,y] \in J(R)$.

Lemma 2.3. (i) If $z \in C$ and $x \in R$, then $(z^{n+1}\text{-}z)\,x \in C$.
(ii) If $z \in C \cap J(R)$, then $z[x,y]=0$ for all $x,y \in R$.

Proof (i). If $zx \in C$, there is nothing to prove. Suppose that $[zx,y] \neq 0$ for some $y \in R$, then from $z \in C$ we deduce that $[x,y] \neq 0$; and by the hypothesis we have

$$(zx)\text{-}(zx)^n\,[zx,y] \in C \text{ and } x\text{-}x^n\,[x,y] \in C. \tag{2.1}$$

Having $z \in C$, (2.1) implies that

$$(zx)\text{-}(z^{n+1}x^n)\,[x,y] \in C \text{ and } (z^{n+1}x)\text{-}(z^{n+1}x^n)\,[x,y] \in C, \tag{2.2}$$

hence $(z^{n+1}\text{-}z)\,x \in C$.

Proof (ii). Let $z \in C \cap J(R)$ and let $x,y \in R$. By part (i),

$$[(z^{n+1}\text{-}z)x,y]=0. \tag{2.3}$$

Since $z \in C$, (2.3) yields that

$$z^{n+1}\,[x,y]=z[x,y]. \tag{2.4}$$

On the other hand, from $z \in J(R)$, we get $z^n \in J(R)$. Hence, z^n is a quasi-regular element in R. Therefore, from (2.4) we deduce that $z[x,y]=0$.

Lemma 2.4. For all $x,y \in R$, $[x,y]$ commutes with both x and y.

Proof. Since $[x,y]=-[y,x]$, it suffices to show that

$$[[x,y],x]=0 \text{ for all } x,y \text{ in } R. \tag{2.5}$$

To prove (2.5), let $x,y \in R$ and set
$a=[x,y]$.

Obviously if $[a,x]=0$ we would be done. Therefore, by the hypothesis placed on R, we may assume that

$$a\text{-}a^n\,[a,x] \in C. \tag{2.6}$$

By the Corollary above $a,[a,x] \in J(R)$; hence

$$a\text{-}a^n\,[a,x] \in C \cap J(R). \tag{2.7}$$

In view of Lemma 2.3 (ii), (2.7) yields that

$$(a\text{-}a^n\,[a,x])\,[x,y]=0. \tag{2.8}$$

Since $a=[x,y]$, from (2.8) we get

$$a^2=a^n\,[a,x]a. \tag{2.9}$$

Having $a^{n-2}\,[a,x]a$ (= $[a,x]a$, if $n=2$) in $J(R)$, (2.9)implies that $a^2=0$. Hence $a \in C$, by (2.6). This proves (2.5) and completes the proof of this Lemma.
With the above Lemmas established we are able to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $x,y \in R$. If $[x,y]=0$ we are done. Therefore, it is enough to show that $x\text{-}x^n\,[x,y] \in C$ also implies that $[x,y]=0$. But having $x\text{-}x^n\,[x,y] \in C$ we get

$$[x\text{-}x^n[x,y],y]=0. \tag{2.10}$$

Since by Lemma 2.4 $[x,y]$ commutes with both x and y, (2.10) yields that

$$[x,y]=[x^n[x,y],y]=[x,y]\,[x^n,y]. \tag{2.11}$$

But by the Corollary $[x^n,y] \in J(R)$, hence $[x^n,y]$ is a quasi-regular element in R. Therefore, from (2.11) we deduce that $[x,y]=0$. Theorem 1.1 is now proved.

Results and Discussion

Remark 3.1. Suppose that for all x,y in R either $[x,y]=0$ or $x\text{-}[x,y] \in C$ then R is commutative. Because if $[x,y] \neq 0$ for some $x,y \in R$ we have $x\text{-}[x,y] \in C$ and $y\text{-}[y,x] \in C$. This would place $x+y$ in C, hence $0=[x+y,y]=[x,y]$ contrary to $[x,y] \neq 0$.

Remark 3.2. Let m be a fixed positive integer such that for all x,y in R either $[x,y]=0$ or $x\text{-}x^m[x,y] \in C$. Suppose that $[x,y] \neq 0$ for some x,y in R, then $x\text{-}x^m\,[x,y] \in C$ and $x \notin C$, hence $x[x,y] \neq 0$ i.e. $[x,xy] \neq 0$. From $[x,xy] \neq 0$ we get $x\text{-}x^m\,[x,xy] \in C$, i.e. $x\text{-}x^{m+1}$

$[x,y] \in C$. Continuing in this way it can be shown that if $[x,y] \neq 0$ then $x\text{-}x^n [x,y] \in C$ for all integers $n \geq m$. Since in any stage of the proof of Theorem 1.1 we just deal with a finite number of elements of R, then in view of the above remarks from Theorem 1.1 we get

Theorem 3.1. Let R be a ring with center C such that for all $x,y \in R$ either $[x,y]=0$ or $x\text{-}x^n[x,y] \in C$ for some non negative integers $n=n(x,y)$ depending on x and y (for $n=0$, $x^n [x,y]= [x,y]$). Then R is commutative.

Remark 3.3. If we replace the hypothesis of Theorem 3.1 by

"$x\text{-}x^n [x,y] \in C$, for all x,y in R".

Then the theorem would be trivial; because for each x in R from $x\text{-}x^n [x,x] \in C$ we get $x \in C$.

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