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MODULE HOMOMORPHISMS ASSOCIATED WITH HYPERGROUP ALGEBRAS

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Abstract

Let X be a hypergroup. In this paper, we study the homomorphisms on certain subspaces of $L(X)^*$ which are weak*-weak* continuous.

Keywords: Homomorphisms; Hypergroup algebras; Weak*-weak* continuous

1. Introduction and Notations

The theory of hypergroups was initiated by Dunkl [3], Jewett [7] and has received a good deal of attention from harmonic analysts. It is still unknown whether an arbitrary hypergroup admits a left Haar measure (for more information see [2]). The lack of the Haar measure and involution presents many difficulties, however, we succeed to get some interesting results. Let X be a hypergroup (for more information see [3] or [10]) with convolution measure algebra M(X) and probability measures $M_p(X)$. Recall that L(X) denotes the set of all measures $\mu\in M(X)$ for which the mapping $x\to|\mu|*\delta_x$ is norm-continuous [6,10]. We assume that X is foundation, i.e. $\cup\{supp(\mu);\ \mu\in L(X)\}$ is dense in X. It is well known that L(X) is an ideal in M(X) and L(X) has a positive bounded approximate identity bounded by 1 ([6], Lemma 1).

The first Arens product on $L(X)^{**}$ is defined in three steps as follows. For $\mu,\nu\in L(X)$, f in $L(X)^*$ and F,G in $L(X)^{**}$, the elements $f\mu$, Ff of $L(X)^*$ and GF of $L(X)^{**}$ are defined by $<f\mu, \nu>=<f, \mu*\nu>$, $<Gf, \mu>=<G, f\mu>$ and $<FG, f>=<F, Gf>$.

Let $B=L(X)^*L(X)$, we know that B is a Banach subspace of $L(X)^*$. The formulas which define the first Arens product in $L(X)^{**}$ can also be used to define a Banach algebra structure on B^* [10]. Finally, for every $\mu\in L(X)$, $\nu\in M(X)$ and $f\in L(X)^*$ we define $<\nu,f\mu> =<f,\mu*\nu>$ and $<f\nu,\mu> =<f,\nu*\mu>$, so that $M(X)\subseteq B^*$. Also, we define $<mf,\nu>=<m,f\nu>$ for any $m\in B^*$, $f\in L(X)^*$ and $\nu\in L(X)$. Most of our notation in this paper coming from [6,10]. In this paper, we will characterize some homomorphisms which are weak*-weak* continuous (see below).

2. Main Results

We remember that the topological centre of B^* is defined by $Z_t(B^*)=\{m\in B^*;$ the mapping $n\to mn$ is weak*-weak* continuous$\}$. When X is a hypergroup with an involution and Haar measure, it is known that $Z_t(B^*)=M(X)$. We do not exactly know $Z_t(B^*)=M(X)$, however, for the following Theorem, we assume that $Z_t(B^*)=M(X)$.

Lemma 2.1. Let $T:B\to L(X)^*$ be a bounded linear map. Then $T\in Hom_{L(X)}(B,L(X)^*)$ (where

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$T\in Hom_{L(X)}(B,L(X)^*)$ means $T(f\mu)=T(f)\mu$ for $f\in B$, $\mu\in L(X)$) if and only if $T(f\delta_x)=T(f)\delta_x$ for $f\in B$ and $x\in X$.

Proof. Let $T\in Hom_{L(X)}(B,L(X)^*)$, and let (e_α) be a bounded approximate identity in L(X) ([6], Lemma 1). For $f\in B$ and $x\in X$, we have $T(f\delta_x)=\lim T(f e_\alpha*\delta_x)=\lim T(f)e_\alpha*\delta_x= T(f)\delta_x$, i.e. $T(f\delta_x)=T(f)\delta_x$.

To prove the converse, let $f\in B$ and $\mu\in L(X)$. Let T^* be adjoint to T. By ([4], Lemma 3.4), for each $\nu\in L(X)$, we can write $<T(f\mu),\nu>=<T^*(\nu),f\mu>=\int<T^*(\nu),f\delta_x>d\mu(x) =\int<\nu,T(f\delta_x)>d\mu(x)=\int<\delta_x*\nu,T(f)>d\mu(x)=<\mu*\nu,T(f)>=<T(f)\mu,\nu>$. This shows that $T(f\mu)=T(f)\mu$. Consequently $T\in Hom_{L(X)}(B,L(X)^*)$.

Notation 2.2. For $\mu\in M(X)$, let ρ_μ be a right multiplier on L(X) defined by $\rho_\mu(\nu)=\nu*\mu$, where $\nu\in L(X)$.

Define $T:C_b(X)\to L(X)^*$ by $<T(\varphi),\mu>=\int\varphi(x)d\mu(x)$ for every $\varphi\in C_b(X)$, $\mu\in L(X)$. Then it is easy to see that $\|T(\varphi)\|=\|\varphi\|$. This shows that we may identify $C_b(X)$ with a subspace of $L(X)^*$.

Theorem 2.3. Let $T:B\to L(X)^*$ be a bounded linear map such that;

1) $T(f\delta_x)=T(f)\delta_x$ for any $f\in B$ and $x\in X$,

2) T is weak*-weak* continuous.

Then $T= \rho_\mu^*$ for some $\mu\in M(X)$. Moreover, μ is unique and $\|\mu\|=\|T\|$.

Proof. By Lemma 2.1, $T\in Hom_{L(X)}(B,L(X)^*)$. It is easy to see that $T^*(\mu*\nu)=\mu T^*(\nu)$ for any $\mu,\nu\in L(X)$. Now, let (n_α) be a net in B^* such that $n_\alpha\to n$ ($n\in B^*$) in the weak*-topology. Since T is weak*-weak* continuous and $n_\alpha f\to nf$ ($f\in L(X)^*$) in the weak*-topology, so that for each $\nu\in L(X)$ we have $<T^*(\nu)n_\alpha,f>\to<T^*(\nu)n,f>$. It follows that $T^*(L(X))\subseteq Z_t(B^*)=M(X)$. On the other hand, L(X) has a bounded approximate identity, and L(X) is an ideal in M(X). Therefore $T^*(L(X))\subseteq L(X)$. By ([6], Proposition 1), there exists a measure $\mu\in M(X)$ such that $T^*(\nu)=\rho_\mu(\nu)$ for all $\nu\in L(X)$. Clearly $T=\rho_\mu^*$. It is easy to see that $\|T\|\le\|\mu\|$. Now, let $\varepsilon>0$ be given. There exists $\varphi\in C_\circ(X)$ with $\|\varphi\|\le1$ such that $|<\varphi,\mu>|\ge\|\mu\|-\varepsilon$. Let (e_α) be a bounded approximate identity with norm 1. Therefore $\|T\|\ge\|T(\varphi)\|\ge|\lim<\rho_\mu^*(\varphi),e_\alpha>|=|<\varphi,\mu>|\ge\|\mu\|-\varepsilon$ (since $C_\circ(X)\subseteq B$ [10]). Consequently $\|T\|=\|\mu\|$. It is easy to see that μ is unique. This completes our proof.

Corollary 2.4. Let G be a locally compact abelian group and $T:LUC(G)\to L^\infty(G)$ (where LUC(G) denote the closed subspace of bounded left uniformly continuous functions on G) be a bounded linear map such that;

1) $T(f\delta_x)=T(f)\delta_x$ for any $f\in LUC(G)$ and $x\in G$,

2) T is weak*-weak* continuous.

Then $T= \rho_\mu^*$ for some $\mu\in M(G)$. Moreover, μ is unique and $\|\mu\|=\|T\|$.

Proof. We know that $L^\infty(G)L(G)=LUC(G)$ and $Z_t(LUC(G)^*)=M(G)$ ([8], Theorem 1). The results follows from Theorem 2.3.

Let A be a Banach algebra with a bounded approximate identity. It is well known that A^{**} and $(A^*A)^*$ with the first Arens product are Banach algebras [1]. In addition, we define $<nf,a>=<n,fa>$ for $n\in(A^*A)^*$, $f\in A^*$ and $a\in A$.

We recall that multiplication in a locally convex algebra A is said to be hypocontinuous, if for every neighbourhood U of zero in A and a bounded subset C of A, there exists a neighbourhood V of zero such that $CV\cup VC\subseteq U$. The following Lemma shows that if multiplication in a Banach algebra A with a bounded approximate identity, is hypocontinuous in the weak-topology, then A^* factors on the left, i.e. $A^*A=A^*$ [9].

Lemma 2.5. Let A be a Banach algebra with a bounded approximate identity, and let the multiplication with weak-topology on A be hypocontinuous. Then A^* factors on the left.

Proof. Let $h\in A^*$ and B_1 be unit ball in A. By assumption, weak-topology on A is hypocontinuous. Therefore there exists a finite subset $\{f_1,f_2,...,f_n\}$ in A^* and $\varepsilon>0$ such that $B_1\{a\in A;\ |<f_i,a>|<\varepsilon$ for any $i\in\{1,2,...,n\}\}\subseteq\{a\in A;\ |<h,a>|<1\}$. Now, let $a\in A$ and $<f_i,a>=0$ for all $i\in\{1,2,...,n\}$. For $b\in B_1$, we have $<h,ba>=0$, and so $hA\subseteq span\{f_1,f_2,...,f_n\}$. By ([11], Theorem 1.21), hA is a closed subspace of A^*. On the other hand, if (e_α) is a bounded approximate identity in A, then $he_\alpha\to h$ in the weak*-topology. Consequently by ([11], Theorem 1.21), $he_\alpha\to h$ in the norm topology. But $he_\alpha\in hA$ and so $h\in hA$. It follows that A^* factors on the left.

Theorem 2.6. Assume X is such that weak-topology on L(X) is hypocontinuous. Let $T:L(X)^*\to L(X)^*$ be a bounded linear map such that $T(f\delta_x)=T(f)\delta_x$ for any $f\in L(X)^*$ and $x\in X$. Then $T\in Hom_{L(X)}(L(X)^*,L(X)^*)$.

Proof. See Lemma 2.1 and Lemma 2.5.

In [4], we have shown that if G is a nondiscrete abelian locally compact group, then there exists a bounded linear map $T:L^\infty(G)\to L^\infty(G)$ such that $T(f\delta_x)=T(f)\delta_x$ ($f\in L^\infty(G)$, $x\in G$) and $T\notin Hom_{L(G)}(L^\infty(G), L^\infty(G))$.

Let A be a Banach algebra with a bounded approximate identity (e_α) bounded by 1. Baker, Lau and Pym [1] have been proved $Hom_A(A^*,A^*)$ (where $T\in Hom_A(A^*,A^*)$ means $T(fa)=T(f)a$ for every $f\in A^*$ and $a\in A$) is isomorphic with the Banach algebra $(A^*A)^*$. Indeed, we can prove that $Hom_A(A^*,A^*)$ is isometric with the Banach algebra $(A^*A)^*$. Let $T\in Hom_A(A^*,A^*)$, there exists a $n\in(A^*A)^*$ such that $T=T_n$ where $T_n(f)=nf$ ($f\in A^*$). Indeed, we define $<n,f>=<E,T(f)>$ ($f\in A^*A$), where E is a right identity for A^{**} (for more information see Theorem 1.1 in [1]). Hence $\|T_n\|\le\|n\|$. Now, let $\varepsilon>0$ be given. There is a functional $f\in A^*A$ with $\|f\|\le1$ such that $|<n,f>|\ge\|n\|-\varepsilon$. It follows that, $\|T_n\|\ge\lim|<T_n(f),e_\alpha>| =\lim|<nf,e_\alpha>|\ge|\lim<nf,e_\alpha>|=|<n,f>|\ge\|n\|-\varepsilon$, i.e. $\|T_n\|=\|n\|$.

Theorem 2.7. Let A be a Banach algebra with a bounded approximate identity bounded by 1, and $T\in Hom_A(A^*,A^*)$. The following statements are equivalent:

1) There exists a $n\in(A^*A)^*$ such that $an\in A$ for all $a\in A$, and $T= T_n$.

2) T is weak*-weak* continuous.

Proof. Let $T=T_n$ and $an\in A$ for any $a\in A$. Let (f_α) be a net in A^* such that $f_\alpha\to f$ $(f\in A^*)$ in the weak*-topology. For $a\in A$, we have $\langle an,f_\alpha\rangle\to\langle an,f\rangle$, and so $\langle T_n(f_\alpha),a\rangle\to\langle T_n(f),a\rangle$. This shows that T is weak*-weak* continuous.

To prove the converse, let $T\in Hom_A(A^*,A^*)$. By ([1], Theorem 1.1), there exists a $n\in(A^*A)^*$ such that $T=T_n$. Now, let $a\in A$. By assumption, T is weak*-weak* continuous, and so $T_n^*(a)\in A^{**}$ is weak*-continuous. It follows that $T_n^*(a)\in A$ ([11], Chapter 3). On the other hand, $\langle T_n^*(a),f\rangle = \langle a,T_n(f)\rangle=\langle a,nf\rangle=\langle an,f\rangle$ where $f\in A^*$, i.e. $T_n^*(a)=an$. Consequently $an\in A$ for any $a\in A$. This completes our proof.

Corollary 2.8. Let A be a Banach algebra with a bounded approximate identity bounded by 1. If all operators T in $Hom_A(A^*,A^*)$ are weak*-weak* continuous, then $(A^*A)^*=Z_t$ where $Z_t =\{n\in(A^*A)^*$; the mapping $m\to nm$ is weak*-weak* continuous$\}$.

Proof. Suppose all operators T in $Hom_A(A^*,A^*)$ are weak*-weak* continuous, and let $n\in(A^*A)^*$. Then $T_n\in Hom_A(A^*,A^*)$ is weak*-weak* continuous. By Theorem 2.7, $an\in A$ for any $a\in A$. A standard argument using the Cohen-Hewitt factorization Theorem shows that $AA=A$. Now, let $m_\alpha\to m$ in the weak*-topology, and let $f\in B$. There exist $g\in B$ and $a\in A$ such that $f=ga$. Therefore $\langle nm_\alpha,f\rangle=\langle anm_\alpha,g\rangle=\langle m_\alpha,gan\rangle$ and $\langle nm,f\rangle =\langle m,gan\rangle$. This shows that $\langle nm_\alpha,f\rangle\to\langle nm,f\rangle$, i.e. $n\in Z_t$. Consequently $Z_t=(A^*A)^*$.

Corollary 2.9. Let G be a locally compact group. Then all operators T in $Hom_{L(G)}(L^\infty(G), L^\infty(G))$ are weak*-weak* continuous if and only if G is compact.

Proof. By ([8], Theorem 1), we have $Z_t(LUC(G)^*)=M(G)$. On the other hand, $LUC(G)^*= M(G)\oplus C_\circ(G)^\perp$ ([5], Lemma 1.1). The results follows from Corollary 2.8.

For some Banach algebras A, the subspace $\{n\in(A^*A)^*; An\subseteq A\}$ of B^* have been studied by Lau and Ulger in [9]. In the following Theorem we will study $\{n\in B^*; L(X)n\subseteq L(X)\}$.

Theorem 2.10. Let X be a hypergroup. Then $\{n\in B^*; L(X)n \subseteq L(X)\}=M(X)$.

Proof. Since L(X) is an ideal in M(X), we have $M(X)\subseteq\{n\in B^*; L(X)n\subseteq L(X)\}$. For the reverse inclusion, let $n\in\{n\in B^*; L(X)n \subseteq L(X)\}$. So the mapping $\nu\to\nu n$ from L(X) into L(X) is a right multiplier. By ([6], Proposition 1), there exists a measure μ in M(X) such that $\nu*\mu=\nu n$ for any $\nu\in L(X)$. Now, let (e_α) be a bounded approximate identity in L(X) and $f\in B$. Then $\langle e_\alpha*\mu,f\rangle=\langle e_\alpha n,f\rangle$ (for all α) implies $\langle\mu,f\rangle=\langle n,f\rangle$, i.e. $\mu=n$. This completes our proof.

Corollary 2.11. Assume X is such that $Z_t(B^*)=M(X)$. Then L(X) is an ideal in B^* if and only if X is compact.

Proof. Let L(X) be an ideal in B^*, and let $n\in B^*$. It is easy to see that the operator T_n is weak*-weak* continuous. Consequently by Corollary 2.8, $B^*=Z_t(B^*)=M(X)$. But $B^*=M(X)\oplus C_\circ(X)^\perp$ ([11], Theorem 4), and so $C_\circ(X)^\perp=\{0\}$, i.e. X is compact.

To prove the converse, let X be compact. Then $B^*=M(X)$, and so the operator T_n $(n\in B^*)$ is weak*-weak* continuous. Theorem 2.7 shows that L(X) is a right ideal in B^*. On the other hand, by definition X is commutative [3,6,10], so that L(X) is an ideal in B^*.

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