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Estimation of AR Parameters in the Presence of Additive Contamination in the Infinite Variance Case

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Abstract

If we try to estimate the parameters of the AR process $\{X_n\}$ using the observed process $\{X_n+Z_n\}$ then these estimates will be badly biased and not consistent but we can minimize the damage using a robust estimation procedure such as GM-estimation. The question is does additive contamination affect estimates of "core" parameters in the infinite variance case to the same extent that it does in the finite variance case? We will see that if the contamination $\{Z_n\}$ has higher tails than the core process $\{X_n\}$, the estimation of parameters of the core process will not be greatly affected; that at least its consistency is preserved.

Keywords: Contamination; Infinite variance; Autoregressive

1. Introduction

Suppose that $\{X_n\}$ is a finite variance AR process and $\{Z_n\}$ is some other stationary stochastic process a common outlier model in time series analysis is the additive model where we observe $X_n + Z_n$ instead of X_n as we had assumed.

Let $\{Y_n\}$ be a stationary process satisfying the following three conditions:

(a) $Y_n = \sum_{k=0}^{\infty} c_k \varepsilon_{n-k}$

(b) $\{\varepsilon_n\}$ are i.i.d random variables which are in the domain of attraction of stable random variables with index $\alpha \in (0,2)$

(c) $\sum_{k=0}^{\infty} |c_k|^{\delta} < \infty$ for some $\delta < \alpha$ and $\delta \le 1$

The almost sure convergence of the infinite series defining Y_n was established by Cline [2] under conditions (b) and (c). The class of processes satisfying (a)-(c) is sufficiently rich to include all stationary ARMA (p,q) processes with innovations in the domain of attraction of a stable random variable.

Let $\{Y_t\}$ be the ARMA (p,q) process satisfying:

$$\begin{aligned} Y_t &= \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \dots \\ &\quad + \alpha_p Y_{t-p} + \varepsilon_t - \beta_1 \varepsilon_{t-1} - \dots - \beta_q \varepsilon_{t-q} \end{aligned}$$

where $\{\varepsilon_t\}$ is an i.i.d sequence of random variables whose common distribution belong to the domain of attraction of a stable law with index $\alpha \in (0,2)$ which we denote by $\varepsilon_0 \in D(\alpha)$ or $\{\varepsilon_t\} \in D(\alpha)$ and $\phi(Z) = 1 - \alpha_1 z - \alpha_2 z^2 - \dots - \alpha_p z^p \neq 0$ for all complex z with $|z| \le 1$. The conditions $P(|\varepsilon_0| > y) = y^{-\alpha} L(y)$

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and $\lim_{y\to\infty} \frac{p(\varepsilon_1 > y)}{p(|\varepsilon_0| > y)} = p$ where L(y) is slowly varying at ∞ and $\alpha > 0$, $0 \le p \le 1$, are necessary and sufficient condition for $\varepsilon_0 \in D(\alpha)$.

Davis and Rensick [3] showed that for all non-negative integers p:

$$\alpha_N^{-2}\left[\sum_{n=1}^{N} Y_n^2, \sum_{n=2}^{N} Y_n Y_{n-1}, \ldots \sum_{n=p+1}^{N} Y_n Y_{n-p}\right] \xrightarrow{d} S\left[\sum_{j=0}^{\infty} c_j^2, \sum_{j=0}^{\infty} c_j c_{j+1}, \ldots \sum_{j=0}^{\infty} c_j c_{j+p}\right]$$

where S is a positive stable random variable with index $\frac{\alpha}{2}$ and α_N is some normal constant as in condition (b). Moreover if we replace each $\sum Y_n Y_{n-k}$ by its mean centered version $\sum (Y_n - \overline{Y})(Y_{n-k} - \overline{Y})$ the same limit law results. As a consequence of this the sample autocorrelation converges in probability to the same limits as in the finite variance case:

$$\frac{\sum_{n=k+1}^{N} Y_n Y_{n-k}}{\sum_{n=1}^{N} Y_n^2} \xrightarrow{p} \frac{\sum_{j=0}^{\infty} c_j c_{j+k}}{\sum_{j=0}^{\infty} c_j^2}$$

and the mean centered versions have the same limits in probability.

Suppose now that $\{Z_n\}$ is another stochastic process such that the bivariate process $\{(Y_n, Z_n)\}$ is stationary and $Z_n \in L^\delta$ for some $\delta > \alpha$. Suppose we observe $\tilde{Y} = Y_n + Z_n$ that is the original process Y_n contaminated by additive noise Z_n. We shall see that the asymptotic properties of the sample autocovariances and sample autocorrelations are largely unaffected. This is in contrast to the finite variance where any arbitrary contamination will lead to asymptotic bias in any parameter estimates.

Theorem 1. For fixed integers $K \ge 0$:

(a)
$$\alpha_N^{-2} \sum_{n=k+1}^{N} \tilde{Y}_n \tilde{Y}_{n-k} = \alpha_N^{-2} \sum_{n=k+1}^{N} Y_n Y_{n-k} + o(1) \ a.s \ as \ N \to \infty$$

(b)
$$\frac{\sum_{n=k+1}^{N} \tilde{Y}_n \tilde{Y}_{n-k}}{\sum_{n=1}^{N} Y_n^2} \xrightarrow{p} \frac{\sum_{j=0}^{\infty} c_j c_{j+k}}{\sum_{j=0}^{\infty} c_j^2}$$

Proof.

$$\alpha_N^{-2} \sum_{n=k+1}^{N} \tilde{Y}_n \tilde{Y}_{n-k} =$$
$$\alpha_N^{-2} \sum_{n=k+1}^{N} (Y_n + Z_n)(Y_{n-k} + Z_{n-k}) =$$
$$\alpha_N^{-2} \sum_{n=k+1}^{N} Y_n Y_{n-k} + \alpha_N^{-2} \sum_{n=k+1}^{N} (Y_{n-k} Z_n + Y_n Z_{n-k} + Z_n Z_{n-k})$$

This is sufficient to show that the second term above tends almost surely to zero. By holder's inequality it is easy to verify that $E(|Y_m \varepsilon_n|^\gamma)$ is finite for $\gamma < \frac{\alpha\delta}{\alpha+\delta}$ for any m and n. Also $E(|Z_m Z_n|^{\frac{\alpha}{2}})$ is finite.

Noting that $\frac{\alpha\delta}{\alpha+\delta} > \frac{\alpha}{2}$ then for some $\gamma > \frac{\alpha}{2}$ there exist absolute moments of order γ for both $Y_m Z_n$ and $Z_m Z_n$. Furthermore we can take $\gamma < 1$ and so by the following theorem (noting that the summands from a stationary and hence identically distributed sequence)

$$N^{-\frac{1}{\gamma}} \sum_{n=k+1}^{N} (Y_{n-k} Z_n + Y_n Z_{n-k} + Z_n Z_{n-k}) \xrightarrow{a.s} 0$$

Since $\alpha_N^{-2} N^{-\frac{1}{r}} \to 0$ part (a) follows and part (b) follows similarly.

Theorem 2. Let $Y, Y_1, \ldots, Y_n$ be a sequence of random variables with $Y \in L^\gamma$ for some $\gamma \in (0,2)$. Suppose that for all x and for all n if either

$$P(|Y_n| > x) \le P(|Y| > x) \quad \text{if} \quad \gamma \ne 1$$

or

$$P(|Y_n| > x \mid Y_1, \ldots, Y_{n-1}) \le P(|Y| > x \mid Y_1, \ldots, Y_{n-1}) \quad \text{if} \quad \gamma = 1$$

then

$$N^{-\frac{1}{\gamma}}\sum_{n=1}^{N}(Y_n-\mu_n)\xrightarrow{a.s}0$$

where $\mu_n=0$ if $\gamma<1$ and $\mu_n=E(Y_n|Y_1,...,Y_{n-1})$ if $1\le\gamma<2$.

One will note that we could replace the o(1) in part (a) of theorem 1 by o($N^{-\sigma}$) for some value of $\sigma>0$ which will depend on α,δ and any independence between $\{X_n\}$ and $\{Z_n\}$ we will illustrate this indirectly considering the p-th order autoregressive AR(p) process defined as follows:

$$X_n=\beta_1X_{n-1}+...+\beta_pX_{n-p}+\varepsilon_n$$

where the AR parameters satisfy the usual stationary constraints. Let $\tilde{X}_n=X_n+Z_n$ where Z_n is defined as before. Consider LS estimates of $\beta_1,...,\beta_p$ defined by the estimating equations:

$$\sum_{j=1}^{p}\hat{\beta}_j[\sum_{n=p+1}^{N}\tilde{X}_{n-j}\tilde{X}_{n-k}]=\sum_{n=p+1}^{N}\tilde{X}_n\tilde{X}_{n-k}\quad k=1,...,p \tag{1}$$

We know that if $\{X_n\}$ is perfectly observed ($Z_n\equiv o$) then for all $\delta>\alpha$ $N^{-\frac{1}{\delta}}(\hat{\beta}_j-\beta_j)\xrightarrow{a.s}0$ provided E(ε_n)=0 if exists. However we will not require this latter condition to hold in what follows.

Theorem 3. Let $Z_n\in L^{\delta}$ for some $\delta>\alpha$ and $\{(X_n,Z_n)\}$ be a stationary bivariate process then for j =1,...,p we have

(a) if $\{X_n\}$ and $\{Z_n\}$ are independent then $N^{\sigma}(\hat{\beta}_j-\beta_j)\xrightarrow{a.s}0$ for $\delta<\frac{2}{\alpha}-\max[\frac{1}{\alpha},\frac{2}{\delta},1]$

(b) otherwise in general $N^{\sigma}(\hat{\beta}_j-\beta_j)\xrightarrow{a.s}0$ for $\delta<\frac{2}{\alpha}-\max[\frac{1}{\alpha}+\frac{1}{\delta},1]$

Proof. Equation (1) can be expressed as follows:

$$\begin{aligned}\sum_{j=1}^{p}[\hat{\beta}_j-\beta_j][\sum_{n=p+1}^{N}(X_{n-j}-Z_{n-j})(X_{n-k}-Z_{n-k})]=\\ \sum_{n=p+1}^{N}\varepsilon_nX_{n-k}+\sum_{n=p+1}^{N}X_nX_{n-k}+\sum X_{n-k}Z_n\\ +\sum_{n=p+1}^{N}Z_nX_{n-k}-\sum_{j=1}^{p}\beta_j[\sum_{n=p+1}^{N}X_{n-j}Z_{n-k}]\\ +\sum_{n=p+1}^{N}X_{n-k}Z_{n-j}+\sum_{n=p+1}^{N}Z_{n-j}Z_{n-k}\end{aligned} \tag{2}$$

For k=1,…,p, if we take $\gamma_1<\min(1,\alpha)$ then $N^{\frac{1}{\gamma_1}}\sum_{n=p+1}^{N}\varepsilon_nX_{n-k}\xrightarrow{a.s}0$ and by Theorem 2 for j,k =1,...,p, $N^{-\frac{1}{\gamma_1}}\sum_{n=p+1}^{N}X_{n-j}X_{n-k}\xrightarrow{a.s}0$ since X_{n-j} and ε_n are independent and $\{X_n\}$ and $\{Z_n\}$ are independent now. Thus

$$N^{-\frac{1}{\gamma_2}}\sum_{n=1}^{N}Z_{n-j}Z_{n-k}\xrightarrow{a.s}0$$

since $Z_{n-j}Z_{n-k}\in L^{\gamma_2}$. Therefore taking $\gamma=\max[\frac{1}{\gamma_1},\frac{1}{\gamma_2}]$ the right hand side of Equation (2) multiplied by $N^{-\gamma}$ tends in probability to zero.

Following Hannan and Kanater [5] we also have that for all $k<\frac{2}{\alpha}$,

$$\min_{\|v\|=1}N^{-k}\sum_{j=1}^{p}\sum_{k=1}^{p}v_jv_k\sum_{n=p+1}^{N}X_{n-j}X_{n-k}\xrightarrow{a.s}\infty$$

where

$$\|v\|^2=\sum_{j=1}^{p}v_j^2$$

Now by taking k sufficiently close to $\frac{2}{\alpha}$ by noting that all terms involving $\{Z_n\}$ tends almost surely to zero (by the arguments used above) we have

$$\lambda_{\min}(N^{-k}C_N) = \min_{\|v\|=1} N^{-k} \sum_{j=1}^{p}\sum_{k=1}^{p} v_j v_k$$

$$\sum_{n=p+1}^{N} (X_{n-j} + Z_{n-j})(X_{n-k} + Z_{n-k}) \xrightarrow{a.s} \infty$$

This implies that $N^{k-\gamma}(\hat{\beta}_j - \beta_j) \xrightarrow{a.s} 0$.

By taking k arbitrary close to $\frac{2}{\alpha}$ and γ arbitrary close to max $[\frac{1}{\alpha}, \frac{2}{\delta}, 1]$ the results of part (a) follows. In general case the same procedure works except we must ensure that $\gamma_1 < \min[\frac{\alpha\delta}{\alpha+\delta}, 1]$.

Conclusions

If $\alpha \le 1$ and the contaminating process Z_n has enough moments then asymptotically the contamination will not affect the estimates of the AR parameters. Specifically if $\{Z_n\}$ is independent of $\{X_n\}$ then the previous statement will be true if $E(Z_n^2) < \infty$ and in general will be true if $E(|Z_n|^r) < \infty$ for all r>0. For $\alpha > 1$ we cannot guarantee that the contamination will not affect the rate of convergence of the parameter estimates.

References

1. Maltz A.L. A central limit Theorem for nonuniform ϕ-mixing random fields with infinite variance. *Statistics and Probability Letters*, **51**: 351-359 (2001).
2. Cline D. Infinite series of random variables with regularly varying tails. Technical report no 83, Iinstitute of Applied Mathematics and Statistics, University of British Colombia (1983).
3. Davis R. and Resnick S. Limit theory for moving averages of random variables with regularly varying tail probabilities. *Ann. Prob.*, **13**: 179-95 (1958a).
4. Surgailis D. Stable limits of empirical processes of moving average with infinite variance. *Stochastic Processes and Their Applications*, **l00**: 255-274 (2002).
5. Hannan E. and Kanater M. Autoregressive processes with infinite variance. *J. App. Prob.*, **14**: 411-15 (1977).
6. Loeve. Michel. Probability Theory. Van Nostrand.
7. Martin R.D. and Yohai V. Robustness in time series and estimating ARMA models. In: E.J. Hannan, P.R. Kariainaiah, and M.M. Rao (Eds.), *Hand Book of Statistics 5*, pp. 119-55 (1985).
8. Mittnik S., Rachev T., and Samorodnitsky G. The distribution of test statistics for outlier detection in heavy-tailed samples. *Mathematical and Computer Modeling*, **34**: 1171-1183 (2001).
9. Davis R.A. Gauss-Newton and M-estimates for AR processes with infinite variance. *Stochastic Processes and Their Applications*, **63**: 75-95 (1996).