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Strong Convergence of Weighted Sums for Negatively Orthant Dependent Random Variables

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Abstract

We discuss in this paper the strong convergence for weighted sums of negatively orthant dependent (NOD) random variables by generalized Gaussian techniques. As a corollary, a Cesaro law of large numbers of i.i.d. random variables is extended in NOD setting by generalized Gaussian techniques.

Keywords: Cesaro mean; Generalized Gaussian; Negatively orthant dependent random variables; Strong convergence; Weighted sum

1. Introduction

Many useful linear statistics based on a random sample which are weighted sums of i.i.d. random variables. Examples include least-squares estimators, nonparametric regression function estimators and jackknife estimators, among other. In this respect, studies of strong convergence for these weighted sums have demonstrated significant progress in probability theory with applications in mathematical statistics. Up to now, various limit properties for sums of i.i.d. random variables have been studied by many authors.

The most commonly studied method of summation is that of Cesaro's. Set, for $\alpha > -1$,

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}$$

$n = 1, 2, \ldots$ and $A_0^\alpha = 1$.

Let $\{X_n, n \geq 0\}$ be a sequence of i.i.d. random variables. One says that $\{X_n, n \geq 0\}$ satisfies Cesaro Law of Large Numbers of order α, $0 < \alpha < 1$, if

$$\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} X_k \text{ converges } a.s. \text{ as } n \to \infty.$$

It is well known that

$$\lim_{n\to\infty}\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} X_k = \mu \quad a.s.$$

If and only if $E|X_0|^{1/\alpha} < \infty$ and $EX_0 = \mu$.

For $\alpha = 1$ this result is the classical Kolmogorove strong law. For $1/2 < \alpha < 1$ the proof is due to Lorentz [10]; for $0 < \alpha < 1/2$ it follows from Chow and Lai [4]. The case $\alpha = 1/2$ was treated by Deniel and Derriennic [5]. Heinkel [6] established a version of this result in a Banach space setting. Li *et al.* [8] studied the convergence rates of Cesaro Law of Large Numbers and pointed out the following result.

Theorem 1.1. Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. For $0 < \alpha < 1/2$, if $Ee^{t|X|} < \infty$ for all $t > 0$, then, as $n \to \infty$,

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$$\frac{1}{A_n^{\alpha}}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}(X_k - EX) = o(n^{-\alpha}\log n) \quad a.s.$$

■

Liang *et al.* [9] derived Theorem 1.1 for a sequence of *negatively associated* random variables. (For definition of *negatively associated* see Jage-Dov and Prochan [7].)

However, many variables are dependent in problems. For example, *negatively orthant dependent* random variables, its definition is as follows:

Definition 1.1. A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively orthant dependent (NOD) if for all real $x_1,\ldots,x_n$,

$$P(X_1 > x_1,\ldots,X_n > x_n) \le \prod_{i=1}^{n} P(X_i > x_i),$$

and

$$P(X_1 \le x_1,\ldots,X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i).$$

An infinite family of random variables is NOD if every finite subfamily is NOD.

In order to extend Theorem 1.1 to NOD setting, we will use generalized Gaussian techniques to provide strong convergence for NOD random variables.

Definition 1.2. (Chow [3]) A random variable X is said to be generalized Gaussian (GG), if there exists $\delta \ge 0$ such that for every real number *t*,

$$Ee^{tX} \le e^{\delta^2 t^2/2}. \tag{1.1}$$

The infimum of those δ satisfying (1.1) is denoted by $\tau(X)$.

■

It is clear that, if X is generalized Gaussian, so are $-X$ and aX (for $a \ne 0$) with $\tau(-X) = \tau(X)$ and $\tau(aX) = |a|\,\tau(X)$.

Remarks 1.1. A) The parameter δ is not unique unless the value is assigned to be the minimum of the values satisfying (1.1). **B)** All moments of a generalized Gaussian random variable are exists and the mean must be zero. **C)** Let X be a generalized Gaussian random variable with $\tau(X) \le \delta$ then,

$$Ee^{t|X|} \le 2e^{\delta^2 t^2/2}.$$

■

By generalized Gaussian techniques, strong convergence has been studied by many authors. For example, Chow [3] for weighted sums of independent random variables, Ouy [12] for m-dependent random variables, Taylor and Hu [11] for bounded and independent random variables, Amini [1] for NOD random variables. This paper is organized as follows. In Section 2, we give our main results. In Section 3, we prove the main results.

2. Main Results

By the following theorem, we extend Theorem 1.1 to NOD setting.

Theorem 2.1. Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of row NOD generalized Gaussian random variables with $\tau(X_{ni}) \le \delta_{ni}$ such that $\{\delta_{ni}, 1 \le i \le k_n, n \ge 1\}$ is an array of positive numbers. Assume that $\{a_{ni}, 1 \le i \le k_n,\ n \ge 1\}$ is an array of real numbers satisfying

$$(i)\quad \max_{1\le i\le k_n}(|a_{ni}|\,\delta_{ni}) = O((\log n)^{-1}),$$

$$(ii)\quad \sum_{i=1}^{k_n} a_{ni}^2\delta_{ni}^2 = o((\log n)^{-1}).$$

Then for all $\varepsilon > 0$ and all $r \ge 2$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| > \varepsilon\right) < \infty.$$

■

Corollary 2.1. In Theorem 2.1, if $\max_{1\le i\le k_n}\delta_{ni} = O(1)$, for $n \ge 1$. (*e.g.* $\delta_{ni} = \delta$, for all $1 \le i \le k_n$, $n \ge 1$, where δ is a positive constant.) we can replace the conditions (i) and (ii) of Theorem 2.1 by

$$(i)'\quad \max_{1\le i\le k_n}|a_{ni}| = O((\log n)^{-1}),$$

$$(ii)'\quad \sum_{i=1}^{k_n} a_{ni}^2 = o((\log n)^{-1}).$$

■

The following corollary extends the result of Li *et al.* [8] in NOD setting by generalized Gaussian techniques.

Corollary 2.2. Let $\{X_n, n \ge 0\}$ be a sequence of NOD generalized Gaussian random variables with $\tau(X_{ni}) \le \delta_{ni}$ such that $\max_{1\le i\le k_n}\delta_{ni} = O(1)$ for $n \ge 1$. Then for $0 < \alpha < 1/2$, as $n \to \infty$, we have,

$$\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} X_k = o(n^{-\alpha}\log n) \quad a.s. \tag{2.1}$$

■

3. Proofs

In this section, $a \ll b$ means $a = O(b)$, $a^+ = \max(0,a)$, $a^- = \max(0,-a)$. Let C be a positive constant whose value is unimportant and may vary at different place. We give some of the NOD properties that will be needed in the proof of the main result. For the proof see Bozorgnia *et al.* [2].

Proposition 3.1. Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables and $\{f_n, n \ge 1\}$ be a corresponding sequence of Borel functions such that, are monotone increasing (or all are monotone decreasing), then $\{f(X_n), n \ge 1\}$ is a sequence of NOD random variables. ■

Proposition 3.2. Let $\{X_i, 1 \le i \le n\}$ be a finite family of NOD random variables and $t_1, \ldots, t_n$ be all nonnegative (nonpositive) real numbers. Then

$$E[\exp\sum_{i=1}^{n} t_i X_i] \le \prod_{i=1}^{n} \exp(t_i X_i).$$

■

Proof of Theorem 2.1. Since $a_{ni} = a_{ni}^+ - a_{ni}^-$, it suffices to show, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{k_n} a_{ni}^+ X_{ni}| > \varepsilon) < \infty, \tag{3.1}$$

$$\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{k_n} a_{ni}^- X_{ni}| > \varepsilon) < \infty. \tag{3.2}$$

We prove only (3.1), the proof of (3.2) is analogous. To prove (3.1), we need only to prove, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} > \varepsilon) < \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} n^{r-2} P(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} < -\varepsilon) < \infty. \tag{3.4}$$

We first prove (3.3). From the Proposition 3.1, we know that $\{a_{ni}^+ X_{ni}, 1 \le i \le k_n, n \ge 1\}$ is still an array of row NOD random variables. Since $e^x \le 1 + x + \frac{1}{2}x^2 e^{|x|}$ for all $x \in R$, hence by using Proposition 3.2 and Remarks 1.1, for $t = M \log n/\varepsilon$, where M is a large constant and will be specified later on, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{r-2} P(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} > \varepsilon) &\le \sum_{n=1}^{\infty} n^{r-2} e^{-\varepsilon t} E e^{t\sum_{i=1}^{k_n} a_{ni}^+ X_{ni}} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{k_n} E e^{t a_{ni}^+ X_{ni}} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{k_n} [1 + \frac{1}{2} t^2 (a_{ni}^+)^2 E X_{ni}^2 e^{t a_{ni}^+ |X_{ni}|}] \\
&\ll \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{k_n} [1 + C(\log n)^2 (a_{ni}^+)^2 \delta_{ni}^2] \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \exp\{C(\log n)^2 \sum_{i=1}^{k_n} (a_{ni}^+)^2 \delta_{ni}^2\} \\
&\le \sum_{n=1}^{\infty} n^{(r+\varepsilon)-(2+M)} < \infty,
\end{aligned}$$

provided $M > (r+\varepsilon) - 1$. Thus, (3.3) is proved.

By replacing X_{ni} by $-X_{ni}$ from the above statement, and noticing $\{a_{ni}^+(-X_{ni}), 1 \le i \le k_n, n \ge 1\}$ is still an arrays of row NOD random variables, we know that (3.4) holds.

■

Proof of Corollary 2.2. Set $a_{nk} = n^\alpha (\log n)^{-1} A_{n-k}^{\alpha-1} / A_n^\alpha$, for $0 \le k \le n$ and $n \ge 1$. Then (2.1) holds if and only if

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_{nk} X_k = 0 \quad a.s. \tag{3.5}$$

Note that, for $\alpha > -1$, $A_n^\alpha \sim n^\alpha \Gamma(\alpha+1)$ as $n \to \infty$. It is easy to see that

$$\max_{0 \le k \le n} |a_{nk}| = O((\log n)^{-1}), \quad \sum_{k=0}^{n} a_{nk}^2 = o((\log n)^{-1}).$$

as $n \to \infty$. (*cf*, Li *et al.* [8])

Now by applying Theorem 2.1 and Corollary 2.1 we get (3.5).

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