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Decomposition of H^*-Algebra Valued Negative Definite Functions on Topological *-Semigroups

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Abstract

In the present paper, among other results, a decomposition formula is given for the *w*-bounded continuous negative definite functions of a topological *-semigroup *S* with a weight function *w* into a proper H^*-algebra *A* in terms of *w*-bounded continuous positive definite *A*-valued functions on *S*. A generalization of a well-known result of K. Harzallah is obtained. An earlier conjecture of the author is also established.

Keywords: H^*-algebra; Positive definite function; Negative definite function; Topological semigroup

Introduction

In this work we introduce the notion of a negative definite function of a topological *-semigroup *S* into a proper H^*-algebra *A*. Through a different method, among other results, we extend a result of K.Harzallah from the case of bounded continuous complex-valued negative definite functions to the case of w-bounded continuous negative-definite *A*-valued functions of *S* with a weight function *w*. It should be noted that the Harzallah's argument heavily depends on the existence of a Haar measure on a topological group. We have also established our earlier conjecture in [14] even in a more general setting.

This paper is organized as follows. The basic results on H^*-algebra valued negative definite functions are given in section one. Section two is devoted to the study of both H^*-valued negative definite and positive definite functions on weighted commutative topological semi-groups. A Lévy-Khinchin formula for the H^*-valued continuous negative definite functions on weighted foundation semigroups is given in this section.

Preliminaries

Throughout this paper, *S* will denote a locally compact, Hausdorff topological semigroup. A semigroup *S* is called a **-semigroup* if there is a continuous mapping $*:S\to S$ such that $(xy)^*=y^*x^*$ and $(x^*)^*=x$ for all $x, y \in S$. A function *w* on a topological *-semigroup *S* with an identity *e* such that $w(e)=1$, $w(x)\geq 0$, $w(x^*)=w(x)$, $w(xy)\leq w(x)w(y)$ $(x,y\in S)$ is called a *weight function* on S. A complex-valued function f on S is called w*-bounded* if there exists $k>0$ such that $|f(x)|\leq kw(x)$ $(x\in S)$. A nonzero mapping $\chi:S\to \mathbf{C}$ such that $\chi(xy)=\chi(x)\chi(y)$ and $\chi(x^*)=\overline{\chi(x)}$ $(x,y\in S)$ is called a $*$-semicharacter on S. A complex-valued function φ on a $*$-semigroup S is called *positive-definite* if

$$\sum_{i=1}^{n}\sum_{j=1}^{n}c_i\overline{c_j}\varphi(x_i x_j^*)\geq 0$$

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for all choices $\{x_1,\ldots,x_n\}$ from S and $\{c_1,\ldots,c_n\}$ from $\mathbb{C}$. For further information on positive definite functions we refer the reader to [3,4,13].

Recall that an H^**-algebra* is a nonzero Banach algebra A whose underlying Banach space is a Hilbert space with an scalar inner product $\langle\, ,\rangle_A$ which induces the norm $\|.\|_A$ on A and for each x in A there is some x^* in A for which the mapping $x \mapsto x^*y$ (resp. $x \mapsto yx^*$) is the adjoint of the mapping $x \mapsto xy$ (resp. $x \mapsto yx$). An H^*-algebra A is called *proper* if the only $x \in A$ for which $xA = \{0\}$ is the zero element. Note that every H^*-algebra with an identity defines a proper H^*-algebra. Let $\tau(A) = \{xy: x,y \in A\}$ be its trace class, then it is well known that $\tau(A)$ is a Banach algebra with respect to a norm $\tau(.)$ which is related to the norm $\|.\|_A$ by the identity $\tau(a^*a) = \|a\|_A^2$ $(a \in A)$ (see [17]). There is a partial ordering defined on $\tau(A)$ by the requirement that $a \geq 0$ if $a = b^*b$ for some $b \in A$. Also there is a trace *tr* defined on $\tau(A)$ such that $tr\, a = \tau(a)$ if $a \geq 0$ and $tr(xy^*) = tr(y^*x) = \langle x,y\rangle_A$. Note that $|tr\, x| \leq \tau(x)$ for all $x \in \tau(A)$.

A right Hilbert module H over A is called a Hilbert A-module if there exists a $\tau(A)$-valued function $(\, ,)$ on $H \times H$ with the following properties:

(i) $(f+g,h) = (f,h) + (g,h)$ for all $f,g,h \in H$.

(ii) $(f,g)^* = (g,f)$ for all $f,g \in H$.

(iii) $(f,ga) = (f,g)a$ for all $f,g \in H$ and each $a \in A$.

(iv) For each non-zero $f \in H$ there exists $a \neq 0$ in A such that $(f,f) = a^*a$.

(v) $|tr(f,g)| \leq \tau(f,f)\tau(g,g)$ for all $f,g \in H$.

(vi) H is complete in the norm $\|f\| = (\tau(f,f))^{1/2} = tr(f,f)^{1/2}$.

The function $(\, ,)$ is called a generalized scalar product. There is a linear structure on H such that H is an ordinary Hilbert space with respect the scalar product $\langle f,g\rangle = tr(g,f)$ $(f,g \in H)$ (see Theorem 1 of [17]). An *A-linear* operator on H is an additive mapping $T: H \to H$ such that $T(fa) = T(f)a$ for all $f \in H$ and $a \in A$; T is called bounded if $\|Tf\| \leq M\|f\|$ for some $M \geq 0$ and all $f \in H$. Each bounded A-linear operator T is linear and its adjoint T^* has the property that $(Tf,g) = (f,Tg)$ for all $f,g \in H$.

For more detail on proper H^*-algebras we refer the reader to [6,17-19].

Let X be a nonempty set. A kernel $\varphi: X \times X \to \tau(A)$ of a proper H^*-algebra A is called *hermitian* if $\varphi(x,y) = [\varphi(y,x)]^*$ $(x,y \in X)$, and is called *positive definite* if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^*\varphi(x_i,x_j) \geq 0$$

for all subsets $\{a_1,\ldots,a_n\}$ of A and $\{x_1,\ldots,x_n\}$ of X, and is called *weakly positive definite* if

$$\sum_{j,k=1}^{n} c_j\overline{c_k}\varphi(s_i,s_j) \geq 0$$

for every choice of $n \in \mathbb{N}$, $s_1,\ldots,s_n \in S$, and $c_1,\ldots,c_n \in \mathbb{C}$.

If S is a *-semigroup then $\varphi: S \to \tau(A)$ is called *positive definite* if the kernel: $(x,y) \mapsto \varphi(x^*y)$ $(x,y \in S)$ is positive definite. A function $\varphi: S \to A$ is called *weakly positive definite* if the kernel: $(x,y) \mapsto \varphi(x^*y)$ $(x,y \in S)$ is weakly positive definite. It is obvious that every positive definite function is weakly positive definite, but the converse is false. For example, if S is any semigroup and $A = M_2(\mathbb{C})$, then the function φ form S into A given by

$$\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

defines an A-valued weakly positive definite function, which is not positive definite. For if

$$a = \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}$$

then

$$a\varphi(ss^*)a^* + a\varphi(s^*s)a^* = \begin{pmatrix} 4 & -i \\ i & 0 \end{pmatrix}$$

which is not a positive element of $M_2(\mathbb{C})$ as $2-(5)^{\frac{1}{2}}$ belongs to its espectrum.

On a non-empty set X a kernel $\psi: X \times X \to \tau(A)$ is called *negative definite* if it is hermitian and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^*\psi(x_i,x_j)a_j \leq 0$$

for all subsets $\{x_1,\ldots,x_n\}$ of X and $\{a_1,\ldots,a_n\}$ of A

with $\sum_{i=1}^{n} a_i = 0$. A kernel $\psi : X \times X \to \tau(A)$ is called *weakly negative definite* if it ishermitian and

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \psi(x_k, x_j) \le 0$$

for every choice of $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$. Note that a kernel ψ on a nonempty set *X* is negative definite if and only if for every positive real number *t*, $e^{-t\psi}$ is positive definite (see Theorem 3.2.2 of [4]).

If *S* is a *-semigroup, then a function $\psi : S \to \tau(A)$ is called *negative definite* (respectively, *weakly negative definite*) if the kernel: $(x, y) \mapsto \psi(x^* y)$ $(x, y \in S)$ is negative definite (respectively, weakly negative definite). Note that if *A* has an identity then every negative definite function is also weakly negative definite but the converse is not true. A mapping $\gamma : S \to A$ is called **-additive* if it is hermitian and $\gamma(xy) = \gamma(x) + \gamma(y)$ for all $x, y \in S$. For every *a* in a proper H^*-algebra *A* we denote $\frac{1}{2}(a + a^*)$ by Re(*a*) and $\frac{i}{2}(-a + a^*)$ by Im(*a*). Note that *a*=Re(*a*) + *i* Im(*a*).

Finally, a mapping *T* from a topological *-semigroup *S* with an identity into the bounded *A*-linear operators on a Hilbert module *H* is called a **-representation* if $T_e = I$ (the identity operators), $T_{x^*} = (T_x)^*$ and $T_{xy} = T_x T_y$ for all $x, y \in S$.

§1 The Basic Results

We start with the following proposition whose proof is omitted, since it can be obtained by a slight modification in the proof of Proposition 4.1.9 on [4].

Proposition 1.1. Let A be a proper H^*-algebra. Let S be a commutative *-semigroup with identity e and $\psi : S \to \tau(A)$ be a hermitian function with $\psi(0) \ge 0$.$ Then ψ is weakly negative definite if and only if the kernel: $(x, y) \mapsto \psi(x) + \psi(y)^* - \psi(xy^*)$ is weakly positive definite on $S \times S$.

The proof of the following lemma is straightforward.

Lemma 1.2. Let A be a proper H^*-algebra and S be a commutative *-semigroup with identity e. Let $\psi : S \to \tau(A)$ be hermitian weakly negative definite. Then the following statements hold.

(*i*) $2\mathrm{Re}\,\psi(xy^*) \ge \psi(xx^*) + \psi(yy^*)$ $(x, y \in S)$.

(*ii*) $2\mathrm{Re}\,\psi(x) \ge \psi(e) + \psi(xx^*)$ $(x \in S)$.

Lemma 1.3. Let A be an H^*-algebra with identity 1 and S be a commutative *-semigroup with identity e. Let $\psi : S \to \tau(A)$ be weakly negative definite. Then ψ is *-additive if and only if $2\mathrm{Re}\,\psi(x) = \psi(e) + \psi(xx^*)$ $(x \in S)$. If this is the case, then $\psi(e) = 0$.

Proof. Since the kernel $k : (x, y) \mapsto \psi(x) + \psi(y)^* - \psi(xy^*)$ $(x, y \in S)$ is weakly positive definite, we conclude that the

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

with $a = k(x, x)$, $b = k(x, y)$, $c = k(y, y)$ is positive definite. Thus $ac - bb^* \ge 0$. Now if $2\mathrm{Re}\,\psi(x) = \psi(e) + \psi(xx^*)$ $(x \in S)$, then $a = 0$. We now prove that this implies $b = 0$. So the problem turns into showing that if a matrix of the form

$$\begin{pmatrix} 0 & b \\ b^* & c \end{pmatrix}$$

is positive definite then $b = 0$. By the definition of positive definiteness we have

$$2\mathrm{Re}(u^* bv) + v^* bv \ge 0$$

for all $u, v \in A$. Replacing *u* with $nu\,(n \in \mathbf{N})$, dividing by *n*, and letting $n \to \infty$, we get

$$\mathrm{Re}(u^* bv) \ge 0$$

Replacing *u* by -*u*, we get $\mathrm{Re}(u^* bv) \le 0$, which combined with the preceding yields

$$\mathrm{Re}(u^* bv) = 0.$$

Replacing *u* by *iu*, we get $\mathrm{Im}(u^* bv) = 0$, so

$$u^* bv = 0.$$

for all $u, v \in A$. Taking $u = 1$ and $v = 1$ we get $b = 0$. That is $\psi(x) + \psi(y^*) = \psi(xy^*)$. Replacing *y* by y^*, we obtain $\psi(x) + \psi(y) = \psi(xy)$. This equality gives $\psi(e) = 0$. The converse is obvious.□

Proposition 1.4. Let $\psi : S \to \tau(A)$ be a *-additive function with Re ψ bounded, *i.e.* there exists a positive real number M such that $\tau(\mathrm{Re}\psi(x)) \le M \quad (x, y \in S)$ Then $\mathrm{Re}\psi = 0$.

Proof. As in the proof of 4.3.9 of [4] one can easily prove that every $\tau(A)$ valued *-additive function is weakly negative definite. By Lemma 1.3 we have $2\mathrm{Re}\psi(x) = \psi(xx^*)$ $(x \in S)$. For every positive integer n we can write $(xx^*)^n = tt^*$ for some $t \in S$. Thus

$$n\tau(\psi(ss^*)) = \tau(n\psi(ss^*)) = \tau(\psi(ss^*)^n)$$
$$= \tau(\psi(tt^*)) = 2\tau(\mathrm{Re}\psi(t)) \le 2M$$

where $M > 0$ is a fixed number such that $\tau(\mathrm{Re}\psi(x)) \le M$ for all $x \in S$. Hence

$$0 \le \tau(\psi(ss^*)) \le \frac{2M}{n}$$

Letting $n \to \infty$, we obtain $\tau(\psi(ss^*)) = 0$. So $\tau(\mathrm{Re}\psi(s)) = 0$. Hence $\mathrm{Re}\psi(s) = 0$ $(s \in S)$.□

The following result is indeed the key lemma to this paper.

Lemma 1.5. Let A be an H^*-algebra with identity. Let S be a topological *-semigroup with identity e and with a weight function w. Let $\psi : S \to \tau(A)$ be a τ-norm w-bounded continuous negative definite function on S. Then there exist a w-bounded *-representation π_ψ of S by bounded A-linear operators on a Hilbert module K_ψ and a norm-continuous mapping $C_\psi : S \to K_\psi$ such that $C_\psi(st) = \pi_\psi(s)C_\psi(t) + C_\psi(s)$ $(s, t \in S)$.

Proof. Let K_1 denote the set of all formal finite sums of the form $f = \sum_{i=1}^n x_i a_i$ with $x_i \in S$, $a_i \in A$ and $\sum_{i=1}^n a_i = 0$ $(n \in \mathbb{N})$. We make K_1 into a right A module by defining $fa = \sum_{i=1}^n x_i a_i a$ for every $f = \sum_{i=1}^n x_i a_i \in K_1$ and every $a \in A$. For $f = \sum_{i=1}^n x_i a_i$ and $g = \sum_{j=1}^m y_j b_j$ in K_1 we define $(f, g)_\psi = -\sum_{i=1}^n \sum_{j=1}^m a_i^* \psi(x_i^* y_j) b_j$. Put

$$N_\psi = \{f \in K_1 : \tau(f, f)_\psi = 0\}.$$

From the fact that $|tr(f, g)| \le \tau(f, f)\tau(g, g)$ for all $f, g \in A$ it follows that N_ψ defines a linear subspace of K_1. Using the fact that $\tau(fa, fa) \le \tau((f.f)aa^*) \le \tau(f, f)\tau(aa^*)$ $(f \in K_1, a \in A)$, we conclude that fa is in hN_ψ for every $f \in N_\psi$ and $a \in A$. So N_ψ defines a right A-module. Let $K_0 = K_1 / N_\psi$. Then for every $f = \sum_{i=1}^n x_i a_i + N_\psi$ and $g = \sum_{j=1}^m y_j b_j + N_\psi$ in K_1 the equation

$$\langle f, g\rangle_\psi = tr\left(-\sum_{i=1}^n \sum_{j=1}^m a_i^* \psi(x_i^* y_j) b_j .\right)$$

defines an inner product $\langle .,. \rangle_\psi$ on K_0. Let K denote the Hilbert space completion of K_0 with respect to this inner product and we denote the corresponding norm on K by $\|\,.\,\|_\psi$. For every $x \in S$ and $f = \sum_{i=1}^n x_i a_i + N_\psi \in K_0$ we define $\pi_\psi(x)f = \sum_{i=1}^n x x_i a_i + N_\psi$. It is clear that $\pi_\psi(xy) = \pi_\psi(x)\pi_\psi(y)$ $(x, y \in S)$ on K_0. We now prove that for every $x \in S$, $\pi_\psi(x)$ defines a w-bounded operator on K_0. To this end, choose $f = \sum_{i=1}^n x_i a_i + N_\psi \in K_0$ and define the complex-valued function h on S by $h(x) = tr\left(-\sum_{i=1}^n \sum_{j=1}^n a_i^* \psi(x_i^* x x_j) a_j\right)$. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $y_1, \ldots, y_m \in S$. Since $\sum_{k=1}^m \sum_{i=1}^n \lambda_k a_i = (\sum_{k=1}^m \lambda_k)(\sum_{i=1}^n a_i) = 0$, we conclude that $\sum_{k=1}^m \sum_{\ell=1}^m \lambda_k \overline{\lambda_\ell} h(y_k y_\ell^*) \ge 0$. So h defines a complex-valued positive definite function on S.

$$|h(x)| \le \tau\left(\sum_{i=1}^n \sum_{j=1}^n a_i^* \psi(x_i^* x x_j) a_j\right)$$
$$\le \sum_{i=1}^n \sum_{j=1}^n \tau(a_i^*)\tau(a_j)\tau(\psi(x_i^* x x_j))$$
$$\le M w(x) \qquad (x \in S),$$

with $M = M_1 \sum_{i=1}^n \sum_{j=1}^n \tau(a_i^*)\tau(a_j) w(x_i) w(x_j)$ where $M_1 > 0$ is chosen so that $\tau(\psi(x)) \le M_1 w(x)$ $(x \in S)$. By Proposition 4.1.12 of [4] we have

$$\|\pi_\psi(x) f\|_\psi^2 = h(xx^*) \le h(e) w(xx^*)$$
$$= \langle f, f\rangle_\psi w(xx^*)$$
$$\le \|f\|_\psi^2 (w(x))^2 \quad (x \in S).$$

Hence $\|\pi_\psi(x)\|_\psi \le \|f\|_\psi w(x)$. By the norm density of K_0 in K, one can easily extend π_ψ to a representation $\tilde{\pi}_\psi$ of S by bounded operators on K such that $\|\tilde{\pi}_\psi(x)\|_\psi \le \|f\|_\psi w(x) \quad (x \in S)$. For simplicity we again denote $\tilde{\pi}_\psi$ by π_ψ.

Now for every $s \in S$ we define $C_\psi : S \to K_0$ by $C_\psi(s) = s1 + e(-1) + N_\psi$. Since for every $s,t \in S$

$$\|C_\psi(s) - C_\psi(t)\|_\psi^2 = tr(-\psi(ss^*)2\operatorname{Re}\psi(st^*) - \psi(tt^*)),$$

from the τ-norm continuity of ψ it follows that C_ψ is also continuous. It is now clear that C_ψ, π_ψ satisfy the equation (1).□

The following theorem is the main result of this paper.

Theorem 1.6. Let S be a commutative topological *-semigroup with identity and with a weight function w. Let A be an H^*-algebra with identity 1. Let ψ be a τ-valued τ-norm w-bounded and τ-norm continuous negative definite function on S such that $\operatorname{Re}\psi$ is bounded. Then there exists a τ-norm continuous w-bounded positive definite function φ on S, and a *-homomorphism $\gamma : S \to \operatorname{Re}(A)$ such that $\psi = \psi(e) - \varphi(e) + i\gamma + \varphi.$

Proof. We construct the Hilbert space K_ψ, the continuous representation π_ψ, and the norm continuous function $C_\psi : S \to K_\psi$ as in the proof of Lemma 1.5. For every $s \in S$ we have

$$\begin{aligned}\|C_\psi(s)\|_\psi^2 &= \langle C_\psi(s), C_\psi(s)\rangle_\psi \\ &= tr(s1 + e(-1) + N_\psi, s1 + e(-1) + N_\psi) \\ &= tr(\psi(ss^*) - 2\operatorname{Re}\psi(s) + \psi(ee^*)) \\ &\le \tau(\psi(ss^*)) + 2\tau(\operatorname{Re}\psi(s)) + \tau(\tau(ee^*)) \\ &< M + 2M + M = 4M,\end{aligned}$$

where $M > 0$ is such that $\tau(\operatorname{Re}\psi(s)) \le M$ for all $s \in S$. Therefore $\|C_\psi(s)\|_\psi \le 2M^{1/2} \quad (s \in S)$. Let K' denote the closed convex hull of the set $\{C_\psi(s) : s \in S\}$ in K_ψ. Since on K_ψ the weak topology coincides with its weak *-topology, from the Banach Alaoglu Theorem and the Krein-Milman Theorem it follows that K' is weakly compact. For every $t \in S$ we define $\tilde{t}(C_\psi(s)) = C_\psi(ts)$. Then we extend $\tilde{t}$ on K' in the obvious way and we denote its extension again by $\tilde{t}$. It is clear that $\|\tilde{t}(\lambda)\|_\psi \le 2M^{1/2}$ for every $\lambda \in K'$. Since S is commutative, from the Markov-Kakutani fixed point theorem (p. 456 of [7]) it follows that there exists $\nu \in K'$ such that $\tilde{t}(\nu) = \nu$ for all $t \in S$. So by (1) we have $\pi_\psi(s)\nu = C_\psi(s) + \nu$ ($s \in S$). Thus for every $s,t \in S$

$$\begin{aligned}\psi(t^*s) - \psi(t^*) - \psi(s) + \psi(e) &= (C_\psi(s), C_\psi(t))_\psi \\ &= (\pi_\psi(s)\nu - \nu, \pi_\psi(t)\nu - \nu)_\psi \\ &= (\pi_\psi(t^*s)\nu, \nu)_\psi - (\pi_\psi(s)\nu, \nu)_\psi \\ &\quad - (\pi_\psi(s)\nu, \nu)_\psi^* + (\nu, \nu)_\psi.\end{aligned}$$

Let $\varphi(s) = (\pi_\psi(s)\nu, \nu)_\psi$ ($s \in S$). From the equality $\|\pi_\psi(s)\nu - \pi_\psi(t)\nu\|_\psi = \|C_\psi(s) - C_\psi(t)\|_\psi$ for every $s,t \in S$ and the continuity of C_ψ it follows that φ is also continuous. It is also clear that φ defines a w-bounded positive definite function. For every $s,t \in S$

$$\begin{aligned}&\psi(t^*s) - \psi(t^*) - \psi(s) + \psi(e) \\ &\quad = (\varphi(t^*s) - a) - (\varphi(t^*s) - a)^* - (\varphi(s) - a)\end{aligned}$$

with $a = (\nu, \nu)_\psi$ which is positive in $\tau(A)$. If we put $b = a - \psi(e)$, then $b \ge 0$, and $\chi = \psi + b + \varphi$ satisfies $\chi(t^*s) = \chi(t)^* + \chi(s) = \chi(t^*) + \chi(s)$ ($s,t \in S$). That is; χ defines a *-homomorphism of S into $\tau(A)$. It is clear that χ is a negative definite. So by Proposition 1.4, $\operatorname{Re}(\chi) = 0$. Thus $\chi = i\gamma$, where $\gamma : S \to \operatorname{Re}(A)$ is a *-homomorphism. Since both ψ, it follows that γ is continuous. Using the fact that $\gamma(e) = 0$, we conclude that $b = \psi(e) - \varphi(e)$. The proof is now complete.□

An application of Theorem 1.6 with the aid of Proposition 1.4 gives the following generalization of the Harzallah result in [10] (see also Proposition 7.13 of [5]) from the case of continuous complex-valued negative definite functions on commutative topological groups to the case of bounded continuous H^*-valued negative definite functions on commutative topological *-semigroups.

Corollary 1.7. Let S be a commutative topological *-semigroup with identity. Let $\psi : S \to \tau(A)$ be a bounded τ-norm continuous negative definite function.

Then there exists a bounded continuous A-valued positive definite function $\varphi : S \to \tau(A)$ such that

$$\psi = \psi(e) - \varphi(e) + \varphi .$$

§2 A Lévy-Khinchin Formula for H^*-valued Negative Definite Functions

In this section we assume that *S* is a commutative topological *-semigroup with identity and with a weight function *w* continuous at the identity.

We denote by S^* the set of all *-semicharacters on *S*. Note that when S^* equipped with the topology of pointwise convergence inherited from $\square^S$, defines a completely regular space. We also note that a *-semicharacter χ is *w*-bounded if and only if $|\chi| \leq w$. Hence Γ_w^*, the space of all *w*-bounded *-semicharacters on *S* is a compact subset of S^*. We denote by $\Gamma^*_{(w,c,e)}$ ($\Gamma^*_{(w,c)}$, respectively) the set of all semicharacters in Γ_w^* which are continuous at *e* (continuous on *S*, respectively).

Let λ be a nonnegative Radon measure on S^*; the generalized Laplace transform of λ whenever it is defined is given by

$$\hat{\lambda}(s) = \int_{S^*} \gamma(s) d\lambda(\gamma) \quad (s \in S).$$

These functions are referred to as moment functions (see, [23]). Note that every moment function is positive definite. We denote by $\mathrm{P}(S,w,c,e)$ ($\mathrm{P}(S,w,c)$, respectively) the set of all *w*-bounded continuous at identity (continuous, respectively) complex-valued positive definite functions on *S*. We denote the complex span of $\mathrm{P}(S,w,c,e)$ by $\langle \mathrm{P}(S,w,c,e)\rangle$. As is shown in Proposition 1 of [15] $\langle \mathrm{P}(S,w,c,e)\rangle$ is translation invariant, that is $\ell_a\varphi \in \langle \mathrm{P}(S,w,c,e)\rangle$ for every $\varphi \in \langle \mathrm{P}(S,w,c,e)\rangle$ and $a \in S$, where $(\ell_a\varphi)(x) = \varphi(ax)$ for all $x \in S$. Let *w* be a weight function on *S*. By the continuity of *w* at *e* there is a fixed neighbourhood V_0 of *e* on which *w* is bounded. Let $\mathcal{V}$ be a basis of neighbourhoods *V* of *e* which are contained in V_0. For $V \in \mathcal{V}$ and $\varphi \in \langle \mathrm{P}(S,w,c,e)\rangle$, set

$$\|\varphi\|_V = \sup\{|\varphi(s)| : s \in V\}.$$

Let $\langle \mathrm{P}(S,w,c,e)\rangle^*$ denote the complex-vector space of all linear functionals *L* on $\langle \mathrm{P}(S,w,c,e)\rangle$ such that for every $V \in \mathcal{V}$ there exists a positive number C_V satisfying

$$|L(\varphi)| \leq C_V \|\varphi\|_V \quad (\varphi \in \langle \mathrm{P}(S,w,c,e)\rangle, V \in \mathcal{V}).$$

The infimum of the constants C_V will be denoted by $\|L\|_V$. Note that $\|\,.\,\|_V$ defines a norm on $\langle \mathrm{P}(S,w,c,e)\rangle^*$. The topology on $\langle \mathrm{P}(S,w,c,e)\rangle^*$ will be the topology induced by the norm $\|\,.\,\|_V$, that is a net (L_α) in $\langle \mathrm{P}(S,w,c,e)\rangle^*$ converges to $L \in \langle \mathrm{P}(S,w,c,e)\rangle^*$ if $\|L_\alpha - L\|_V \to 0$ for every $V \in \mathcal{V}$. A functional $L \in \langle \mathrm{P}(S,w,c,e)\rangle^*$ is called *nonnegative on* $V \in \mathcal{V}$ if $L(\varphi) \geq 0$ for every $\varphi \in \langle \mathrm{P}(S,w,c,e)\rangle$ with $\varphi \geq 0$ on *V*. A topological $*-$semigroup *S* is called *admissible with respect to a weight w* if for each $V \in \mathcal{V}$, there exists an element $L = L_V = \langle \mathrm{P}(S,w,c,e)\rangle^*$ which is nonnegative on *V* and $s \to \ell_s L$ from *S* into $\langle \mathrm{P}(S,w,c,e)\rangle^*$ is continuous at *e*, where $\$(\ell_s L)(\varphi) = L(\ell_s(\varphi))$ ($\varphi \in \langle \mathrm{P}(S,w,c,e)\rangle$). For further information on admissible topological semigroups with respect to a weight we refer the reader to [23].

Theorem 2.1. (Generalized Bochner's Theorem). Let S be a commutative topological $*-$semigroup admissible with respect to a weight w and let A be a proper H^*-algebra. Let φ be a $\tau(A)-$valued and $\tau-$norm w-bounded, $\tau-$continuous at the identity and positive definite function on S. Then there exists a unique A-valued spectral measure (c.f.p. 118 of [20]) $P : \Delta \to P(\Delta)$ defined on the $\sigma-$algebra of Borel subsets of Γ_w^* such that $P(K) = 0$ for every compact subset K of $\Gamma_w^* \setminus \Gamma^*_{(w,c,e)}$ and

$$\varphi(x) = \int_{\Gamma^*_{(w,c)}} \chi(x) dP(\chi) \quad (x \in S).$$

Proof. By Theorem 1 of [19], there exists a *w*-bounded *-representation *T* of *S* by *A*-linear operators on a Hilbert module *K* with some vector $\xi_0 \in K$ such that $\varphi(x) = tr(\xi_0, T_x\xi_0)$ $(x \in S)$. So to every vector $\xi \in K$ the function φ_ξ where $\varphi_\xi(x) = tr(\xi, T_x\xi)$ $(x \in S)$. defines a *w*-bounded, continuous at the identity and complex-valued positive definite function on *S*. So by Theorem B of [23] there exists a positive regular measure μ_ξ such that

$$\varphi_\xi(x) = \int_{\Gamma_w^*} \chi(x) d\mu_\xi(\chi) \quad (x \in S).$$

Now an argument similar to the proof of Theorem 3.5, of [13] (see also, [16]) shows that there exists a spectral measure E from $B(\Gamma_w^*)$ (the $\sigma-$algebra of all Borel subsets of Γ_w^* into the bounded operators on the Hilbert space $(K,\langle\ ,\ \rangle)$ $(\langle \nu,\eta\rangle = tr(\nu,\eta)\ (\nu,\eta\in K))$ such that

$$\langle \nu, T_x\eta\rangle = \int_{\Gamma_w^*} \chi(x)\, d\langle \nu, E(\chi)\eta\rangle \quad (x\in S, \nu,\eta\in K).$$

Now if we define the generalized spectral measure P on $B(\Gamma_w^*)$ by

$$(P(\Delta)\xi,\eta) = (E(\Delta)\xi,\eta) \quad (\nu,\eta\in K, \Delta\in B(\Gamma_w^*))$$

then we obtain

$$\varphi(x) = \int_{\Gamma_w^*} \chi(x)\, dP(\chi) \quad (x\in S).$$

Thus the theorem is established.□

A combination of Theorems 2.1 and 1.6 gives the following type of the Lévy-Khinchin formula for the $\tau(A)$-valued negative definite functions (see, page 271 of [3]). It also establishes our conjecture in [14] even in a more general setting. Note that the proof of Theorem 1.6 shows that if ψ is continuous at the identity then so is γ.

Theorem 2.2. Let S be a commutative *-semigroup admissible with respect to a weight function w. Let A be an H^*-algebra with identity. Suppose that ψ is a τ-norm w-bounded and τ-continuous at identity and negative definite function of S into $\tau(A)$ such that $\mathrm{Re}\psi$ is bounded. Then there exists a unique A-valued spectral measure $P:\Delta\to P(\Delta)$ defined on the σ-algebra of Borel subsets of Γ_w^* such that $P(K)=0$ for every compact subset K of $\Gamma_w^*\setminus\Gamma_{(w,c,e)}^*$ and a continuous at the identity *-homomorphism $\gamma: S\to \mathrm{Re}(A)$ with

$$\psi(x) = \psi(e) + i\,\gamma(x) + \int_{\Gamma_{(w,c)}^*} [1-\chi(x)]\, dP(\chi) \quad (x\in S).$$

Before turning the next result, we shall first recall that (see [2,13,15]) on a topological semigroup S the algebra $M_a(S)$ denotes the space of all measures $\mu\in M(S)$ (the Banach algebra of bounded regular complex measures on S) such that the mappings: $x\mapsto \delta_x * |\mu|$ and $x\mapsto |\mu| * \delta_x$ (δ_x denotes the Dirac measure at x) from S into $M(S)$ are weakly continuous. S is called a *foundation semigroup* if $\bigcup \mathrm{supp}\{\mu : \mu\in M_a(S)\}$ is dense in S. As well as from weighted topological groups, topological *-groups for which the involution * is not necessarily the same as the inversion, and weighted discrete semigroups there are many other examples of weighted foundation semigroups. For example, S_1, the semigroup with underlying space the subset $[1,3]\times[1,3]$, of $\mathbb{R}^2$ and multiplication defined as follows:

$$(a,b)(c,d) := (\min(ac,3), \min(ad+b,3))$$

for all $a,b,c,d\in[1,3]$, and S_2 with the underlying space also $[1,3]\times[1,3]$, but multiplication defined by

$$(a,b)(c,d) := (\min(ac,3), \min(bc+d,3))$$

for all $a,b,c,d\in[1,3]$, whenever both S_1 and S_2 are endowed with restriction topology of $\mathbb{R}^2$ are foundation semigroups. For more details see [22]. It is also easy to see that $S_3=[0,1]$ with the restriction topology of $\mathbb{R}$ and multiplication defined by $xy=\min(x+y,1)$ for all $x,y\in[0,1]$ is a foundation semigroup. For further examples we refer the interested reader to [22].

Recall that if S is foundation *-semigroup with identity then it is admissible with respect to any weight w which is continuous at the identity, moreover $\Gamma_{(w,c)}^* = \Gamma_{(w,c)}^*$ (see,[15]). So in the case from Theorem 2.1 we obtain the following generalization of Theorem 5.3 of [13].

Theorem 2.3. (Generalized Bochner's Theorem on foundation semigroups). Let S be a commutative foundation topological *-semigroup with identity and with a weight function w. Let A be a proper H^*-algebra and φ be a τ-norm w-bounded and τ-continuous positive definite function of S into $\tau(A)$. Then there exists a unique A-valued spectral measure $P:\Delta\to P(\Delta)$ defined on the σ-algebra of Borel subsets of Γ_w^* such that

$$\varphi(x) = \int_{\Gamma_{(w,c)}^*} \chi(x)\, dP(\chi) \quad (x\in S).$$

In the particular case that S is a foundation semigroup with identity, an application of Theorem 2.2 with the aid of Theorem 2.3 gives the following Lévy-Khinchin formula for the τ-norm w-bounded τ-norm continuous negative definite functions on S.

Theorem 2.4. Let S be a commutative foundation topological *-semigroup with identity and with a weight function w. Let ψ be a τ-norm w-bounded τ-norm continuous negative definite function of S into $\tau(A)$ of an H^*-algebra A with identity. If $\mathrm{Re}\psi$ is bounded, then there exists a unique A-valued spectral measure $P:\Delta \to P(\Delta)$ defined on the σ-algebra of Borel subsets of Γ_w^* such that $P(K)=0$ for every compact subset K of $\Gamma_w^* \setminus \Gamma_{(w,c,e)}^*$ and a $\mathrm{Re}(A)$-valued continuous *-homomorphism γ on S with

$$\psi(x)=\psi(e)+i\,\gamma(x)+\int_{\Gamma_{(w,c)}^*}[1-\chi(x)]dP(\chi) \quad (x \in S).$$

As an application of the above result we obtain the following generalization of Theorem 3 of [19] from the case of locally compact groups to the case of locally compact *-groups for which the involution * is not necessarily the same as the inversion. Note that the space of *w*-bounded *-characters on *G* is denoted by G_w^*.

Theorem 2.5. Let G be a locally compact group with a continuous involution * and with a weight function w. Let φ be a τ-norm w-bounded and τ-continuous positive definite function of G into $\tau(A)$ of a proper H^*-algebra A. Then there exists a unique A-valued spectral measure $P:\Delta \to P(\Delta)$ defined on the σ-algebra of Borel subsets of Γ_w^* such that

$$\varphi(x)=\int_{G_w^*}\chi(x)dP(\chi) \qquad (x \in G).$$

A result similar to that of Theorem 2.4 can be proved for locally compact *-groups. We have omitted even the statement of the theorem.

In the following we give an example of an H^*-algebra *A* together with a weighted foundation semigroup *S* and with a $\tau(A)$-valued continuous positive definite function on *S*.

Example. Let *S* be a commutative foundation *-semigroup with identity and with a weight function *w*. Let (X,μ) be a probability measure space and λ be a positive measure in $M(\Gamma_w^*)$. By the Example 1 on page 368 of [1], $A=L^2(X\times\Gamma_w^*,\mu\times\lambda)$ defines an H^*-algebra. Take a positive element *a* in *A*. So $a\in\tau(A)$. Define $\varphi:S\to\tau(A)$ by

$$\varphi(s)(x,\chi)=a(x,\chi)\chi(s) \quad (x\in X,\chi\in\Gamma_w^*,s\in S)$$

It is obvious that φ defines a *w*-bounded $\tau(A)$-valued positive definite function on *S*. To prove the continuity of φ we first prove that it is continuous at *e*. To see this, by the continuity of *w* at *e* we can take a fixed compact neighbourhood *V* of *e* on which *w* is bounded. Suppose that $k=\sup\{w(s):s\in V\}$. For every positive ε, by the regularity of λ, there exists a compact subset *K* of G_w^* such that $\lambda(\Gamma_w^*\setminus K) < \frac{\varepsilon}{4k^2}$. From Ascoli's theorem (11, p.233, Theorem 17) it follows that *K* is equicontinuous. Therefore there exists an open neighbourhood *W* of *e* such that $|\chi(s)-\chi(e)|<\varepsilon^{1/2}$ ($\chi\in K, s\in W$). Let $U=W\cap V$. Then for every $s\in U$ we have

$$\begin{aligned}
&\|\varphi(s)-\varphi(e)\|_2^2\\
&=\int_{X\times\Gamma_w^*}|a(x,\chi)|^2|\chi(s)-\chi(e)|^2\,d(\mu\times\lambda)(x,\chi)\\
&\le \|a\|_2^2\int_{\Gamma_w^*}|\chi(s)-\chi(e)|^2\,d\lambda(\chi)\\
&=\|a\|_2^2\Big(\int_{\Gamma_w^*\setminus K}|\chi(s)-\chi(e)|^2\,d\lambda(\chi)\\
&\quad+\int_K|\chi(s)-\chi(e)|^2\,d\lambda(\chi)\Big)\\
&\le \|a\|_2^2\,(4k^2\lambda(\Gamma_w^*\setminus K)+\varepsilon\lambda(K))\\
&<\varepsilon\|a\|_2^2\,\|\lambda\|.
\end{aligned}$$

Thus φ is continuous at *e*. Since *S* is a foundation semigroup with identity, an argument similar to that in the proof of Lemma 3 of [15] proves that φ is continuous on the whole of *S*.

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