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MIXED BOUNDARY VALUE PROBLEM FOR A QUARTER-PLANE WITH A ROBIN CONDITION

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Abstract

We consider a mixed boundary value problem for a quarter-plane with a Robin condition on one edge. We have developed two procedures, one based on the advanced theory of dual integral equations and the other, in our opinion simpler technique, relying on conformal mapping. Both of the procedures are of interest, because the former may be easier to adapt to other boundary value problems.

Keywords: Mixed boundary value problems; Dual integral equations; Fredholm's integral equation of second kind

Introduction

Problem. Find the solution of the following mixed boundary value problem.

P.D.E: $\Delta\Phi = 0$, $\quad x > 0,\ y > 0.$

B.C. 1) $\Phi_x(0,y) + h(y)\,\Phi(0,y) = q(y)$, $\quad y > 0.$

2) $\Phi_y(x,0) = f(x)$, $\quad 0 < x < 1.$

3) $\Phi(x,0) = 0$, $\quad x > 1.$

Note. Our strategy is to create a sequence of tractable boundary value problems leading to standard integral equations which may be solved numerically.

Solution. We first find the solution to our auxiliary problem by replacing the Robin condition by $\Phi_x(0,y) = \theta(y)$, where $\theta(y)$ is an admissible boundary function.

Let $\Phi(x,y)$ be the desired function, then $\Phi(x,y)$ has the following integral representation:

$$\Phi(x,y) = \int_0^\infty A(\xi)\exp(-x\xi)\sin(y\xi)d\xi + \int_0^\infty B(\xi)\exp(-y\xi)\cos(x\xi)d\xi . \tag{1.1}$$

Let

$$\frac{\partial}{\partial x}\Phi(0,y) = -\int_0^\infty \xi A(\xi)\sin(y\xi)d\xi = \theta(y). \tag{1.2}$$

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From Fourier's inversion-formula for the sine transform

$$\xi A(\xi) = -\frac{2}{\pi}\int_0^\infty \theta(s)\sin(s\xi)ds \,. \tag{1.3}$$

From (1.3) we obtain the following equation:

$$\int_0^\infty \xi A(\xi)\exp(-x\xi)d\xi = -\frac{2}{\pi}\int_0^\infty \frac{s\theta(s)}{s^2+x^2}ds \,. \tag{1.4}$$

The conditions on y = 0 lead us to the pair of dual integral equations

$$\int_0^\infty \xi A(\xi)\exp(-x\xi)d\xi - \int_0^\infty \xi B(\xi)\cos(x\xi)d\xi = f(x) \,, \qquad 0<x<1, \tag{1.5}$$

$$\int_0^\infty B(\xi)\cos(x\xi)d\xi = 0 \,, \qquad x>1, \tag{1.6}$$

The equation (1.6) is satisfied by [1,3,4]:

$$B(\xi) = \frac{2}{\pi}\int_0^1 tg(t)J_0(t\xi)dt \,,$$

where g (t) is determined to satisfy (1.5). Thus (1.5) may be rewritten as

$$-\frac{d}{dx}\int_0^\infty B(\xi)\sin(x\xi)d\xi - \frac{2}{\pi}\int_0^\infty \frac{s\theta(s)}{s^2+x^2}ds = f(x) \,, \qquad 0<x<1. \tag{1.7}$$

Following Sneddon's elementary solution [4], we have

$$g(t) = -\int_0^t \frac{f(x)dx}{\sqrt{t^2-x^2}} - \int_0^\infty \frac{\theta(s)ds}{\sqrt{s^2+x^2}} \,, \tag{1.8}$$

$$\Phi(0,y) = \int_0^\infty A(\xi)\sin(y\xi)d\xi - \int_0^\infty B(\xi)\exp(-y\xi)d\xi \,, \tag{1.9}$$

from (1.3) and [2]

$$\Phi(0,y) = \frac{-1}{\pi}\int_0^\infty \theta(s)\ln\left|\frac{y+s}{y-s}\right|ds + \frac{2}{\pi}\int_0^1 \frac{tg(t)}{\sqrt{t^2+y^2}}dt \,. \tag{1.10}$$

Now we turn to the Robin condition, which gives us the equation

$$\theta(y) = q(y) - h(y)\left[\frac{-1}{\pi}\int_0^\infty \theta(s)\ln\left|\frac{y+s}{y-s}\right|ds + \frac{2}{\pi}\int_0^1 \frac{tg(t)}{\sqrt{t^2+y^2}}dt\right]. \tag{1.11}$$

Further simplification is possible. We have

$$\int_0^1 \frac{tg(t)}{\sqrt{t^2+y^2}}dt = -\int_0^1 f(x)\left[\int_x^1 \frac{tdt}{\sqrt{(s^2+t^2)(t^2+y^2)}}\right]dx - \int_0^\infty \theta(s)\left[\ln\frac{\sqrt{1+y^2}+\sqrt{1+s^2}}{y+s}\right]ds \,.$$

Thus, θ (y) is determined by the Fredholm's integral equation of the second kind:

$$\theta(y) - h(y)\int_0^\infty \theta(s)\ln\left|\frac{y-s}{\sqrt{1+y^2}+\sqrt{1+s^2}}\right|ds = F(y) \qquad 0<y<\infty,$$

where

$$F(y)=q(y)-h(y)\frac{2}{\pi}\int_0^1 f(x)\ln\frac{\sqrt{x^2+y^2}}{\sqrt{1-x^2}+\sqrt{1+y^2}}dx\,.$$

Putting the pieces together, θ(y) determines g(t) which in turn allows us to compute

$$\Phi(x,y)\,.$$

Note. In the above integral equation we used the following Integrals:

a)

$$\int_0^1\frac{tdt}{\sqrt{t^2+y^2}}\int_0^t\frac{f(x)dx}{\sqrt{t^2-x^2}}=$$

$$\int_0^1 f(x)dx\{\int_x^1\frac{tdt}{\sqrt{(t^2-x^2)(t^2+y^2)}}\}\,,$$

but the inner integral of right side can be written after changing in variables:

$$t^2=x^2\cos^2\alpha+\sin^2\alpha$$

and

$$dt=(1-x^2)\sin\alpha\cos\alpha\; d\alpha$$

$$\int_x^1\frac{tdt}{\sqrt{(t^2-x^2)(t^2+y^2)}}=$$

$$\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-x^2}\cos\alpha}{\sqrt{(x^2+y^2)+(1-x^2)\sin^2\alpha}}d\alpha\,.$$

Using another change of variable of the form $\sqrt{1-x^2}\sin\alpha=u$ one gets,

$$\int_x^1\frac{tdt}{\sqrt{(t^2-x^2)(t^2+y^2)}}=\int_0^{\sqrt{1-x^2}}\frac{du}{\sqrt{(x^2+y^2)+u^2}}=$$

$$-\ln\frac{\sqrt{x^2+y^2}}{\sqrt{1-x^2}+\sqrt{1+y^2}}\,.$$

b)

$$\int_0^1\frac{tdt}{\sqrt{(t^2+y^2)(t^2+s^2)}}=$$

$$\int_0^1\frac{tdt}{\sqrt{(t^2+\frac{y^2+s^2}{2})^2-(\frac{y^2-s^2}{2})^2}}\,.$$

Let

$$t^2+\frac{y^2+s^2}{2}=\left|\frac{y^2-s^2}{2}\right|u$$

$$\int_0^1\frac{tdt}{\sqrt{(t^2+\frac{y^2+s^2}{2})^2-(\frac{y^2-s^2}{2})^2}}=$$

$$\ln\frac{\sqrt{1+y^2}+\sqrt{1+s^2}}{y+s}\,.$$

Next we show how the conformal mapping w = 2sin^{-1}z may be gainfully used to solve the problem.

Let

w = 2sin^{-1}z, x + iy = $\sin\frac{u+iv}{2}$

implies that,

x = $\sin\frac{u}{2}\cosh\frac{v}{2}$, y = $\cos\frac{u}{2}\sinh\frac{v}{2}$

Let

$$\Phi(x,y)\;=\;\Psi(u,v)\,.$$

The boundary conditions change to:

$$\Delta\Psi(u,v)=0$$

1- $\Psi_u(0,v)\sec hv+h(\sinh\frac{v}{2})\Psi(0,v)=q(\sinh\frac{v}{2})$,

2- $\Psi_v(u,0)\sec hv=f(\sinh\frac{u}{2})\cos\frac{u}{2}$, $0<u<\pi$,

3- $\Psi(\pi,v)=0$, $v>0$,

We formulate an auxiliary problem for Laplace's equation

$$\Delta\Psi(u,v)=0$$

1- $\Psi_u(0,v)=\theta(v)$, $v>0$,

2- $\Psi_v(u,0)=f(\sinh\frac{u}{2})\cos\frac{u}{2}$, $0<u<\pi$,

3- $\Psi(\pi,v)=0$, $v>0$.

Since we have a non-homogenous boundary condition, $\Psi(u,v)$ has the following integral representation:

$$\Psi(u,v)=\sum_{n=1}^{\infty}a_n\exp[-(n-\frac{1}{2})v]\cos(n-\frac{1}{2})u+\int_0^{\infty}c(\xi)\sinh(\pi-u)\xi\cos(v\xi)d\xi\ . \quad (1.12)$$

From boundary conditions (1) and (2) we obtain

$$\int_0^{\infty}c(\xi)\sinh(\pi-u)\xi\cos(v\xi)d\xi=-\theta(v)\,, \quad (1.13)$$

$$\sum_{n=1}^{\infty}(n-\frac{1}{2})a_n\cos(n-\frac{1}{2})u=f(\sinh\frac{u}{2})\cos\frac{u}{2}\,, \quad (1.14)$$

or,

$$(n-\frac{1}{2})a_n=-\frac{2}{\pi}\int_0^{\pi}f(\sin\frac{u}{2})\cos\frac{u}{2}\cos(n-\frac{1}{2})udu\ . \quad (1.15)$$

From the above relation one gets

$$a_n=-\frac{4}{\pi(2n-1)}\int_0^{\pi}f(\sin\frac{u}{2})\cos\frac{u}{2}\cos(n-\frac{1}{2})udu\ . \quad (1.16)$$

Then

$$\Psi_u(0,v)=\sum_{n=1}^{\infty}a_n\exp[-(n-\frac{1}{2})v]+\int_0^{\infty}c(\xi)\sinh\pi\xi\cos(v\xi)d\xi\ , \quad (1.17)$$

and,

$$\Psi_u(0,v)=-\int_0^{\infty}c(\xi)\cosh\pi\xi\cos(v\xi)d\xi=\theta(v)\ . \quad (1.18)$$

We need to express $\Psi(0,v)$ in terms of θ and f. Therefore from (1.18) we get

$$\xi\ c(\xi)\cosh\pi\xi=\frac{-2}{\pi}\int_0^{\infty}\theta(s)\cos(\xi s)ds\ . \quad (1.19)$$

The above relation yields:

$$\int_0^{\infty}c(\xi)\sinh\pi\xi\cos(v\xi)d\xi=\frac{-2}{\pi}\int_0^{\infty}\theta(s)\,ds\int_0^{\infty}\frac{\cos\xi s\cos\xi v\sinh\pi\xi}{\xi\cosh\pi\xi}d\xi\ . \quad (1.20)$$

The inner integral at right side can be written as follows [2]:

$$\frac{1}{\pi}\int_0^\infty \frac{\tanh \pi\xi}{\xi}[\cos\xi(v+s)+\cos\xi(v-s)]d\xi = \frac{-1}{\pi}\log[\coth(\frac{v+s}{4})\coth(\frac{v-s}{4})] \quad (1.21)$$

As in previous case, the Robin condition will give a Fredholm's integral equation of the second kind with a complicated kernel that can be solved

$$\theta(v)\sec hv + h(\sinh\frac{v}{2})\Psi(0,v) = q(\sinh\frac{v}{2}) \quad (1.22)$$

with,

$$\Psi(0,v) = \sum_{n=1}^{\infty} a_n \exp[-(n-\frac{1}{2})v] + \int_0^\infty c(\xi)\sinh\pi\xi\cos(v\xi)d\xi ,$$

$$\int_0^\infty c(\xi)\sinh\pi\xi\cos(v\xi)d\xi = 2\int_0^\infty \theta(s)[\log\coth\frac{v+s}{4}\coth\frac{v-s}{4}]ds ,$$

$$a_n = -\frac{4}{\pi(2n-1)}\int_0^\pi f(\sin\frac{u}{2})\cos\frac{u}{2}\cos(n-\frac{1}{2})udu .$$

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