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UNIQUENESS OF SOLUTION FOR A CLASS OF STEFAN PROBLEMS

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Abstract

This paper deals with a theoretical mathematical analysis of one-dimensional solidification problem, in which kinetic undercooling is incorporated into the This temperature condition at the interface. A model problem with nonlinear kinetic law is considered. We prove a local result intimate for the uniqueness of solution of the corresponding free boundary problem.

Keywords: Free boundary problem; Nonlinear integral equation

Introduction

It is well known that in many industrial areas, the solidification process plays a significant role. Mathematical models of solidification including interface kinetics effects have been considered for quite some time (see [1], and references therein). This class of free boundary problems, which arises in a number of physical situations, is that of on equilibrium problems, in which the phase - change temperature is dependent on the velocity of the front at which the phase-change occurs (for more physical problems, see [3-7]). Here, we study a model problem with nonlinear kinetic law at the interface in the one-dimensional case. Specifically, let the curve with s(0)=b(0<b<1) be defined as the interface that separates the liquid and solid phases. With u denoting temperature (scaled so that is vanishes at equilibrium), we may write the system of equations as

$$u_t = K_l u_{xx} \text{ in } Q_1 = \{(x,t) | 0 < x < s(t), 0 < t \le T\}, \tag{1.1}$$

$$u_t = K_s u_{xx} \text{ in } Q_1 = \{(x,t) | 0 < x < s(t), 0 < t \le T\}, \tag{1.2}$$

and on the interface $x = s(t)$ as

$$u_1 = u_2 = g_1(V(f)), \tag{1.3}$$

$$Ku_x^+ - Ku_x^- = g_2(V(t)), \tag{1.4}$$

$$s(0) = b, \; 0 < b < 1, \tag{1.5}$$

where K_l and K_s are thermal diffusivities of a liquid and a solid respectively, $L > 0$ is the latent heat and the superscripts + and – denote, respectively the right-hand and left-hand limits with respect to the special variable x. These equations are subject to the initial and boundary conditions

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$$u(x,0)=\varphi_1(x) \quad 0\le x\le b\,, \tag{1.6}$$

$$u(x,0)=\varphi_2(x) \quad b\le x\le 1\,, \tag{1.7}$$

$$u(i-1,t)= f_i(t) \quad t\ge 0, \quad (i=1,2) \tag{1.8}$$

where

$$V(t)=\frac{ds(t)}{dt} \tag{1.9}$$

is the propagation velocity of the free boundary. The free boundary problem considered here was formulated in [1], where reduction the problem to an integral equation was given. In the context of solid fuel combustion, s(t) represents the boundary between the unburnt and burnt material, and u_1, u_2, are the nondimensionalized temperature in the unburnt and burnt material respectively, (see [3-7] and references therein). The temperature at the free boundary controls its velocity $V(t)=g_1^{-1}(u_1(s(t),t))$. The heat exchange between the unburnt $(x<s(t))$ and burnt material is modeled by the boundary condition in (1.4) which, in principle, may be nonlinear.

Main Results

Theorem. Consider the problem (1.1)-(1.9). Suppose that the kinetic function and initial and boundary data satisfy the assumption $(H_1)-(H_3)$ in [1]. Then the problem (1.1)-(1.9) has not more than one solution.

To prove uniqueness for $t<\sigma$ suppose that $u_0=(u_{01},u_{02}), s_0$ is another solution of (1.1)-(1.9) for $t<\sigma$ and $v_0(t)=(v_{01}(t),v_{02}(t))^T$ is another solution of integral equations (26) and (27) in [1]. It suffices to prove uniqueness, for any $\overline{\sigma}<\sigma$.

Let

$$\overline{M}=Max\{M,\ \underset{0\le t\le\overline{\sigma}}{l.u.b}|v_0(t)|\}$$

where M introduced in section 4.2 in [1], and let be any positive number satisfying

$$\sqrt{\overline{\sigma}}<$$

$$Min\left\{\left(C_2\|\varphi_1'\|+C_3\|f_1'\|+C_4+C_5\overline{M}+C_6+C_7\right)^{-1}\frac{\overline{M}}{2},\right.$$

$$\left.\left(D_2+D_3+D_4\overline{M}+D_5+D_6\|\varphi_2'\|+D_7\|f_2'\|\right)^{-1}\frac{\overline{M}}{2}\right\}$$

where the constants C_i and D_i, i=2,3,…,7 are simple combination of $\pi, b, \frac{1}{b}, M, M', M_2, M_2', K$. Then by the same calculations in [1] which were used to prove that T maps $B_{M,\sigma}$ into itself (where T and $B_{M,\sigma}$ introduced in subsection 4.2 in [1]) and is a contraction one shows that T maps $B_{\overline{M},\overline{\sigma}}$ into itself and is a contraction. Hence, there exists at most one fixed point of T in $B_{\overline{M},\overline{\sigma}}$. It follows that $v(t)=v_0(t)$ for $0\le t\le\overline{\sigma}$, where $v(t)$ is solution of integral equations (26) and (27) in [1]. Hence also $s(t)=s_0(t)$,$u(x,t)=u_0(x,t)$ if $0\le t\le\overline{\sigma}$, $0\le x\le s(t)$ and $s(t)\le x\le 1$. We next consider the system (1.1)-(1.9) for $t>\overline{\sigma}$, i.e. (1.1)-(1.5), (1.8), (1.9) are considered for $t>\overline{\sigma}$ (instead of $t\ge 0$) where as (1.6), (1.7) are replaced by $u_1(x,\overline{\sigma})=u_1(x,\overline{\sigma})$ for $0\le x\le s(\overline{\sigma})$, $u_2(x,\overline{\sigma})=u_2(x,\overline{\sigma})$ for $s(\overline{\sigma})<x<1$.

This problem can again be transformed into integral equations (26), (27) in [1] extend to the present integral equation provided M is replaced by M_0 where

$$M_0=\underset{\sigma<t<\sigma}{l.u.b}\ |V(t)g_1(V(t))|$$

Similarly to section 4 in [1], we reduce the problem (1.1)-(1.9) for u_0, s_0 in the interval $\overline{\sigma}\le t<\sigma$ to an integral equation. Since $u_1(x,\overline{\sigma})=u_1(x,\overline{\sigma})$, $u_2(x,\overline{\sigma})=u_2(x,\overline{\sigma})$, the integral equation for $v(t)$ and $v_0(t)$ coincide. Repeating now the same argument as before we conclude that for $v(t)=v_0(t)$ for any $\tilde{\sigma}$ satisfying

$$\sqrt{(\tilde{\sigma}-\overline{\sigma})}<$$

$$Min\left\{\left(C_2\|\varphi_1'\|+C_3\|f_1'\|+C_4+C_5\overline{M_0}+C_6+C_7\right)^{-1}\frac{\overline{M_0}}{2},\right.$$

$$\left.\left(D_2+D_3+D_4\overline{M_0}+D_5+D_6\|\varphi_2'\|+D_7\|f_2'\|\right)^{-1}\frac{\overline{M_0}}{2}\right\}$$

$$(\overline{M}_0 = Max\{M_0, \underset{\overline{\sigma}\le t\le\sigma}{l.u.b}|v_0(t)|\})$$

We can now proceed in the same manner as before in [2] step by step, nothing that in each step the time interval can be taken to be $\ge\varepsilon$ where satisfies

$$\sqrt{\varepsilon} <$$

$$Min\left\{\left(C_2\|\varphi_1'\| + C_3\|f_1'\| + C_4 + C_5\overline{M_1} + C_6 + C_7\right)^{-1}\frac{\overline{M_1}}{2},\right.$$

$$\left.\left(D_2 + D_3 + D_4\overline{M_1} + D_5 + D_6\|\varphi_2'\| + D_7\|f_2'\|\right)^{-1}\frac{\overline{M_1}}{2}\right\}$$

where

$$\overline{M}_1 = Max\{\underset{\overline{\sigma}<t<\sigma}{l.u.b}\ |V(t)g_1(V(t))|, \underset{\overline{\sigma}<t<\sigma}{l.u.b}\ |v_0(t)|\}$$

Having proved existence and uniqueness for all $t<\sigma$ where σ is any positive number satisfying (36) in [1]. Let us stress that the previous proof (see (38), (39) in [1]) shows also the following:

If instead of (1.1)-(1.9) for t > 0 we consider (1.1)-(1.9) for $t>\lambda$, i.e., (1.1)-(1.5), (1.8), (1.9) hold for $t>\lambda$ and (1.6), (1.7) replaced by $u_1(x,\lambda)=u_1(x,\lambda)$ for $0<x<s(\lambda)$ and $u_2(x,\lambda)=u_2(x,\lambda)$ for $s(\lambda)<x\le 1$ respectively, and if

$$|V(\lambda)g_1(V(\lambda))|,\ s(\lambda),\ \frac{1}{s(\lambda)}$$

are bounded independently of λ, then there exists a unique solution for the problem in an interval $\lambda\le t\le\lambda+\varepsilon$, where ε is some positive number independent of λ.

Since for any solution of (1.1)-(1.9) the function s(t) is monotone non-decreasing, $\frac{1}{s(\lambda)}\le\frac{1}{b}$. To complete the proof of theorem it suffices to prove the following statement:

For every $t_0>0$ there exists an $\varepsilon>0$ such that if the system (1.1)-(1.9) has a unique solution for all $t<t_0$, then it also has a unique solution for all $t<t_0+\varepsilon$ in view of the previous remarks it suffices to show: If $u(x,t), s(t)$ is a solution of (1.1)-(1.9) for all $t<t_0$, then for all $\eta>0$ sufficiently small, the functions

$$l.u.b|V(t_0-\eta)g_1(V(t_0-\eta))|,\ s(t_0-\eta) \qquad (2.1)$$

are bounded independently of η. If we prove that

$$l.u.b|v(t)|<\infty,$$

then from (28) in [1] follows the boundedness of s(t) for $t<t_0$. Consequently, if we prove (2.2) then the proof of theorem is completed.

Proof of (2.2). We use for $v(t)$ the integral equation which corresponds to the system (1.1)-(1.9) in the interval $t_0-\mu<t<\mu$ (μ sufficiently small) in [1]. Since

$$u_1(0,t_0-\mu)=f_1(t_0-\mu), u_2(0,t_0-\mu)=f_2(t_0-\mu),$$

the equations are

$$v_1(t) = -g_1(V(t))V(t) +$$

$$2\int_0^{s(t_0-\mu)} u_{1\xi}(\xi,t_0-\mu)N(s(t),t;\xi,t_0-\mu)d\xi$$

$$+\int_{t_0-\mu}^{t} v_1(\tau)G_x(s(t),t;s(\tau),\tau)d\tau$$

$$-2\int_{t_0}^{t} f_1'(\tau)N(s(t),t;0,\tau)d\tau +$$

$$2g_1(V(t_0-\mu))N(s(t),t;s(t_0-\mu),t_0-\mu)$$

$$+2\int_{t_0-\mu}^{t} g_1'(V(\tau))N(s(t),t;s(\tau),\tau)d\tau$$

$$-2\int_{t_0-\mu}^{t} g_1(V(\tau))V(\tau)G_x(s(t)-1,t;s(\tau),\tau)d\tau$$

$$v_2(t) = -g_1(V(t))V(t)$$

$$-2g_1(V(t_0-\mu))N(s(t)-1,t;s(\tau)-1,t_0-\mu)$$

$$+\int_{t_0-\mu}^{t} g_1'(V(\tau))N(s(t)-1,t;s(\tau)-1,\tau)d\tau$$

$$-2\int_{t_0-\mu}^{t} v_2(t)G_x(s(t)-1,t;s(\tau)-1,\tau)d\tau$$

$$+2\int_{s(t_0-\mu)}^{1} u_{2\xi}(\xi,t_0-\mu)N(s(t)-1,t;\xi,t_0-\mu)d\xi$$

$$+\int_{t_0}^{t} f_2'(\tau)N(s(t)-1,t;0,\tau)d\tau$$

In section 4 in [1] we proved $v_1(t)$ and $v_2(t)$ are bounded functions, we obtain that

$$\underset{0<t<t_0}{l.u.b}\,|v(t)| < \infty$$

therefore we established theorem.

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