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A Property of the Haar Measure of Some Special LCA Groups

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Abstract

The Euclidean group $(\mathbb{R}^n,+)$ where $(n \in \mathbb{N})$, plays a key role in harmonic analysis. If we consider the Lebesgue measure $d\mu_{\mathbb{R}^n}(x)$ as the Haar measure of this group then $\frac{1}{2}d\mu_{\mathbb{R}^n}(2x) = d\mu_{\mathbb{R}^n}(x)$. In this article we look for LCA groups *K*, whose Haar measures have a similar property. In fact we will show that for some LCA groups *K* with the Haar measure μ_K, there exists a constant $C_K > 0$ such that $\mu_K(A) = C_K \mu_K(A^2)$ for every measurable subset *A* of *K*. Moreover we will characterize this constant for some special groups.

Keywords: Locally compact abelian (LCA) group; Haar measure; Dual group; Fourier transform

1. Introduction and Preliminaries

In harmonic analysis the additive group $\mathbb{R}$ is one of the most famous and important groups. It provides orientation for further development [1]. In this paper we concentrate on the property $\frac{1}{2}d\mu_{\mathbb{R}^n}(2x) = d\mu_{\mathbb{R}^n}(x)$ of the Euclidean group $(\mathbb{R}^n,+)$ and want to know that when this property holds for an LCA group *K*. On the other hand, is there a concrete relation between $d\mu_K(x^2)$ and $d\mu_K(x)$? First we review some notations.

A left Haar measure on an LCA group *K* is a nonzero Radon measure μ_K on *K* that satisfies $\mu_K(xE) = \mu_K(E)$ for every Borel set $E \subseteq K$ and every $x \in K$. The set of all continuous characters (group homomorphisms from *K* to $\mathbb{T} = \{z \in \mathbb{C} ; |z| = 1\}$) of *K* is called the dual group of *K*, when it is equipped with the topology of uniform convergence on compact subsets of G, it is a locally compact abelian group, and denoted by $\hat{K}$. We will denote by $d\omega := d\mu_{\hat{K}}(\omega)$ the Haar measure of LCA group $\hat{K}$. Also by $L^2(K)$ we mean, the set of all measurable functions $f : K \to \mathbb{C}$ such that $\int_K |f(x)|^2 \, d\mu_K(x) < \infty$. We shall continue to write the group operation as multiplication. One must note that, the group law, is addition in many of the common abelian groups, such as $\mathbb{R}$ and $\mathbb{Z}$. Also for every subset *A* of *K*, we define $A^2 = \{x^2 ; x \in A\}$. In Section 2, by a well-known theorem, we show that $d\mu_K(x^2) = \frac{1}{C_K} d\mu_K(x)$ for some $C_K > 0$. Moreover by applying Plancherel theorem (4.25 of [3]) we will prove that for some special LCA groups, C_K is a power of 2^{-1}, similar to the Euclidean group $\mathbb{R}^n$. In Section 3, we introduce two infinite compact groups and show that the multiple C_K must be 1 for these groups.

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2. Main Results

For an LCA group K with Haar measure μ_K, we observe that the mapping $T(x)=x^2$ is a continuous homomorphism from K onto $K^2:=T(x)$, a subgroup of K. This map is not open in general. For example let $K_1=(\prod_{i=1}^{\infty}\mathbb{Z}_2)$ be endowed with the product topology. Then consider $K=(\prod_{i=1}^{\infty}\mathbb{Z}_4)$ and regard K_1 as a subgroup of K, where we identify the elements of $\mathbb{Z}_2$ with the elements of $\mathbb{Z}_4$ of order ≤ 2. We endow K with the topology such that K_1 is an open and compact subgroup of K. Obviously $(K_1)^2=\{0\}$ and $K^2=K_1$. Hence T is not open. Nevertheless, a reasonably large class of continuous homomorphisms are automatically open, as Theorem 5.29 of [5] shows. If K is a σ-compact locally compact abelian group such that K^2 is closed, then T is open. Recall that, if K^2 is open, it is automatically closed. Also it is worthwhile to note that for every LCA group K with open component of identity, K^2 is open, (combine Theorems 24.25 and 24.30 of [5]). In particular, K^2 is open if K is an LCA Lie group.

Throughout this section assume that K is homeomorphic (topologically isomorphic) with its subgroup $K^2=\{x^2;\ x\in K\}$. In this situation by the uniqueness of Haar measure ([2], 3.8), there exists a positive constant C_K such that for all measurable subsets A of K we have;

$$\mu_K(A)=C_K\,\mu_K(A^2)\,. \tag{1}$$

We call C_K the expansion factor of square isomorphism $x\mapsto x^2$ on K. For example $C_{\mathbb{R}^n}=\frac{1}{2^n}$. Also we can find easily C_K for some other groups as follow.

Proposition 2.1.
(i) $C_K=1$ if K is discrete.
(ii) $C_K\ge 1$ if K has a compact subgroup with positive measure; in particular K is compact.
(iii) $C_H=C_K$ if K contains an open subgroup H.

Proof. Take $A=\{1_K\}$, then (i) follows immediately from (1). To get (ii) consider a compact subgroup L of K with positive measure, then $L^2\subseteq L$ is a compact subgroup also. Now by using (1) we have $C_K=\dfrac{\mu_K(L)}{\mu_K(L^2)}\ge 1$. (iii) is clear. □

We now turn to the characterization of C_K for all groups K. First according to the following theorem we split K to a multiplication of two subgroups such that one of them doesn't have any compact open subgroup e.g. the copies of Euclidean group $\mathbb{R}^n$, and the other one contains a compact open subgroup.

Theorem 2.2. ***(Hewitt and Ross [5])*** *Every LCA group K is topologically isomorphic with $\mathbb{R}^n\times K_0$, where n is a non-negative integer and K_0 is an LCA group containing a compact open subgroup. If K is also topologically isomorphic with $\mathbb{R}^n\times K_1$ and LCA group K_1 contains a compact open subgroup, then $m=n$.*

The nonnegative integer n using in this theorem is called the covering dimension of K as a topological space ([7], chapter 3). It should be mentioned that the covering dimension of $\mathbb{R}^n$ and any n-dimensional Manifold is n. By combining the above theorem and the fact that group topological isomorphism carries the Haar measure ([2], 3.8), we obtain C_K as following:

Theorem 2.3. *Let K be an LCA group, topologically isomorphic with $\mathbb{R}^n\times K_0$, where n is a non-negative integer and K_0 an LCA group containing a compact open subgroup L. Then $C_K=\frac{1}{2^n}.C_L$.*

Proof. Let $\mu_{\mathbb{R}^n}$ and μ_{K_0} be the Haar measures of $\mathbb{R}^n$ and K_0, respectively, and T be the topological isomorphism between $\mathbb{R}^n\times K_0$ and K. Then by 3.8 of [2] there exists a constant $C>0$ such that

$$(\mu_{\mathbb{R}^n}\times\mu_{K_0})(E)=C\,.\,\mu_K(TE)$$

for all measurable subsets E of $\mathbb{R}^n\times L$. Put $E=\prod_{i=1}^{n}[a_i,b_i]\times L$. Then we have:

$$\begin{aligned}C\,.\mu_K((TE)^2&=C\,.\mu_K(TE^2)\\&=(\mu_{\mathbb{R}^n}\times\mu_{K_0})(E^2)\\&=\mu_{\mathbb{R}^n}(\prod_{i=1}^{n}[2a_i,2b_i]).\mu_{K_0}(L^2)\\&=2^n\frac{1}{C_L}C\,.\mu_K(TE)\end{aligned}$$

The last equality follows from (1) and Proposition 2.1(iii). Hence $C_K=(\frac{1}{2}^n)\,C_L$. □

Let L be a compact group and $\hat{L}$ its dual group which according to ([3], 4.4) it is discrete. Assume that $C_{\hat{L}}$ is well-defined. Then we are going to show that $C_L = C_{\hat{L}} = 1$. First we consider a more general case.

Theorem 2.4. *Let K be an LCA group and $\hat{K}$ be its dual group. Also let the mapping $T : a \to a^2$ on both K and $\hat{K}$ be topological isomorphism. Then $C_K = C_{\hat{K}}$.*

Proof. Recall that $f \circ T \in L^2(K)$ for all $f \in L^2(K)$. Also a straightforward calculation shows that $(f \circ T)\hat{}(\omega^2) = C_K \hat{f}(\omega)$. Now by using Plancherel theorem we have

$$\|f\|_2^2 = \frac{1}{C_K}\int_K \left| f\left(x^2\right)\right|^2 d\mu_K(x)$$

$$= \frac{1}{C_K C_{\hat{K}}}\int_{\hat{K}} \left|(f \circ T)^{\wedge}\left(\omega^2\right)\right|^2 d\omega$$

$$= \frac{(C_K)^2}{C_K C_{\hat{K}}}\int_{\hat{K}} \left|\hat{f}(\omega)\right|^2 d\omega$$

$$= \frac{C_K}{C_{\hat{K}}}\|f\|_2^2 . \qquad \square$$

Now we summarize our results as a direct consequence from previous theorems in the following

Corollary 2.5. Let K be an LCA group such that C_K and $C_{\hat{K}}$ exist. Then $C_K = C_{\hat{K}} = 1$ if K is discrete or has a compact open subgroup, otherwise $C_K = \frac{1}{2^n}$ for some $n \in \mathbb{N} \cup \{0\}$.

3. Examples

Theorem 2.3 shows that if we can find C_L for compact groups L then C_K will be determined for all LCA groups K. But in the following examples it will be shown that $C_L = 1$ for those infinite compact groups. This leads us to the following conjecture, "$C_L = 1$ for every compact group L."

Example 3.1. The Group of p-Adic Integers

Fix a prime number $p > 2$ in $\mathbb{Z}$. Then the p-adic norm on $\mathbb{Q}$ is given by

$$|a|_p = \begin{cases} 0 & \text{if } a = 0 \\ p^{-k} & \text{if } a \neq 0, \text{ where } k \in \mathbb{Z} \text{ and } a = p^k \frac{n}{m} \\ & \text{with } n, m \in \mathbb{Z} \text{ prime to p.} \end{cases}$$

The group of p-adic numbers $\mathbb{Q}_p$ is the completion of additive group $\mathbb{Q}$ with respect to the norm $|.|_p$. This is a locally compact, totally disconnected group. The closure of $\mathbb{Z}$ in $\mathbb{Q}_p$, denoted by $\mathbb{O}_p$, is called the group of p-adic integers. In [6] chapter 12, it is shown that $\mathbb{O}_p$ is a compact group and $\mathbb{O}_p = \{a \in \mathbb{Q}_p ;\ |a|_p \le 1\}$. To find the Haar measure of $\mathbb{O}_p$, first state that there is a standard model for p-adic integers as following ([6], 12.3.6);

$$a = \sum_{n=0}^{\infty} \tilde{a}(n) p^n \tag{2}$$

with $\tilde{a}(n) \in \{0,1,2,\ldots,p-1\}$ for all $n \in \mathbb{N} \cup \{0\}$. Let a,b be in $\mathbb{O}_p$ and take $c = a + b$ where $\tilde{C}(n)$ for $n \ge 0$ is given by

$$\tilde{c}(n) + t_n p = \tilde{a}(n) + \tilde{b}(n) + t_{n-1} \quad \text{for } t_{-1} \text{ and } t_n \in \{0,1\}. \tag{3}$$

The reader can easily check that this agrees with addition in $\mathbb{N} \cup \{0\}$ under (2). We naturally normalize Haar measure λ on $\mathbb{O}_p$ so that $\lambda(\mathbb{O}_p) = 1$. By translation invariance, the set

$$\left\{a \in O_p ;\ \tilde{a}(0) \in S \subseteq \{0,1,2,\ldots,p-1\}\right\},$$

must have Haar measure $\frac{card(S)}{p}$. So if $A = \{a \in \mathbb{O}_p ;\ \tilde{a}(0) = 0\}$ then $\lambda(A) = \frac{1}{p}$ and it is easy to see that $p > 2$ implies that $2A = A^2 = A$. Hence by (1) we have $C_{(\mathbb{O}_p)} = 1$.

Example 3.2. $(\prod_1^{\infty})\mathbb{Z}_p)$

Assume that $p \ge 3$ is a prime number and let $\mathbb{Z}_p$ be the set of all integers mod p. By $K := (\prod_1^{\infty})\mathbb{Z}_p)$ we mean the product of an infinite sequence of copies of $\mathbb{Z}_p$. Its Haar measure on each factor assigns measure $\frac{1}{p}$

to each of the p point $0,1,2,\ldots,p-1$. The elements of the compact group K are sequences $(a_1, a_2, \ldots)$ where each $a_j \in \mathrm{Z}_p$. Consider the map $\phi : K \to [0,1]$ that assigns to such a sequence the real number $\sum_{j=1}^{\infty} a_j p^{-j}$. This map is a group homomorphism. However ϕ is measurable and maps Haar measure μ on K to the Lebesgue measure λ on [0,1]; in fact $\mu(E) = \lambda(\phi(E))$ for all Borel subset E of K. In particular $\mu(K) = \lambda([0,1]) = 1$. On the other hand it is obvious that the mapping $x \to x^2$ is one to one on K. Furthermore, $K^2 = K$ and hence again $C_K = 1$ as a consequence of (1).

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