Journal of Sciences, Islamic Republic of Iran 17(3): 265-272 (2006) http://jsciences.ut.ac.ir
University of Tehran, ISSN 1016-1104

Relationships between Darboux Integrability and Limit Cycles for a Class of Able Equations

H. Shariati[1] and H.M. Mohammadi Nejad[2,*]

[1]*Department of Mathematics, Faculty of Sciences, Sistan and Baluchestan University, Zahedan, Islamic Republic of Iran*
[2]*Department of Mathematics, University of Birjand, Birjand, Islamic Republic of Iran*

Abstract

We consider the class of polynomial differential equation $\dot{x} = P_n(x,y)+P_{n+m}(x,y)+P_{n+2m}(x,y)$, $\dot{y} = Q_n(x,y)+Q_{n+m}(x,y)+Q_{n+2m}(x,y)$. For $m,n \geq 1$ where P_i and Q_i are homogeneous polynomials of degree i. Inside this class of polynomial differential equation we consider a subclass of Darboux integrable systems. Moreover, under additional conditions we proved such Darboux integrable systems can have at most 1 limit cycle.

Keywords: Limit cycles; Darboux integrable; Homogeneous polynomial; Abel equations; Bernoulli equation

1. Introduction

In 1878 Darboux showed how the first integrals of planar polynomial vector fields can be constructed possessing sufficient invariant algebraic curves. He proved that if a polynomial system of degree m has at least $m(m+1)/2$ invariant algebraic curves, then it has either a first integral or an integrating factor of the form $\prod_{i=1}^{q} f_i^{\lambda_i}(x,y)$, for suitable $\lambda_i \in C$ not all zero and where $f_i(x,y)=0$ are algebraic invariant curves of system. The above function is called either a Darboux first integral or a Darboux integrating factor. Jouanolou in 1979 showed that if the number of invariant algebraic curves of a planar polynomial vector field of degree m is at least $[m(m+1)/2]+2$, then the vector field has a rational first integral, and consequently all its solutions are invariant algebraic curves. Cozma and Suba proved that a weak focus of a polynomial system of degree $m \geq 3$ having the first Liapunov constant zero and $m(m+1)/2-2$ algebraic invariant curves has a Darboux first integral or a Darboux integrating factor. Probably the three main open problems in the qualitative theory differential systems in R^2 are the determination of the number of the limit cycles and their distribution in the plane; the distinction between a center and a focus, called the center problem (see for instance [21]); and the determination of their first integrals. (see for instance [4]). Limit cycles of planar vector fields were defined by Poincaré [22], and started to be studied intensively at the end of 1920s by Van der Pol [23], Liénard [18] and Andronov [1]. A ***limit cycle*** is a periodic orbit of the planar differential system isolated in the set of all periodic orbits. One of the classical ways to produce limit cycles is perturbing a system which has a center. In such a way that limit cycles bifurcate in the perturbed system from some of

2000 Mathematics Subject Classification. Primary 34C05; secondary 34C07.
**E-mail: hmohmmadin@yahoo.com*

the periodic orbits of the original system, see for instance Pontrjagin [24].

2. Statement and Preliminary Results

We say that a function $R \in C^k(U)$ with $k \geq 1$, not identically null in U, is an, ***Integrating factor*** of system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ in U if $\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}$. In this case a first integral $H(x,y)$ is given by this integrating factor R,

$$H(x,y) = \int R(x,y)P(x,y)dy + f(x)$$

where $\frac{\partial H}{\partial x} = -RQ$. Let U be an open set of *R*, we say the function $V \in C^k(U)$ with $k \geq 1$, not identically null in U, is an ***inverse integrating factor*** of system in U if it satisfies the following linear equation in partial derivatives:

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial Y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V \ .$$

We notice that this function V is a particular solution of system

$$\dot{x} = P(x,y),\ \dot{y} = Q(x,y). \qquad (2\text{-}1)$$

The expression $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is called ***divergence*** of system (2-1). This function V is very important because $R = \frac{1}{V}$ defines in $U \setminus \{V = 0\}$ an integrating factor of system (2-1) which let us determine a first integral for system (2-1) in $U \setminus \{V = 0\}$. In [16] it is proved that function V must be null over all the limit cycles contained in U.

Darboux showed that a polynomial system of degree m with at least $\frac{m(m+1)}{2}+1$ invariant algebraic curves has a first integral which can be expressed by means of these algebraic curves, Darboux idea consists on looking for a first integral of system (2-1) with the form $H(x,y) = \prod_{i=1}^{q} f_i^{\lambda_i}(x,y)$ where $\lambda_i \in C$, $f_i(x,y) = 0$ are invariant algebraic curve of system (2-1). The former first integral is called a ***Darboux first integral***, in general, a Darboux first integral is a weak first integral. Jouanoulou [17] showed in 1976 that if the number of algebraic solutions for a polynomial system of degree m is at least $\frac{m(m+1)}{2}+2$, then the system has a rational first integral and all the solutions of system are algebraic. Prelle and Singer [25] showed in 1983 that if a polynomial system has an elementary first integral then this integral can be computed using the algebraic solutions of the system; in particular they showed that this polynomial system admits an integrating factor which is a rational function with coefficients in C. Later on, Singer [26] in 1992 showed that if a polynomial system has a Liovillian first integral then the system has an integrating factor of the form:

$$R(x,y) = \exp\left(\int_{(x_0,y_0)}^{(x,y)} U(x,y)dx + V(x,y)dy\right)$$

where U and V are rational functions which verify $\frac{\partial U}{y} = \frac{\partial V}{\partial x}$. We called ***generalized Darboux functions*** to the functions of the form $H = \mathrm{e}^g \prod f_i^{\lambda_i}$ where g is a rational function, f_i are polynomials and $\lambda_i \in C$. We say system (2-1) is ***Darboux integrable*** if the system has a first integral or an integrating factor which is a Darboux function.

Llibre in [11] shows that the problem of finding first integrals or integrating factors is reduced to a question of linear algebra on the set of cofactors. We introduce the following concepts introduced in [1]. Let *X* be a vector field of degree d, and $S \subset C^2$ a finite set of points. The restricted cofactor space with respect to *S*, $\sum_S$, is defined by $\Sigma_S = \bigcap_{p \in S} m_p \cap C_{d-1}[x,y]$, where m_p is the maximal ideal of $C[x,y]$ corresponding to the point *p*. If *S* consists of *q* points, then we say that they are independent with respect to $C_{d-1}[x,y]$ if

$$\sigma = \dim \Sigma_S = \dim C_{d-1}[x,y] - q = 1/2(d+1)(d+2) - q$$

With this notation in [1] prove the following result.

Proposition. Let X be a vector field of degree d. Assume that X has r distinct invariant algebraic curves $f_i = 0$, $i = 1,2,...,r$ (all irreducible and reduced) of multiplicity m_i, and let $N = \sum_i m_i$. Suppose, furthermore, that there are q critical points p_1, p_2, …, p_q which are independent with respect to

$C_{d-1}[x,y]$, and $f_j(p_k) \neq 0$ for $j = 1,...,q$ and $k = 1,...,r$ then the following statements hold.

i) If $N \geq \sigma + 2$, then X has a rational first integral.

ii) If $N \geq \sigma + 1$, then X has a Darboux first integral.

iii) If $N \geq \sigma$, and $div(X)$ vanishes at the p_i, then X has either a Darboux first integral or a Darboux integrating factor.

3. The Main Results

N.G. Lloyd in the studies limit cycles consider systems with homogeneous nonlinearities the forms

$$\dot{x} = \lambda x + y + P_n(x,y), \quad \dot{y} = -x + \lambda y + Q_n(x,y) \tag{3-1}$$

where P_n and Q_n are homogeneous polynomials of degree n. In the polar form (3-1) is:

$$\dot{r} = \lambda r + f(\theta) r^n, \ \dot{\theta} = -1 + g(\theta) r^{n-1}$$

where f and g are homogeneous polynomials of degree $n+1$ in $\cos\theta$ and $\sin\theta$. Now let $\rho = r^{n-1}(1 - r^{n-1} g(\theta))^{-1}$, a little calculation show that ρ satisfies the first order non autonomous equations:

$$\frac{d\rho}{d\theta} = A(\theta)\rho^3 - B(\theta)\rho^2 - \lambda(n-1)\rho. \tag{3-2}$$

where $A(\theta)$ and $B(\theta)$ are homogeneous polynomials in $\cos\theta$ and $\sin\theta$ of degree $2(n+1)$ and $n+1$, respectively. In the case $n = 2$ this transformation was introduced by Lins Neto (see [19]). The connection between (3-1) and (3-2) was explained in [21], where it was used to calculate the focal values for (3-1). In the [5] Chengzhi, Li and Weigu considered system (3-1) as $\lambda = 0$, *i.e.*

$$\dot{x} = -y + P_n(x,y), \ \dot{y} = x + Q_n(x,y) \tag{3-3}$$

where P_n and Q_n are homogeneous polynomials of degree n. Inside this class they consider a new subclass that having a center at the origin, and give a new method to study the limit cycles which bifurcate from their periodic orbits when they perturb this subclass inside the class of all systems (3-3), see [5]. In the [12], Jaume Giné and Jaume Llibre considered the class of polynomial differential equations

$$\dot{x} = P_n(x,y) + P_{n+1}(x,y) + P_{n+2}(x,y), \quad \dot{y} = Q_n(x,y) + Q_{n+1}(x,y) + Q_{n+2}(x,y) \tag{3-4}$$

where P_i and Q_i are homogeneous polynomials of degree i. Inside this class, consider a new subclass of Darboux integrable systems, such that some of them having a degenerate center, *i.e.*, a center with linear part identically zero. In this paper we consider systems of the form:

$$\dot{x} = P_n(x,y) + P_{n+m}(x,y) + P_{n+2m}(x,y), \quad \dot{y} = Q_n(x,y) + Q_{n+m}(x,y) + Q_{n+2m}(x,y) \tag{3-5}$$

where P_i and Q_i are homogeneous polynomials of degree i. Inside this class, we consider new subclasses of Darboux integrable systems, and some of them having a degenerate center, *i.e.*, a center with linear part identically zero. In the polar coordinates (r,θ), defined by $x = r\cos\theta$, $y = r\sin\theta$ system (3-5) becomes

$$\begin{aligned} \dot{r} &= f_{n+1}(\theta) r^n + f_{n+m+1}(\theta) r^{n+m} + f_{n+2m+1}(\theta) r^{n+2m} \\ \dot{\theta} &= g_{n+1}(\theta) r^{n-1} + g_{n+m+1}(\theta) r^{n+m-1} + g_{n+2m+1}(\theta) r^{n+2m-1} \end{aligned} \tag{3-6}$$

where

$$f_i(\theta) = \cos\theta \ \mathrm{P}_{i-1}(\cos\theta, \sin\theta) + \sin\theta \ \mathrm{Q}_{i-1}(\cos\theta, \sin\theta)$$

$$g_i(\theta) = \cos\theta \ \mathrm{Q}_{i-1}(\cos\theta, \sin\theta) - \sin\theta \ \mathrm{P}_{i-1}(\cos\theta, \sin\theta)$$

where f_i and g_i are homogeneous trigonometric polynomials in the variable $\cos\theta$ and $\sin\theta$ having degree in the set $\{i, i-2, i-4, ...\} \bigcap N$, where N is the set of non-negative integers. So it is possible that $f_i(\theta)$ can be of the form $(\cos^2\theta + \sin^2\theta)^s \overline{f}_{i-2s}$ with $\overline{f}_{i-2s}$ a trigonometric polynomial of degree $i - 2s \geq 0$. A similar situation occurs for $g_i(\theta)$.

If suppose that $g_{n+m+1}(\theta) = g_{n+2m+1}(\theta) = 0$ and $g_{n+1}(\theta)$ either $\succ 0$ or $\prec 0$ for all θ, and do the change $r = \rho^m$, then system (3-6) becomes the Abel differential equation

$$\frac{d\rho}{d\theta} = \frac{m}{g_{n+1}(\theta)} \left[f_{n+1}(\theta)\rho + f_{n+m+1}(\theta)\rho^2 + f_{n+2m+1}(\theta)\rho^3 \right] \tag{3-7}$$

These kind of differential equations appeared in the studies of Abel on theory of elliptic functions. For more details on Abel differential equations, see [7] or [9].

We say that all systems (3-5) with $g_{n+m+1}(\theta) = g_{n+2m+1}(\theta) = 0$ and $g_{n+1}(\theta)$ either $\succ 0$ or $\prec 0$ for all θ define the **class** F if for some $a \in \mathrm{R}$

$$g_{n+1}(\theta)\left(f'_{n+2m+1}(\theta) f_{n+m+1}(\theta) - f_{n+2m+1}(\theta) f'_{n+m+1}(\theta)\right) = a f^{3}_{n+m+1}(\theta) - f_{n+1}(\theta) f_{n+m+1}(\theta) f_{n+2m+1}(\theta)$$

where $' = \dfrac{d}{d\theta}$.

Since $g_{n+1}(\theta)$ either $\succ 0$ or $\prec 0$ for all θ, It follows that the polynomial differential systems (3-5) in the class F must satisfy that $n+1$ is even.

We shall prove that all polynomial differential systems (3-5) in the class F are Darboux integrable. Using similar techniques in [10] for finding a new Darboux integrable systems.

Theorem 1. *For polynomial differential systems (3-5) in the class F the following statements hold.*

(a) *If $f_{n+1}(\theta) f_{n+m+1}(\theta) f_{n+2m+1}(\theta)$ is not identically, then the system is Darboux integrable with the first integral $\tilde{H}(x,y) = H(\rho,\theta)$ obtained from*

$$\frac{\rho \exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right) \exp\left[-\frac{1}{\sqrt{4a-1}} \arctan\left[\frac{(1+2\rho f_{n+2m+1}(\theta)/f_{n+m+1}(\theta))}{\sqrt{4a-1}}\right]\right]}{\sqrt{\rho^2 f^2_{n+2m+1}(\theta)/f^2_{n+m+1}(\theta) + \rho f_{n+2m+1}(\theta)/f_{n+m+1}(\theta) + a}},$$

$$\text{if } a \succ \frac{1}{4}$$

$$\frac{\rho \exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right) \exp\left(\frac{1}{1+2\rho f_{n+2m+1}(\theta)/f_{n+m+1}(\theta)}\right)}{1+2\rho f_{n+2m+1}(\theta)/f_{n+m+1}(\theta)},$$

$$\text{if } a = \frac{1}{4}$$

$$\frac{\rho \exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right) \left(\sqrt{1-4a} + 1 + \frac{2\rho f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right)^{\frac{1}{2}\left(-1+\frac{1}{\sqrt{1-4a}}\right)}}{\left(\sqrt{1-4a} - 1 - \frac{2\rho f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right)^{\frac{1}{2}\left(1+\frac{1}{\sqrt{1-4a}}\right)}},$$

$$\text{if } a \neq 0 \prec \frac{1}{4}$$

$$\frac{\rho \exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right) f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}, \text{ if } a = 0$$

Through the change of variables $x = r\cos\theta$, $y = r\sin\theta$ with $\rho = r^m$.

(b) *If $f_{n+1}(\theta)$ is not identically zero, $a = 0$ and $f_{n+m+1}(\theta) f_{n+2m+1}(\theta)$ is identically zero, then the system is Darboux integrable with the first integral $\tilde{H}(x,y) = H(r,\theta)$ obtained from*

$$\frac{\exp\left(\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right)}{\rho} + \int \frac{\exp\left(\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right) f_{n+m+1}(\theta)}{g_{n+1}(\theta)} d\theta$$

$$\text{if } f_{n+2m+1}(\theta) = 0$$

$$\frac{\exp\left(2\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right)}{\rho^2} + 2\int \frac{\exp\left(2\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right) f_{n+2m+1}(\theta)}{g_{n+1}(\theta)} d\theta$$

$$\text{if } f_{n+m+1}(\theta) = 0$$

Through the change of variables $x = r\cos\theta$, $y = r\sin\theta$ and $r = \rho^m$.

Proof of Theorem 1(a). We do the change of variable $(\rho,\theta) \to (\varphi,\xi)$ defined by $\rho = u(\theta)\varphi(\xi)$ where $u(\theta) = \exp\left(\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta\right)$ and $\xi = \int \frac{u(\theta) f_{n+m+1}(\theta)}{g_{n+1}(\theta)} d\theta$. This transformation writes the Abel differential equation (3-7) into the form

$$\varphi'(\xi) = g(\xi)[\varphi(\xi)]^3 + [\varphi(\xi)]^2 \qquad (3\text{-}8)$$

where $g(\xi) = \dfrac{u(\theta) f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}$ and $' = \dfrac{d}{d\xi}$. If we do change $\xi \to t$ into the independent variable defined by $\xi' = \dfrac{-1}{t\,\varphi(\xi)}$ where now $' = \dfrac{d}{dt}$, Equation (3-8) takes the form

$$t^2 \xi''(t) + g(\xi(t)) = 0 . \qquad (3\text{-}9)$$

Note that a polynomial differential systems (3-5) with $g_{n+m+1}(\theta) = g_{n+2m+1}(\theta) = 0$ and $g_{n+1}(\theta)$ either $\succ 0$ or $\prec 0$ for all θ define the class F if and only if for some $a \in \mathrm{R}$ we have

$$g_{n+1}(\theta)\left(f'_{n+2m+1}(\theta)f_{n+m+1}(\theta)-f_{n+2m+1}(\theta)f'_{n+m+1}(\theta)\right)=$$
$$af^{3}_{n+m+1}(\theta)-f_{n+1}(\theta)f_{n+m+1}(\theta)f_{n+2m+1}(\theta) \qquad (*)$$

or equivalent for some $a\in \mathrm{R}$ we get

$$\frac{d}{dt}\left(\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right)+\frac{f_{n+1}(\theta)f_{n+2m+1}(\theta)}{g_{n+1}(\theta)f_{n+m+1}(\theta)} =a\frac{f_{n+m+1}(\theta)}{g_{n+1}(\theta)} \qquad (3\text{-}10)$$

Now if for some $a\in \mathrm{R}$ we have $g(\xi)=a\xi$ then of defines $g(\xi)$, ξ we have $u(\theta)\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}=a\int\frac{u(\theta)f_{n+m+1}(\theta)}{g_{n+1}(\theta)}d\theta$, and if derivating with respect to θ we get

$$u(\theta)\frac{d}{d\theta}\left(\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right)=au(\theta)\frac{f_{n+m+1}(\theta)}{g_{n+1}(\theta)} -u'(\theta)\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}$$

Since $u'(\theta)=\frac{u(\theta)f_{n+1}(\theta)}{g_{n+1}(\theta)}$, so we obtain:

$$\frac{d}{d\theta}\left(\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right)=a\frac{f_{n+m+1}(\theta)}{g_{n+1}(\theta)}-\frac{f_{n+1}(\theta)f_{n+2m+1}(\theta)}{g_{n+1}(\theta)f_{n+m+1}(\theta)}$$

which is equivalent to condition (*). So a polynomial differential system (3-5) such that $f_{n+1}(\theta)$ $f_{n+m+1}(\theta)f_{n+2m+1}(\theta)\neq 0$ in the class F if and only if for some $a\in \mathrm{R}$ we have $g(\xi)=a\xi$. If in the equation (3-9) we set $g(\xi)=a\xi$ then equation (3-9) is an Euler equation, therefore if we doing the change $t=\exp\tau$ in the independent variable equation (3-9) becomes the linear ordinary differential equation with constant coefficients

$$\xi''(\tau)-\xi'(\tau)+a\xi(\tau)=0 \qquad (3\text{-}11)$$

where here $'=\frac{d}{d\tau}$. Equation (3-11) has the characteristic equation $\lambda^2-\lambda+a=0$ and are the two roots of characteristic equation $\lambda_{1,2}=\frac{1\pm\sqrt{1-4a}}{2}$, so if $a=1/4$ its general solution is $\xi(\tau)=c_1\exp(\frac{\tau}{2})$ $+c_2\exp(\frac{\tau}{2})$, and if $a\neq 1/4$ its general solution is $\xi(\tau)=c_1\exp(\lambda_1\tau)+c_2\exp(\lambda_2\tau)$ where λ_1 and λ_2 are the two roots of the characteristic equation. Going back to the independent variable $t=\exp(\tau)$ the solution of the Euler differential equation is $\xi(t)=$ $c_1\sqrt{t}+c_2\sqrt{t}\ln(t)$ if $a=1/4$, and $\xi(t)=c_1t^{\lambda_1}+c_2t^{\lambda_2}$ if $a\neq 1/4$. Finally, going back to the variable (ρ,θ) with $\rho=r^m$ and taking into account if the roots and λ_2 are real or complex, after some tedious computations we obtain the first integrals of statement (a) according with the values of a.

Now we prove that systems of statement (a) are Darboux integrable. It is easy to check that for systems (1-5) in the class F with $f_{n+1}(\theta)f_{n+m+1}(\theta)f_{n+2m+1}(\theta)$ not identically zero $V(\rho,\theta)=\rho(\rho^2f^{2}_{n+2m+1}(\theta)/$ $f^{2}_{n+m+1}(\theta)$ + $\rho f_{n+2m+1}(\theta)/f_{n+m+1}(\theta)+a)$ with $\rho=r^m$ is an inverse integrating factor for its Abel differential equation (3-7). As this inverse integrating factor $V(\rho,\theta)$ is an elementary function in Cartesian coordinates, and so systems (3-5) in the class F with $f_{n+1}(\theta)f_{n+m+1}(\theta)f_{n+2m+1}(\theta)$ not identically zero have a Liouvillian first integral according with results of singer, see [24], and this completes the proof of (a).●

Proof of Theorem 1(b). If $f_{n+2m+1}(\theta)$ is identically zero or $f_{n+m+1}(\theta)$ is identically zero, the Abel differential equation (3-7) becomes the Bernoulli differential equation

$$\frac{d\rho}{d\theta}=\rho^2f_{n+m+1}(\theta)/g_{n+1}(\theta)+\rho f_{n+1}(\theta)/g_{n+1}(\theta)$$

or

$$\frac{d\rho}{d\theta}=\rho^3f_{n+2m+1}(\theta)/g_{n+1}(\theta)+\rho f_{n+1}(\theta)/g_{n+1}(\theta)$$

respectively. Solving these Bernoulli equations we obtain the first integrals of statement (b).●

Question. *Is it possible to find other integrable subclasses from the well-known integrable cases of the Abel differential equation*? For answering this question if in the Abel differential equation (3-7) we do the change of variables $(\rho,\theta)\to(\eta,\xi)$ defined by $\rho=u(\theta)\eta(\xi)-\frac{f_{n+m+1}(\theta)}{3f_{n+2m+1}(\theta)}$ where

$$u(\theta)=\exp\left(\int\left[\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}-\frac{f^2_{n+m+1}(\theta)}{3f_{n+2m+1}(\theta)g_{n+1}(\theta)}\right]d\theta\right)$$

and

$$\xi=\int\frac{u^2(\theta)f_{n+2m+1}(\theta)}{g_{n+1}(\theta)}d\theta .$$

This transformation writes the Abel equation (3-7) into form

$$\varphi'(\xi)=[\varphi(\xi)]^3+\eta(\theta) \qquad (3\text{-}12)$$

where

$$\eta(\theta)=\frac{g_{n+1}(\theta)}{f_{n+2m+1}(\theta)u^2(\theta)}\left[\frac{d}{d\theta}\left(\frac{f_{n+m+1}(\theta)}{3f_{n+2m+1}(\theta)}\right)-\frac{f_{n+1}(\theta)f_{n+m+1}(\theta)}{3f_{n+2m+1}(\theta)g_{n+1}(\theta)}+\frac{2f^3_{n+m+1}(\theta)}{27f^2_{n+2m+1}(\theta)g_{n+1}(\theta)}\right].$$

From the definition of $u(\theta)$ we have

$$\ln|u(\theta)|=\int\left[\left[\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}-\frac{f^2_{n+m+1}(\theta)}{3f_{n+2m+1}(\theta)g_{n+1}(\theta)}\right]d\theta=\right. \qquad (3\text{-}13)$$

$$\int\frac{f_{n+m+1}(\theta)}{f_{n+2m+1}(\theta)}\left[\frac{f_{n+1}(\theta)f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)g_{n+1}(\theta)}-\frac{f_{n+m+1}(\theta)}{3g_{n+1}(\theta)}\right]d\theta.$$

If $a\neq 0$, using (3-10) in (3-13) we obtain

$$\frac{-1}{3a}\int\frac{\frac{d}{d\theta}\left(\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right)}{\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}}d\theta+(1-\frac{1}{3a})\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta=$$

$$-\frac{1}{3a}\ln\left|\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right|+(1-\frac{1}{3a})\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta.$$

Using this result we get

$$u(\theta)=\left|\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right|^{\frac{-1}{3a}}\exp\left[\left(1-\frac{1}{3a}\right)\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right]$$

and therefore

$$\eta(\theta)=\left[\frac{2-9a}{27}\right]\left(\frac{f_{n+2m+1}(\theta)}{f_{n+m+1}(\theta)}\right)^{(1-3a)/a}\exp\left[\frac{1-3a}{a}\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right]$$

for $a=2/9$ and for $a=1/3$ we have $\eta(\theta)=0$ and $\eta(\theta)=-1/27$, respectively. For these cases, the differential equation (3-12) is of separable variables and we can obtain the associated first integrals. But $\eta(\theta)=0$ and $\eta(\theta)=-1/27$ imply that equality (*) holds with $a=2/9$ and for $a=1/3$ that we obtain cases already studied.

Theorem 2. *For a polynomial differential system (3-5) in the class* F *the following statements hold.*

(a) *If* $f_{n+1}(\theta)f_{n+m+1}(\theta)f_{n+2m+1}(\theta)$ *is not identically zero, then in the domain of definition of the inverse integrating factor*

$$V(\rho,\theta)=\rho(\rho^2f^2_{n+2m+1}(\theta)/f^2_{n+m+1}(\theta)+\rho f_{n+2m+1}(\theta)/f_{n+m+1}(\theta)+a)$$

System (3-5) has no limit cycles.

(b) *If* $f_{n+1}(\theta)f_{n+2m+1}(\theta)$ *is not identically zero,* $a=0$ *and* $f_{n+m+1}(\theta)$ *is identically zero, then the maximum number of its limit cycles contained in the domain of definition of the inverse integrating factor*

$$V(\rho,\theta)=\rho/2+\rho^3\exp\left(-2\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)\int\frac{\exp\left(2\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)}{g_{n+1}(\theta)}d\theta$$

is one.

(c) *If* $f_{n+1}(\theta)f_{n+m+1}(\theta)$ *is not identically zero,* $a=0$ *and* $f_{n+2m+1}(\theta)$ *is identically zero, then in the domain of definition of the inverse integrating factor*

$$V(\rho,\theta)=\rho+\rho^2\exp\left(-\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)\int\frac{\exp\left(\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)f_{n+m+1}(\theta)}{g_{n+1}(\theta)}d\theta$$

System (3-5) has no limit cycles.

We note that, a ***limit cycle*** of system (3-5) is a periodic orbit isolated in the set of periodic orbits of system (3-5). In order to study the existence and non-existence of the limit cycles of system (3-5) we shall use the following result.

Theorem 3. *Let (P, Q) be a C^1 vector field defined in the open subset U of R^2. Let $V = V(x, y)$ be an inverse integrating factor of vector field (P, Q),* i.e. *a C^1 solution of the linear partial differential equation $P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V$ defined in U. If γ is a limit cycle of (P, Q) in the domain of definition U, then γ is contained in $\{(x, y) \in U : V(x, y) = 0\}$.*

Proof. See Theorem 3 of [12] or [8].

We recall that if $V \in C^k(U)$ with $k \geq 1$ not identically null in U, is a inverse integrating factor of vector field (P, Q) in U then function $R = 1/V$ defines in $U \setminus \{V = 0\}$ is an integrating factor of system which lets us determine a first integral for vector field in $U \setminus \{V = 0\}$. In [15] it is proved that function V must be null over all the limit cycles contained in U.

Proof of Theorem 2(a). For systems (3-5) in the class F with $f_{n+1}(\theta) f_{n+m+1}(\theta) f_{n+2m+1}(\theta)$ not identically zero, it is easy to check that an inverse integrating factor of its associated Abel differential equation (3-7) is given by

$$V(\rho, \theta) = \rho(\rho^2 f^2_{n+2m+1}(\theta) / f^2_{n+m+1}(\theta) + \rho f_{n+2m+1}(\theta) / f_{n+m+1}(\theta) + a).$$

By Theorem 3, If system (3-5) and consequently its associated Abel equation (3-7) have limit cycles, then must limit cycles contained in the set $\{V(\rho, \theta) = 0\}$. From expression of the inverse integrating factor, the unique possible limit cycles must be given by

$$\rho(\theta) = \begin{cases} (-1 \pm \sqrt{1-4a}) f_{n+m+1}(\theta) / 2f_{n+2m+1}(\theta), & \text{if } a \prec 1/4 \\ -f_{n+m+1}(\theta) / 2f_{n+2m+1}(\theta), & \text{if } a = 1/4 \end{cases}$$

with $\rho = r^m$. Since $n+1$ is even, the function $f_{n+m+1}(\theta)$ has zeroes (for some $m = odd$) therefore the above expressions of $\rho(\theta)$ cannot be positive for all θ. consequently, there are no limit cycles in the domain of definition of V.

Proof of Theorem 2(b). The case (b) and (c) of Theorem 2 are identical with the cases (b) and (c) of Theorem 2 of [12].

Systems (3-5) with $n = 3$ and $m = 1$ satisfying $g_5(\theta) = g_6(\theta) = 0$ inside class F linearly zero singular point at the origin. The following corollary provides some quintic polynomial systems which belong to the class F, see [12].

Corollary 4. *Systems (3-5) with $n = 3$ and $m = 1$ satisfying $g_5(\theta) = g_6(\theta) = 0$ belong to the class F if one of the following statements holds.*

(a) $\alpha = \beta = \gamma = \delta = 0$

(b) $b_{30} = b_{12} = b_{03} = 0$, $b_{21} = a\beta^2 / C$, $\alpha = \gamma = \delta = 0$ *and* $A = B = D = E = 0$.

(c) $A = B = C = D = E = 0$ *and* $a = 0$.

The systems (a) and (c) are Darboux integrable with the first integral given by Theorem 1(b) with $n = 3$ and $m = 1$ where $f_5(\theta) = f_6(\theta) = 0$, respectively. The system (b) is Darboux integrable with the first integral given by Theorem 1(a) with $n = 3$ and $m = 1$. Consequently, these quintic systems with a linearly zero singular point at the origin are Darboux integrable, see [13].

References

1. Andronov A.A. Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues. *C.R. Acad. Sci. Paris*, **189**: 559-561 (1929).
2. Arnold V.I. and Ilyashenko Y.S. Dynamical Systems I, Ordinary Differential Equations, Encyclopaedia of Mathematical Sciences. Vol. 1, Springer Verlag (1988).
3. Andreev A. Investigation of the behavior of the integral curves of a system of two differential equations in the neighborhood of a singular points. Translation of AMS 8, 187-207 (1958).
4. Chavarriga J., Giacomini H., Giné J., and Llibre J. On the integrability of two dimensional flows. *J. Differential Equations*, **157**: 163-182 (1999).
5. Chavarriga J., Llibre J., and Sotomayor J. Algebraic solutions for polynomial vector fields with emphasis in the quadratic case. *Exposition. Math.*, **15**: 161-173 (1997).
6. Cairo L., Feix M.R., and Llibre J. Integrability and invariant algebraic curves for planar polynomial differential systems with emphasis on the quadratic systems. *Resenhas de Universidade de Sao Paulo*, **4**: 127-161 (1999).
7. Gasull A. and Llibre J. Limit cycles for a class of Abel equation. *SIAM J. of Math. Anal.*, **21**: 1235-1244 (1990).
8. Giacomini H., Giné J., and Llibre J. The problem of distinguishing between a center and a focus for nilpotent

and degenerate analytic systems. Preprint (2005).
9. Giné J. Sufficient conditions for a center at completely degenerate critical point. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **12**: 1659-1666 (2002).
10. Giné J. and Llibre J. Integrability and algebraic limit cycles for polynomial differential systems with homogeneous nonlinearities. *J. Differential Equations*, **197**: 147-161 (2004).
11. Giné J. and Llibre J. Integrability, degenerate centers and limit cycles for a class of polynomial differential systems. *Ibid.*, **194**: 116-139 (2003).
12. Giné J. Analytic integrability and characterization of centers for generalized nilpotent singular points. *Appl. Math. and Comput.*, **184**: 849-868 (2004).
13. García I.A. and Giné J. Non-algebraic invariant curves for polynomial planar vector fields. *Discrete Continuous Dynamical System*, **10**: 755-768 (2004).
14. Chavarriga J., García I.A., and Giné J. On integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **11**: 711-722 (2001).
15. Giacomini H., Llibre J., and Viano M. On the nonexistence, existence, and uniqueness of limit cycles. Nonlinearity **9**: 501-516 (1996).
16. Kamke E. Differentialgleichungen losungsmethoden und losungen, Col. Mathematik und ihre anwendungen, 18, Akademische Verlagsgesellschaft Becker und Erler Kom-Ges., Leipzig (1943).
17. Jouanoulou J.P. Equations de Pfaff algebriques. Lectures Notes in Mathematics, 708, Springer Verlag (1979).
18. Liénard A. Etude des oscillations entretenues. *Rev. Generale de I'Electricite*, **23**: 901-912 (1928).
19. Lins Neto A. On the number of solutions of the equations $dx/dt = \sum_{j=0}^{n} a_j(t)t^j$, $0 \le t \le 1$, for which $x(0) = x(1)$. *Invent. Math.*, **59**: 67-76 (1980).
20. LIoyd N.G. Small amplitude limit cycles of polynomial differential equations. In: Everitt W.N. and Lewis R.T. (Eds.), Ordinary Differential Equations and Operators. Lecture Notes in Mathematics, No. 1032, Springer-Verlag 346-357 (1982).
21. Manosa V. On the center problem for degenerate singular points of planar vector fields. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **12**: 687-707 (2002).
22. Pol van der B. On relaxation-oscillations. *Phil. Mag.*, **2**: 978-992 (1926).
23. Pontrjagin L.S. Uber autoschwingungs system, die den hamiltonschen nahe liegen. *Phyrikalische Zeitschrift der Sowjetunion*, **6**: 25-28 (1934).
24. Prelle M.J. and Singer M.F. Elementary first integrals of differential equations. *Trans. Amer. Math. Soc.*, **279**: 215-229 (1983).
25. Singer M.F. Liouvillian first integrals of differential equations. *Ibid.*, **333**: 673-688 (1992).